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Modulational Interactions of Two Monochromatic Waves and Packets of Random Waves

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Abstract

The modulational instability of Langmuir waves in unmagnetised plasmas is reviewed for the cases when a pump consist of two monochromatic or a large number of random modes. It is demonstrated that the correct theory for the modulational instability operates with 'renormalised' equations for the linear dielectric function as well as for the effective third-order plasma response. This renormalisation is due to so-called interference terms. The appearance of interference terms is a specific feature of the multi-mode modulational instability in comparison with the well-known instability of a single mode. All calculations use a simple and universal formalism including new methods developed for description of the modulational effects in arbitrary media. The modulational instability of two pump Langmuir modes is considered for the case of comparatively small instability rates, when 'renormalised' expressions for linear and nonlinear plasma polarisation responses provide the maximum effect on the instability development. For instabilities of the broad spectra of random waves, the integral equations are presented for perturbations of wave field correlation functions. In the description of the modulational instability of random wave packets these equations play the same role as the set of coupled equations for the fields of modulational perturbations in the case of two monochromatic pumps. Rates and thresholds of the instabilities are found in various limits.

1. Introduction

The modulational self-action of a wave propagating in a nonlinear medium is one of the fundamental effects which influences many processes in the medium, such as development and evolution of turbulence, soliton formation and self-organisation. In a plasma, modulational effects (first described by Vedenov and Rudakov 1965, who introduced the term 'modulational instability', and Gailitis 1964, 1965) play a significant role: for example, the modulational instability of Langmuir waves, which develops in a weak-turbulent state (when the phases of turbulent oscillations are random), leads to formation of strong Langmuir turbulence (Rudakov and Tsytovich 1978; Thornhill and ter Haar 1978; Goldman 1984; Shapiro and Shevchenko 1984) characterised by (chaotic) interaction of entities (solitons, collapsing caverns, etc.) containing strongly correlated wave packets.

Experimental evidence of the modulational interaction has been found in laboratory, ionospheric and space plasmas. Soliton formation in plasmas due to development of the modulational instability has been observed in many experiments (see Wong and Quon 1975; Ikezi *et al.* 1976; Antipov *et al.* 1978, 1979, 1980, 1981; Cheung *et al.* 1989). In the above experiments electron beams have been used to generate Langmuir waves in plasmas and to create soliton-like structures. Self-compression of the Langmuir waves has been detected by Wong and Quon (1975) and Wong (1979). Formation of ionisation-wave solitons due to the modulational instability of the so-called ionisation waves (known as discharge striations) which exist in glow discharge, has been observed in the experiment by Ohe and Hashimoto (1984). Measurements of the parameters of the solar wind carried out by the Voyager-1 and -2 spacecraft on the flight to Jupiter in 1979 gave evidence (Gurnett and Anderson 1977; Gurnett *et al.* 1981) of the formation of Langmuir envelope solitons (i.e. evidence of the modulational instability) in the solar wind plasma. Furthermore, numerous stimulated scattering phenomena which directly connect with development of modulational instabilities have been observed in laboratory as well as ionospheric modification experiments (see Wong and Taylor 1971; Carlson *et al.* 1972; Kim *et al.* 1974; Wong and Quon 1975; Wong *et al.* 1981; Migulin and Gurevich 1985; Fejer *et al.* 1989; Scales and Kinter 1990; Stubbe *et al.* 1992; Stocker *et al.* 1992).

For many situations consideration of modulational interactions of different plasma modes allows us to account for data obtained in experiments. For example, in many laboratory experiments, such as in controlled fusion installations (Golant and Fedorov 1986), or in interactions of external radio-frequency or laser radiation with plasmas (Popel and Tsytovich 1990*a*, 1990*b*, 1991) where the external magnetic field is present, the important nonlinear process is the modulational interaction of the so-called lower-hybrid waves. In controlled fusion devices when current drive (Fisch 1987; Kolesnichenko *et al.* 1989) is generated by the lower-hybrid waves, the effects of the modulational instability of these waves for comparatively high pump level can provide the filling of the so-called 'spectral gap' by the lower-hybrid waves, providing the solution of an important problem in current drive theory (Popel and Tsytovich 1992; Tsytovich *et al.* 1992). The modulational interactions of the lower-hybrid waves are also important in astrophysical applications when flows of ions across the external magnetic field are present (e.g. the problem of the structure of the transverse shock wave where the flow is formed by the ions reflected from the shock wave front). Such flows can excite the lower-hybrid waves; the modulational interactions of them result in a decrease of their phase velocities and therefore in their effective interaction with ions and electrons (Marchenko 1989). Plasma particles can be accelerated as a result of such interactions, the maximum energy of the accelerated electrons exceeding the energy of the flow ions.

Formation of soliton-like structures in the region of a lower-hybrid resonance as a result of modulational interactions has been observed in a series of experiments (see Gekelman and Stenzel 1975; Gromov *et al.* 1976, 1978).

Consideration of the modulational processes in inhomogeneous plasmas enables us to explain some observations of the plasma in the Earth's environment. The modulational interaction is the most important nonlinear process for the oscillations excited as a result of the lower-hybrid drift instability initiated by an electron current (see Sotnikov *et al.* 1981; Bingham *et al.* 1991; Popel *et al.* 1994*a*). This interaction determines the saturation mechanism for the lower-hybrid drift instability, which is associated with the cascade of the waves with high phase velocities (along the external magnetic field) to the waves with lower phase velocities which are parallel to the magnetic field and close to the electron thermal velocity. This saturation mechanism provides the effective electron collision frequency in inhomogeneous plasmas and permits an estimate of the width of the Earth's magnetopause which is in satisfactory agreement with observations (Sotnikov *et al.* 1981). Consideration of the modulational instability of the lower-hybrid drift waves for the case of the AMPTE experiment (Bingham *et al.* 1991) (which involved the release of barium atoms in the solar wind) is essential for an explanation of the magnetic structures, particle spectra, and electric field amplitudes observed in the experiment. The modulational instability can also be important in experiments on critical ionisation velocity phenomena (see e.g. Torbert and Newell 1986; Swenson *et al.* 1990; Brenning *et al.* 1991).

Thus, we see that the modulational processes also play a significant role in the explanation and interpretation of the results of experiments and observations carried out in laboratory and space plasmas, as well as for plasmas in the Earth's environment.

Historically, the modulational instability of a single monochromatic Langmuir pump has been extensively researched for the simple case of homogeneous unmagnetised collisionless plasmas (see Rudakov and Tsytovich 1978; Thornhill and ter Haar 1978; Goldman 1984; Shapiro and Shevchenko 1984), as well as for the more complicated cases of magnetised, inhomogeneous, collisional, etc., plasmas (see e.g. Veriaev and Tsytovich 1979; Tsytovich *et al.* 1992; Vladimirov *et al.* 1992, 1993, 1994; Vladimirov and Krivitsky 1993; Popel *et al.* 1994*a*). As a result the instability of a single monochromatic pump is now fairly well investigated. It was demonstrated that the nature of the instability is the nonresonant coupling of the Langmuir pump with its sidebands via low-frequency perturbations, which in the resonant limit correspond to forced ion sound waves. Most of the theories on stimulated scattering rely on the ponderomotive force nonlinearity where low-frequency density modulations appear as a result of ponderomotive force effects. In this case, waves refract into density wells in a plasma, which are regions of high refractive index. The ponderomotive force pushes aside the plasma, deepening the density well, raising the refractive index, and leading to further focusing of waves into the region.

The broad wave packet can be modelled (in some approximation) by consideration of two pump monochromatic waves. However, two monochromatic modes in general are not the exact solution of the nonlinear equations. At the same time, investigation of the modulational interaction of broad wave packets is interesting in the description of the transition from weak turbulence, where the broad spectra automatically appear, to strongly turbulent states. In the weak turbulence theory, turbulent spectra are steady-state solutions of the corresponding nonlinear equations. Thus the spectra of random waves are of great interest, since they are the exact solutions and their stationarity can be provided by the processes of nonlinear pumping, spectral flow and, finally, damping of the oscillations.

The problem of instability thresholds is also interesting because it gives criteria for the system transformation from weak to strong turbulent states. Gailitis (1964, 1965) was the first to obtain the threshold for modulational instability; however, an explicit expression for the growth rate of the instability was not derived. This was done by Vedenov and Rudakov (1965) where the instability of the gas of plasmons, i.e. of the packet of random waves was investigated. However, the growth rates obtained by Vedenov and Rudakov describe the case of a narrow packet; moreover, the growth rates which have been found near the threshold there are not correct. Thus the correct expression for the growth rate of the modulational instability was actually obtained by Vedenov and Rudakov only for the case when the packet of random waves gives the same result as a sufficiently narrow packet of regular waves. The correct growth rate for the case of the broad wave spectrum was found by Tsytovich (1970) (see also Komilov *et al.* 1979) for the one-dimensional situation (pump wave and modulational perturbations propagate in the same directions). A generalisation of these results has been given by Komilov *et al.* (1976, 1978) for the case of three-dimensional perturbations. Thus the characteristic effects for the broad spectrum of random waves are given in Gailitis (1964, 1965), Tsytovich (1970), and Komilov *et al.* (1976, 1978, 1979), although the article by Vedenov and Rudakov (1965) is formally devoted to the instability of the packet of random waves. Agreement of the results by Vedenov and Rudakov with the results for monochromatic pumping

occurs for the following reason: they have used the approach and equations which are valid only if the inequality $\gamma \gg \delta\omega$ is fulfilled (where $\delta\omega$ is the spectral width of the packet of random waves). For two (and more) monochromatic pump waves the modulational interaction is described by a system of equations which consists of an infinite number coupled with each other (Vladimirov and Tsytovich 1990). For the case of broad wave spectra this system is transformed to an integral equation (see Tsytovich 1970; Komilov *et al.* 1976, 1978, 1979; Popel *et al.* 1994*b*).

The reason why the modulational instability of broad spectra is less understood than that of a single monochromatic pump is in the complexity of its investigation. First, there are fundamental difficulties. Indeed, when we consider modulational *instabilities* (but not more complicated modulational *interactions*) we imply the existence of the initial steady state. This steady state can refer to the exact nonlinear solution of equations describing the modulational instability. For example, a Langmuir soliton or a monochromatic wave (for the latter we must take into account the nonlinear frequency shift) can constitute this solution (Rudakov and Tsytovich 1978; Thornhill and ter Haar 1978; Goldman 1984). However, for some wave spectra the situation is more complicated: these spectra do not necessarily need to be an exact solution of the corresponding equations. As we already mentioned, two monochromatic pumps are not in general an exact solution of the nonlinear equations. This problem, however, does not arise for the modulational interaction of broad random wave spectra which can automatically appear as a result of the development of weak turbulence.

Furthermore, a description of the modulational instability of broad spectra itself is considerably more complex than that of a monochromatic pump. Even in the case of two pump waves (see Vladimirov and Tsytovich 1990, 1992, 1993*a*, 1993*b*, 1993*c*; Tsytovich and Vladimirov 1992), the instability is described by a set, which consists of an infinite number of equations coupled with each other. For its solution one should introduce simplifying assumptions. It is natural that for the case of broad wave spectra the situation is further complicated, and the modulational instability is described by integral equations which are a generalisation of the above set of coupled equations.

The purpose of this review is to present a formalism as well as theoretical investigations based on this formalism to describe the modulational instability of two monochromatic waves as well as packets of random Langmuir waves. The nonlinear processes considered can be responsible for numerous stimulated scattering phenomena in space and astrophysical plasmas (collisionless shocks, pulsar emission, solar bursts, solar flares, solar wind, cosmic rays, etc.), in plasmas of the Earth's ionosphere and planetary atmospheres, as well as in laboratory plasmas (nuclear fusion devices, laser plasmas, plasmas in modern particle accelerators, etc.).

Modulational interactions will be considered on the basis of a simple and universal approach developed for description of the modulational effects in arbitrary media. This theory enables us to visualise the physics of the modulational processes and to incorporate various auxiliary effects self-consistently.

The structure of this review is as follows: in Section 2 the general nonlinear formalism to calculate modulational effects in arbitrary nonlinear media is presented and the relevant formulas are derived. The fundamental features

of the modulational interaction of two pump waves (e.g. appearance of the interference terms) are discussed in Section 3. In Section 4, the interaction of two monochromatic pumps with a large frequency gap (which significantly exceeds the characteristic modulation frequency) is considered, and the corresponding general equations for arbitrary nonlinear media are derived. In Section 5, the rates of the modulational instability of two pump Langmuir waves are calculated. The general theory of modulational interactions of broad packets of random waves is presented in Section 6. In Section 7, the rates of the instability of packets of random Langmuir waves are found for different cases.

The formulas throughout this paper are written in cgs units.

2. General Nonlinear Formalism

The general formalism suitable for description of the effects of strong phase correlations (the modulational instability, solitons, etc.) and the effects of weak phase correlations (such as decays, scattering) was introduced some time ago (see e.g. Rudakov and Tsytovich 1978; Tsytovich 1977). The formalism developed is applicable to the case of a sufficiently weak nonlinearity

$$\epsilon = W/nT \ll 1\,, \tag{1}$$

where W is the energy density of plasma oscillations, n is the plasma density, and T is the plasma temperature in energy units (which by definition incorporates the Boltzmann factor k_B).

To obtain the evolution equations, we use the Poisson equation (for longitudinal waves; for transverse waves the complete set of Maxwell equations should be used) in which terms up to the third order in the electric field $\mathbf{E}$ are taken into account. In Fourier components

$$A_k = \frac{1}{(2\pi)^4}\int A(\mathbf{r},t)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})d\mathbf{r}dt \tag{2}$$

we have

$$\varepsilon_k E_k = \int S_{kk_1k_2}E_{k_1}E_{k_2}d^{(2)} + \int \Sigma_{kk_1k_2k_3}E_{k_1}E_{k_2}E_{k_3}d^{(3)}, \tag{3}$$

where ω and $\mathbf{k}$ are the frequency and the wavevector, $\varepsilon_k = \varepsilon(\omega,\mathbf{k})$ is the linear dielectric permittivity, $d^{(2)} = \delta(k-k_1-k_2)dk_1dk_2$, $d^{(3)} = \delta(k-k_1-k_2-k_3)dk_1dk_2dk_3$, $k = \{\omega,\mathbf{k}\}$, and $S_{kk_1k_2}$ and $\Sigma_{kk_1k_2k_3}$ are the nonlinear responses of the second- and third-order-in-field $\mathbf{E}$. The expressions for the nonlinear responses can be obtained using plasma kinetic and/or hydrodynamical equations (Rudakov and Tsytovich 1978; Tsytovich 1977).

As a consequence of inequality (1), equation (3) determines the peaks near the frequencies $\pm\omega_{\mathbf{k}}^{\sigma}$ in the spectrum of E_k (here $\omega_{\mathbf{k}}^{\sigma}$ is the linear dispersion of plasma oscillations of the kind σ). Nonlinear interactions lead to smoothing of the linear peaks at frequency $\Delta\omega_{\mathbf{k}}^{\sigma}$ which is much less than $\omega_{\mathbf{k}}^{\sigma}$. Thus, in addition

to the small parameter defined in (1) there is another given by

$$\frac{\Delta\omega_{\mathbf{k}}^{\sigma}}{\omega_{\mathbf{k}}^{\sigma}} \ll 1. \tag{4}$$

Note that strong inequality (4) is closely connected with inequality (1). However, the magnitude $\Delta\omega_{\mathbf{k}}^{\sigma}$ is determined not only by nonlinear effects, but by the linear wave dispersion.

The presence of the peaks at frequencies close to $+\omega_{\mathbf{k}}^{\sigma}$ and $-\omega_{\mathbf{k}}^{\sigma}$ in the spectrum of the electric field allows us to distinguish the positive-frequency E_k^+ and negative-frequency E_k^- harmonics. In the low-frequency region ($|\Delta\omega_{\mathbf{k}}^{\sigma}| \ll \omega_{\mathbf{k}}^{\sigma}$) we have the field of forced (beat) oscillations E_k^v, as well as the analogous fields on the positive and negative double-frequency harmonic, E_k^{++} ($|\Delta\omega_{\mathbf{k}}^{\sigma} - 2\omega_{\mathbf{k}}^{\sigma}| \ll \omega_{\mathbf{k}}^{\sigma}$) and E_k^{--} ($|\Delta\omega_{\mathbf{k}}^{\sigma} + 2\omega_{\mathbf{k}}^{\sigma}| \ll \omega_{\mathbf{k}}^{\sigma}$). Below, we call *virtual waves* the fields of the forced oscillations, stressing that these are not the *real* waves which can propagate in the plasma considered (this is analogous to terminology accepted in quantum electrodynamics). Keeping in mind that the condition of frequency synchronism should be fulfilled, we can obtain from (3) the equations for the fields E_k^+, E_k^v and E_k^{++}. Thus, the equation for the positive-frequency part E_k^+ of the wave field is given by

$$\begin{aligned}\varepsilon_k E_k^+ = {} & 2\int S_{kk_1k_2} E_{k_1}^+ E_{k_2}^v d^{(2)} + 2\int S_{kk_1k_2}^{++} E_{k_1}^- E_{k_2}^{++} d^{(2)} \\ & + 2\int \Sigma_{kk_1k_2k_3} E_{k_1}^+ E_{k_2}^+ E_{k_3}^- d^{(3)} + \int \Sigma_{kk_1k_2k_3}^{++} E_{k_1}^- E_{k_2}^+ E_{k_3}^+ d^{(3)} .\end{aligned} \tag{5}$$

Note that in equation (5), for convenience, we have introduced the symmetric second-order responses

$$S_{kk_1k_2} \equiv \tfrac{1}{2}\left(S_{kk_1k_2} + S_{kk_2k_1}\right) , \tag{6}$$

as well as the third-order responses which are symmetric on the two last indices,

$$\Sigma_{kk_1k_2k_3} \equiv \tfrac{1}{2}\left(\Sigma_{kk_1k_2k_3} + \Sigma_{kk_1k_3k_2}\right) . \tag{7}$$

The r.h.s. of (5) contains terms corresponding to the charge density which are quadratic- and cubic-in-field. The equation for the low-frequency virtual field E_k^v has the following form (note that only a second-order nonlinear charge density should be taken into account here):

$$\varepsilon_k E_k^v = 2\int S_{kk_1k_2} E_{k_1}^+ E_{k_2}^- d^{(2)}. \tag{8}$$

The equation for the virtual field E_k^{++} at double-harmonic frequency is given by

$$\varepsilon_k E_k^{++} = \int S_{kk_1k_2} E_{k_1}^+ E_{k_2}^+ d^{(2)} . \tag{9}$$

Thus, from (5)–(9) we finally obtain

$$\varepsilon_k E_k^+ = \int \Sigma^{\text{eff}}_{kk_1k_2k_3} E^+_{k_1} E^+_{k_2} E^-_{k_3} d^{(3)} + \int \Sigma^{++,\text{eff}}_{kk_1k_2k_3} E^-_{k_1} E^+_{k_2} E^+_{k_3} d^{(3)}\,, \tag{10}$$

where

$$\Sigma^{\text{eff}}_{kk_1k_2k_3} = 2\Sigma_{kk_1k_2k_3} + 4S_{k-k_1k_2k_3} g_{k-k_1} S_{kk_1k-k_1}\,, \tag{11}$$

$$\Sigma^{++,\text{eff}}_{kk_1k_2k_3} = \Sigma^{++}_{kk_1k_2k_3} + 2S^{++}_{k-k_1k_2k_3} g^{++}_{k-k_1} S^{++}_{kk_1k-k_1}\,. \tag{12}$$

In (11) and (12), g_k is the Green function of the longitudinal electric field,

$$g_k = \frac{1}{\varepsilon_k}, \tag{13}$$

whose frequency corresponds to that of the virtual wave considered.

To find the dynamical evolution equation for the slowly varying amplitude of the high-frequency field $\mathbf{E}(\mathbf{r},t)$, we use the inverse Fourier transform

$$\mathbf{E}(\mathbf{r},t) = \int E_k^+ \frac{\mathbf{k}}{|\mathbf{k}|} \exp(i\omega_0^\sigma t - i\omega_{\mathbf{k}}^\sigma t + i\mathbf{k}\cdot\mathbf{r}) \frac{d\omega d\mathbf{k}}{(2\pi)^4}\,, \tag{14}$$

where $\omega_0^\sigma = \omega_{\mathbf{k}_0}^\sigma$ is a characteristic frequency (for a fixed $\mathbf{k}_0$) in the spectrum of the high-frequency field. Therefore we can rewrite (10) in the form

$$\nabla\cdot\left[i\left.\frac{\partial\varepsilon_k}{\partial\omega}\right|_{\omega=\omega_0^\sigma,\mathbf{k}=\mathbf{k}_0} \frac{\partial\mathbf{E}(\mathbf{r},t)}{\partial t} + \hat{D}\mathbf{E}(\mathbf{r},t)\right] = i\int |\mathbf{k}|\Sigma^{\text{eff,full}}_{kk_1k_2k_3} E^+_{k_1} E^+_{k_2} E^-_{k_3}$$
$$\times\exp(i\omega_0^\sigma t - i\omega_{\mathbf{k}}^\sigma t + i\mathbf{k}\cdot\mathbf{r}) d^{(3)} \frac{d\omega d\mathbf{k}}{(2\pi)^4}\,, \tag{15}$$

where

$$\Sigma^{\text{eff,full}}_{kk_1k_2k_3} = \Sigma^{\text{eff}}_{kk_1k_2k_3} + \Sigma^{++,\text{eff}}_{kk_3k_1k_2}\,, \tag{16}$$

and $\hat{D}$ is an operator, which is determined by the dispersion $\omega_{\mathbf{k}}^\sigma$. Equation (15) is the dynamical evolution equation for the slowly varying amplitude of the high-frequency field.

Equations (10) and (15) have been derived under the assumption that all fields in the nonlinear process are longitudinal. The condition, under which the low-frequency virtual fields can be considered longitudinal, is given by (see e.g. Ginzburg 1975)

$$|\omega_-|^2 \ll |\mathbf{k}_-|^2 c^2\,, \tag{17}$$

where $\omega_- = \omega_{\mathbf{k}}^\sigma - \omega_{\mathbf{k}_1}^\sigma$, $\mathbf{k}_- = \mathbf{k} - \mathbf{k}_1$. For further reference, we present here also the condition when interaction via a virtual wave at double-harmonic frequency

can be neglected (Vladimirov and Tsytovich 1988, 1991; Khakimov and Tsytovich 1976):

$$|\omega_-| \ll \omega_{pi}\,, \tag{18}$$

where $\omega_{pi} = (4\pi n e^2/m_i)^{\frac{1}{2}}$ is the ion plasma frequency (we assume that the ions of mass m_i have the charge e).

Thus, below we consider only interactions of longitudinal waves and assume that conditions (17) and (18) are fulfilled. Therefore, all virtual waves which we take into account are also longitudinal, and we neglect all higher plasma nonlinearities such as electron, relativistic, etc. (Vladimirov and Tsytovich 1988, 1991; Khakimov and Tsytovich 1976; Kuznetsov 1976). In particular, interactions via double-harmonic frequency virtual waves as a rule will not be considered. This means that for the effective third-order plasma response we use (unless this is specified) expression (11).

3. Interaction of Two Monochromatic Pumps

In this section, general questions of modulational interactions of two monochromatic pump waves are considered. One of the most important features which qualitatively distinguishes the two-pump modulational interaction from the modulational instability of one monochromatic pump mode is the appearance of the so-called interference terms. These terms are due to nonlinear coupling of two pump waves which produces low-frequency density perturbations at the beat frequency. Their appearance leads to a non-stationary distribution even in the zero approximation. In the equations for modulated perturbations, coupling between modulational fields and the low-frequency density variations leads to an infinite system of coupled equations which cannot be solved in its general form without simplifying assumptions.

3.1 Interference Terms

The very formulation of the problem of the modulational instability of two waves requires some refinement (for definiteness, we will discuss plasma waves in a homogeneous and isotropic plasma). Before we begin the stability analysis of some state of a physical system, we need to specify this state. Moreover, it should be a steady state. Note that a stability analysis can also be carried out for so-called quasi-steady systems, in which the (expected) rise time of a possible instability is much shorter than the time scale of variations in the state of the system without this instability. As is well known (Rudakov and Tsytovich 1978; Thornhill and ter Haar 1978; Goldman 1984), the nonlinear cubic equation (10) (for simplicity, we do not take into account all electron nonlinearities including interaction through a double-harmonic frequency virtual wave) has an equilibrium solution in the case of a single monochromatic pump wave (in this case we will not take up the natural questions of the history of the 'switching on' the wave fields—see Rudakov and Tsytovich 1978). The field amplitude of this stationary solution is spatially uniform and oscillates in time:

$$E^+(t) = E_0 \exp\left(i\,\frac{\omega_{pe}t}{2}\Sigma_0^{\text{eff}}|E_0|^2\right)\,. \tag{19}$$

For simplicity, we have written this equation for $\mathbf{k}_0 = 0$; however, it is clear that the expression for the equlibrium solution in the case $\mathbf{k}_0 \neq 0$ will not differ in any fundamental way.

In a study of the stability of small deviations from this solution, we obtain the usual dispersion relations which yield the growth rate of the modulational instability. In the particular case (which takes place for plasma waves)

$$\Sigma_0^{\text{eff}} = \text{const}(\mathbf{k}, \omega)|_{\mathbf{k}=\mathbf{k}_0, \omega=\omega_{pe}}\,, \tag{20}$$

the change caused in the wave frequency by the plasma nonlinearity can be 'assigned to' a renormalised density (which is the same in all plasma volume occupied by the wave)

$$\tilde{n}_e = n_e \left(1 + \frac{\delta n}{n_e}\right), \tag{21}$$

where

$$\frac{\delta n}{n_e} = -\Sigma_0^{\text{eff}} |E_0|^2\,. \tag{22}$$

The steady-state (or quasi-steady-state) nature of the density perturbation δn justifies our definition of solution (19) as a 'steady-state' (or 'quasi-steady-state') of the system.

Let us examine qualitatively the modulational instability of two monochromatic plasma waves with frequencies ω_0 and ω_1, $\omega_0 \neq \omega_1$. In this case, so-called interference terms (Vladimirov and Tsytovich 1990) arise in the expression for δn [since the superposition principle does not hold for the nonlinear equation (3)]:

$$\frac{\delta n}{n_e} = -\Sigma_0^{\text{eff}} |E_0|^2 - \Sigma_1^{\text{eff}} |E_1|^2 - \Sigma_{0,1}^{\text{eff}} E_0 E_1^* \exp(-i\delta\omega t) - \Sigma_{1,0}^{\text{eff}} E_1 E_0^* \exp(i\delta\omega t)\,, \tag{23}$$

where $\delta\omega \equiv \omega_0 - \omega_1$. It is thus generally not possible to even correctly formulate the problem of the stability of two monochromatic waves with respect to modulational perturbations, since the spectrum specified in this manner is not steady, with a restructuring time scale of order $\delta\omega^{-1}$. However, in certain special cases such an analysis can be carried out (Vladimirov and Tsytovich 1990, 1992, 1993*a*, 1993*b*, 1993*c*).

For example, since the nonlinear responses $\Sigma_{0,1}^{\text{eff}}$ and $\Sigma_{1,0}^{\text{eff}}$ depend on the angle between the propagation directions of the plasma waves with ω_0 and ω_1 (for electromagnetic waves, this would be a dependence on the polarisations of the waves with ω_0 and ω_1), in the case $\mathbf{E}_0 \cdot \mathbf{E}_1 = 0$ we have $\Sigma_{0,1}^{\text{eff}} = \Sigma_{1,0}^{\text{eff}} = 0$. Then, the interference terms drop out of (23), and the formulation of the problem of the modulational instability of two monochromatic waves is not fundamentally different from that for a single monochromatic wave.

We might also assume that the phases of the waves ω_0 and ω_1 are random and that only the density variation averaged over phase, $\langle \delta n / n_e \rangle$, influences the interaction of these waves. In this case, however, we could no longer regard

each of the high-frequency waves as monochromatic, because each would be smeared over a frequency interval $\tau_{\rm corr}^{-1}$, where $\tau_{\rm corr}$ is the time scale of the phase disruption of the wave. In this case the problem essentially reduces to a study of the interaction of two wave packets, but under the condition $\tau_{\rm corr}^{-1} \ll \delta\omega$ and $\tau_{\rm corr}^{-1} \ll \gamma^{\rm mod}$, where $\gamma^{\rm mod}$ is the instability growth rate, the waves can be regarded as 'nearly monochromatic'. Finally, the exact expression (23) might be replaced by some approximate expression in a situation in which the instability develops rapidly, i.e. with $\gamma^{\rm mod} \gg \delta\omega$. Assuming then that we are interested in the behaviour of the system at times $\tau \approx (\gamma^{\rm mod})^{-1}$, we can replace the exponential function in (23) by unity and study the instability of the quasi-steady spectrum specified in this manner. It is clear on the basis of general considerations that the nature of the modulational instability would be analogous to that of a single 'slightly smoothed' mode.

All the examples listed above are valid for nondecay situations, in which $\delta\omega$ does not coincide with the frequency of a natural low-frequency mode of the medium (in the case at hand, these would be ion-acoustic waves; for two electromagnetic waves they might be electron plasma waves in addition to ion-acoustic waves). It is well known (Tsytovich 1971) that two plasma waves excite ion sound if

$$|\delta\omega| > \omega_s \approx |\delta\mathbf{k}| v_s \,, \tag{24}$$

where $\delta\mathbf{k} = \mathbf{k}_0 - \mathbf{k}_1$, $v_s = (T_e/m_i)^{\frac{1}{2}}$ is the speed of (collisionless) ion sound, and m_i is the ion mass. In this case, it is also necessary to introduce the fields of the resonantly excited sound. In this formulation, the original state is not a steady state simply because, at resonance, the amplitude of the low-frequency (ion-acoustic) wave which is excited depends on the time by virtue of the effect of the high-frequency (plasma) waves. If we set

$$\gamma^{\rm mod} \tau_{\rm dec} \gg 1 \,, \tag{25}$$

however, where the time scale of the decay $l \to l + s$ is (Vladimirov and Tsytovich 1990)

$$\tau_{\rm dec} \sim \omega_{pe}^{-1} \frac{n_e T_e}{W} \frac{m_i}{m_e} \mathbf{k}_0^2 r_{De}^2 \tag{26}$$

(we are assuming $|\mathbf{k}_0| \sim |\mathbf{k}_1|$ and $|E_0| \sim |E_1|$), then the initial state can be regarded as quasi-steady, at least from this point of view. In this case it is necessary either to derive a theory of modulational interactions involving pre-existing ion sound or, under the assumption that the ion-acoustic waves have not yet had time to appear, go over to a nonresonant analysis.

3.2 Steady-state Solutions

Under the assumption that decays are forbidden, let us examine the steady-state solutions for two monochromatic waves. Thus, we suppose that the pump field $\mathbf{E}$ is a sum of wave fields with frequencies $\omega_0 \equiv \omega(\mathbf{k}_0)$ and $\omega_1 \equiv \omega(\mathbf{k}_1)$ (note that $\delta\omega \equiv \omega_0 - \omega_1 \ll \omega_{0,1}$), and amplitudes $\mathbf{E}_{0,1}$. Owing to the real character of the fields $\mathbf{E}_{0,1}$, each of their spectra contains two lines: near $+\omega_{0,1}$ and near $-\omega_{0,1}$.

We therefore can distinguish positive-frequency ($\mathbf{E}_k^+$) and negative-frequency ($\mathbf{E}_k^-$) parts of the Fourier components of these two pump waves in the following way [compare with (14)]:

$$\frac{\mathbf{k}}{k}E_{0,1;k}^{\pm} = \mathbf{E}_{0,1;k}^{\pm} = \frac{1}{(2\pi)^4}\int d\mathbf{r}dt\mathbf{E}_{0,1}(\mathbf{r},t)\exp\left(\mp i\omega_{0,1}t + i\omega t - i\mathbf{k}\cdot\mathbf{r}\right) . \quad (27)$$

In the zeroth approximation we set

$$E_k^{\pm} = \pm E_0^{(*)}\delta(k \mp k_0) \pm E_1^{(*)}\delta(k \mp k_1) , \quad (28)$$

where the asterisk denotes complex conjugation, and the minus sign of the negative-frequency component is connected with the longitudinal character of the field considered.

After substitution of (28) into (10) and neglecting all electron, and other plasma nonlinearities, we find

$$\begin{aligned}\varepsilon_0 E_0\delta(k-k_0) + \varepsilon_1 E_1\delta(k-k_1) = \\ -\Sigma_{0,0,-0}^{\text{eff}}|E_0|^2 E_0\delta(k-k_0) - \Sigma_{0,1,-1}^{\text{eff}}|E_1|^2 E_0\delta(k-k_0) \\ -\Sigma_{1,0,-1}^{\text{eff}}|E_1|^2 E_0\delta(k-k_0) - \Sigma_{0,0,-1}^{\text{eff}}(E_0)^2 E_1^*\delta(k-2k_0+k_1) + (0\leftrightarrow 1) , \quad (29)\end{aligned}$$

where $\Sigma_{i,j,l}^{\text{eff}} \equiv \Sigma_{k_{i+j+l},k_i,k_j,k_l}^{\text{eff}}$, $\varepsilon_i \equiv \varepsilon_{k_i}$, and $i,j,l = 0,1$. The last term on the right side of (29) (as well as the term found from it through the interchange $0 \leftrightarrow 1$) is an interference term and corresponds to beats in the density variation (23). In the case

$$\Sigma_{0,0,-1}^{\text{eff}} = \Sigma_{1,1,-0}^{\text{eff}} = 0 \quad (30)$$

there are no such beats, and from equation (29) we find an equation for the steady state of the wave E_0,

$$\varepsilon_0 + \Sigma_{0,0,-0}^{\text{eff}}|E_0|^2 + \Sigma_{1,0,-1}^{\text{eff}}|E_1|^2 + \Sigma_{0,1,-1}^{\text{eff}}|E_1|^2 = 0 , \quad (31)$$

as well as, via the interchange $0 \leftrightarrow 1$, the corresponding equation for E_1.

We now set $\mathbf{E}_0 \cdot \mathbf{E}_1 = 0$. In other words, we assume that the waves $\mathbf{E}_0$ and $\mathbf{E}_1$ propagate perpendicularly to one another. Using the approximate expression for the effective third-order response (Rudakov and Tsytovich 1978; Vladimirov and Tsytovich 1990)

$$\Sigma_{1,2,3}^{\text{eff}} = -\frac{1}{4\pi n_e T_e}\frac{\mathbf{k}\cdot\mathbf{k}_1}{|\mathbf{k}\parallel\mathbf{k}_1|}\frac{\mathbf{k}_2\cdot\mathbf{k}_3}{|\mathbf{k}_2\parallel\mathbf{k}_3|}\frac{(\mathbf{k}-\mathbf{k}_1)^2 v_s^2}{(\omega-\omega_1)^2-(\mathbf{k}-\mathbf{k}_1)^2} , \quad (32)$$

we can easily see that under the condition $\mathbf{k}_1 \parallel \mathbf{k}_2 \perp \mathbf{k}_3$ this response is equal to zero. Moreover, as it has been demonstrated by Vladimirov and Tsytovich (1990), the above statement is correct because, even if we incorporate the electron nonlinearities (Khakimov and Tsytovich 1976; Vladimirov and Tsytovich

1988, 1991) in $\Sigma^{\text{eff}}_{1,2,3}$, its value will be zero if two Langmuir waves propagate perpendicular to each other. However, it would hardly be possible to make expressions like (32) exactly equal to zero. In particular, the small terms of next higher order in the electron nonlinearities, which are proportional to (Vladimirov and Tsytovich 1991)

$$\mathbf{k}_0^4 r_{De}^4 \frac{|E_0|^2}{4\pi n_e T_e}, \tag{33}$$

may turn out to be nonzero. Their contribution, however, can be assumed small enough to be ignored (for example, at the level of the terms of fifth order in the field amplitude).

When $\mathbf{k}_0 \perp \mathbf{k}_1$ we have along with (30)

$$\Sigma^{\text{eff}}_{1,0,-1} \approx \Sigma^{\text{eff}}_{0,1,-0} \approx 0\,, \tag{34}$$

which is correct also within electron nonlinearities. Consequently, the contribution from, for example, the response $\Sigma^{\text{eff}}_{1,0,-1}|E_1|^2$, which is of order

$$\mathbf{k}_1^2 r_{De}^2 \frac{|E_1|^2}{4\pi n_e T_e} \ll \frac{|E_1|^2}{4\pi n_e T_e}, \tag{35}$$

can definitely be ignored in comparison with, say,

$$\Sigma^{\text{eff}}_{0,1,-1}|E_1|^2 \sim \frac{|E_1|^2}{4\pi n_e T_e}\,. \tag{36}$$

In the zeroth approximation, the solutions are thus

$$\varepsilon_0 + \Sigma^{\text{eff}}_{0,0,-0}|E_0|^2 + \Sigma^{\text{eff}}_{0,1,-1}|E_1|^2 = 0\,, \tag{37}$$

$$\varepsilon_1 + \Sigma^{\text{eff}}_{1,1,-1}|E_1|^2 + \Sigma^{\text{eff}}_{1,0,-0}|E_0|^2 = 0\,. \tag{38}$$

Because of the structure of the responses we have (Vladimirov and Tsytovich 1990)

$$\Sigma^{\text{eff}}_{0,0,-0} = \Sigma^{\text{eff}}_{0,1,-1} = \Sigma^{\text{eff}}_{1,0,-0} = \Sigma^{\text{eff}}_{1,1,-1} = 2\Sigma^{\text{eff}}_0 \equiv \frac{1}{4\pi n T_e}\,. \tag{39}$$

In general, the equations (39) depend on the way in which the fields E_0 and E_1 are 'turned on', holding if the two fields are turned on in the same way (quasi-statically, for example). Under conditions (39), the influence of nonlinear effects reduces in zeroth approximation to a simple density renormalisation

$$\frac{\delta n_0}{n} = -2\Sigma^{\text{eff}}_0|E_0|^2 - 2\Sigma^{\text{eff}}_0|E_1|^2\,, \tag{40}$$

which is the same through all the plasma volume and the same for each wave.

3.3 Modulational Matrix

Now, we examine the stability of a steady-state solution of equations (37) and (38) with respect to small perturbations of the wave field. That is, in place of (28) we adopt

$$E_k^{\pm} = \pm E_0^{(*)}\delta(k \mp k_0) \pm E_1^{(*)}\delta(k \mp k_1) + \delta E_k^{\pm}\,, \tag{41}$$

where $|\delta E| \ll |E_{0,1}|$. Then, after substitution of (41) into (10) and subsequent linearisation with respect to δE, we find

$$\begin{aligned}\Big[\varepsilon_{\Delta+0} + 2\Sigma^{\rm symm}_{\Delta+0,0,-0}|E_0|^2 + 2\Sigma^{\rm symm}_{\Delta+0,1,-1}|E_1|^2\Big]\,\delta E^+_{\Delta+0}\\ +\Sigma^{\rm eff}_{0,\Delta+1,-1}E_0E_1^*\delta E^+_{\Delta+1} - \Sigma^{\rm eff}_{0,0,\Delta-0}(E_0)^2\delta E^-_{\Delta-0} - 2\Sigma^{\rm symm}_{0,1,\Delta-1}E_0E_1\delta E^-_{\Delta-1}\\ +\Sigma^{\rm eff}_{1,\Delta+0+\delta,-0}E_0^*E_1\delta E^+_{\Delta+0+\delta} - \Sigma^{\rm eff}_{1,1,\Delta-1+\delta}(E_1)^2\delta E^-_{\Delta-1+\delta} = 0\,,\end{aligned} \tag{42}$$

where

$$\Sigma^{\rm symm}_{i,j,l} \equiv \tfrac{1}{2}\left(\Sigma^{\rm eff}_{i,j,l} + \Sigma^{\rm eff}_{j,i,l}\right)\,, \qquad \delta E_{\Delta+0+\delta} \equiv \delta E_{\Delta k+k_0+\delta k}\,, \tag{43}$$

etc. Note that we have introduced in (42) in a standard way (see Rudakov and Tsytovich 1978; Vladimirov and Tsytovich 1990) the modulational perturbations of frequency $\Delta\omega$ and wave vector $\Delta\mathbf{k}$ associated with the modulational interaction.

To close the system of equations, we need, in addition to (42), equations containing $\varepsilon_{\Delta-0}$, $\varepsilon_{\Delta\pm1}$, as well as $\varepsilon_{\Delta+0+\delta}$ and $\varepsilon_{\Delta-1+\delta}$. If it (hypothetically) became necessary to add only equations which do not contain a dielectric constant with frequencies (and wave vectors) $\Delta k + k_0 + \delta k$ and $\Delta k - k_1 + \delta k$, then the system describing the modulational interactions of two monochromatic pump waves would be closed. The dispersion relation for the modulational interaction would follow in this case from the condition of zero determinant of the (4×4) matrix of the equation coupling four perturbations at the frequencies $\Delta\omega \pm \omega_0$ and $\Delta\omega \pm \omega_1$.

The real situation is more complicated, however. Even in equation (42) there are perturbations of the fields at the frequencies $\Delta\omega + \omega_0 + \delta\omega$ and $\Delta\omega - \omega_1 + \delta\omega$. Thus, if we write four equations containing $\varepsilon_{\Delta\pm0}$ and $\varepsilon_{\Delta\pm1}$, we will see that perturbations arise in them at the frequencies $\Delta\omega \pm (\omega_0 + \Delta\omega)$ and $\Delta\omega \pm (\omega_1 - \delta\omega)$, for which we must also write corresponding equations. It ultimately turns out that, in the development of the modulational instability of two pump waves, satellites are generated not only for modes with frequencies ω_0 and ω_1, but also for waves absent in the zeroth approximation (in our case $\mathbf{k}_0 \perp \mathbf{k}_1$). These waves are at frequencies $\omega_0 + n\delta\omega$, where $n = 1, \pm2, \pm3, \ldots$ or, equivalently, at frequencies $\omega_1 - n\delta\omega$ since by definition we have $\omega_0 - 2\delta\omega = \omega_1 - \delta\omega$, etc.

We thus obtain a system of a (generally infinite) number of equations which describes an infinite number of coupled amplitudes. Physically, however, it is clear that such a system will be finite, if only because the condition $\delta\omega \ll \omega_{0,1}$ holds by virtue of the dispersion relation of the plasma waves. The (infinite) matrix of this system is block-diagonal (Vladimirov and Tsytovich 1990), and to find its determinant in its general form is a rather difficult problem. Thus, below

we investigate the modulational instability of two mochromatic pump waves using some simplifying assumptions.

The most important assumptions which allow us to simplify the corresponding equations significantly follow from comparison of the beat frequency of the two pumps, $\delta\omega$, and the characteristic frequency of the modulational interaction, $\Delta\omega$. From general considerations, it is clear that if $\Delta\omega \gg \delta\omega$, then the modulational instability has a character similar to that of the instability of one monochromatic pump mode. Thus, the most interesting case is when $\Delta\omega \leq \delta\omega$. However, the case $\Delta\omega \sim \delta\omega$ seems to be more difficult for analysis, and this is the reason why this investigation has not been done yet. At the same time the case $\Delta\omega \ll \delta\omega$ is now well investigated (Vladimirov and Tsytovich 1992, 1993*a*, 1993*b*, 1993*c*). In the next two sections, this situation will be considered in detail, and the corresponding instability rates will be presented.

4. Two Monochromatic Pumps with Large Frequency Gap

In this section, we examine the modulational interactions of two monochromatic plasma waves when their frequency difference is large compared with the instability growth rate. General frequency analysis is presented which allows us to introduce a set of virtual fields (forced oscillations) and to obtain eventually a finite matrix of the modulational interactions.

4.1 Frequency Analysis

As we have already mentioned in the preceeding section, in a nonlinear medium the presence of two wave fields leads to oscillations at beat and sum frequencies. In our model of a cubic nonlinear medium (3) it is evident that the beat oscillations are at a near-zero frequency $0 = \omega_0 - \omega_0 = \omega_1 - \omega_1$ and at the beat frequency $\pm\delta\omega$ due to the quadratic nonlinearity. Also, the quadratic nonlinearity gives rise to oscillations at double frequencies $\pm 2\omega_0$, $\pm 2\omega_1$ and $\pm(\omega_0 + \omega_1)$. The cubic nonlinearity causes oscillations at near-pump frequencies $\pm(\omega_0 + \delta\omega)$ and $\pm(\omega_1 - \delta\omega)$, as well as at triple frequencies $\pm 3\omega_0$, $\pm 3\omega_1$, $\pm(2\omega_0 + \omega_1)$ and $\pm(2\omega_1 + \omega_0)$. Furthermore, we consider all these fields as virtual by which we assume their intensities to be small in comparison with the intensities of the real pump fields. This supposition also means that all the above frequencies are not in resonance with waves that can propagate in the medium (including low-frequency, in comparison with the pump frequencies, oscillations). If a resonance appears, such as when for any $\mathbf{k}$ the equality

$$\delta\omega = \omega(\mathbf{k}) \tag{44}$$

is satisfied, where $\omega(\mathbf{k})$ is the frequency of (low-frequency) waves in the medium, the approximation considered is incorrect. However, if the generation rate of the low-frequency mode is sufficiently small compared with the rate of the modulation process investigated, we can ignore the resonance (44). In this case the low-frequency oscillations (if they are initially present or have had sufficient time to develop) are simply taken into account by introducing a low-frequency field, in exactly the same way as we have introduced high-frequency pump waves. However, below, for the sake of simplicity, we shall not consider such low-frequency waves.

In investigating the modulational instability of a one pump mode, virtual fields due to cubic nonlinearity are not taken into account. They are small because of their higher order in the field in comparison with virtual waves due to quadratic nonlinearity—with fields on zero and double frequencies. However, in the case of two pump waves some of the 'cubic' virtual fields have comparatively large intensities. These are fields at frequencies $\pm(\omega_0 + \delta\omega)$ and $\pm(\omega_1 - \delta\omega)$, near to the pump-wave frequencies $\omega_{0,1}$ (note that $\omega_0 - \delta\omega = \omega_1$ and $\omega_1 + \delta\omega = \omega_0$). The higher intensities of these fields in comparison with virtual fields at triple frequencies originate from the large values of the Green functions (13), i.e. small values of the denominators $\varepsilon(\omega_0 + \delta\omega)$ and $\varepsilon(\omega_1 - \delta\omega)$ near the resonances $\varepsilon(\omega_{0,1}) = 0$ (if $|\delta\omega| \ll \omega_{0,1}$; we suppose here that near triple frequencies there are no resonances with eigenwaves of the medium).

Thus, we consider the full electric field in (3) consisting of real and virtual waves as follows:

$$\begin{aligned} E = {} & E_0^+ + E_0^- + E_1^+ + E_1^- + E_\Delta^v + E_\delta^v + E_{-\delta}^v + \left(E_{2\delta}^v + E_{-2\delta}^-\right) \\ & + E_{\delta+0}^v + E_{-\delta-0}^v + E_{-\delta+1}^v + E_{\delta-1}^v + E_{2\cdot 0}^v + E_{-2\cdot 0}^v + E_{2\cdot 1}^v + E_{-2\cdot 1}^v \\ & + E_{0+1}^v + E_{-0-1}^v + \left(E_{2\cdot 0+\delta}^v + E_{-2\cdot 0-\delta}^v + E_{2\cdot 1-\delta}^v + E_{-2\cdot 1+\delta}^v\right) , \end{aligned} \tag{45}$$

where the following notation is used:

$$\begin{gathered} \Delta = \Delta k, \quad \delta = \delta k, \quad \delta + 0 = \delta k + k_0, \quad \delta - 1 = \delta k - k_1, \\ 2\cdot 0(1) = 2k_{0(1)}, \quad 0 + 1 = k_0 + k_1, \quad 2\cdot 0(1) \pm \delta = 2k_{0(1)} \pm \delta k . \end{gathered} \tag{46}$$

The virtual field E_Δ^v is the field at zero frequency, which is 'smoothed' over the frequency $\Delta\omega$ by the modulational interaction (note that lines near other frequencies are smoothed analogously). The fields in parentheses in equation (45) are generated by one real wave at frequency $\pm\omega_{0,1}$ and one virtual wave at frequency $\pm(\omega_0 + \delta\omega)$ or $\pm(\omega_1 - \delta\omega)$.

With regard to the introduction of (45), the following points should be noted. First, our splitting of electric fields assumes that all transition processes (for an initial-value problem) or all near-boundary processes (for a boundary-value problem) are not essential. This means that we are interested in effects with characteristic times much greater than the times of transition processes (for an initial-value problem) or effects in the bulk medium far from the boundaries. Secondly, we may distinguish the (low-frequency) beat frequency $\delta\omega$ and the modulational smoothing frequency $\Delta\omega$ if

$$|\gamma^{\mathrm{mod}}| < |\delta\omega| . \tag{47}$$

When the opposite inequality holds, the modulational processes completely smooth the beat frequency $\delta\omega$, and another approximation is necessary.

4.2 Basic Expressions

On substituting (45) into (3) and choosing fields at frequency ω_0, we obtain

$$\varepsilon_0 E_0^+ = 2\int S(E_0^+ E_\Delta^v + E_1^+ E_\delta^v + E_0^- E_{2\cdot 0}^v + E_1^- E_{0+1}^v + E_{\delta+0}^v E_{-\delta}^v + E_{\delta-1}^v E_{2\cdot 1}^v) d^{(2)}$$

$$+\int \Sigma[2E_0^+(E_0^+ E_0^- + E_1^+ E_1^-) + 2E_1^+(E_0^+ E_1^- + E_{\delta+0}^v E_0^- + E_1^+ E_{\delta-1}^v)$$

$$+2E_1^- E_0^+ E_1^+ + 2E_{\delta+0}^v E_1^+ E_0^- + E_0^-(E_0^+ E_0^+ + 2E_1^+ E_{\delta+0}^v) + E_{\delta-1}^v E_1^+ E_1^+] d^{(3)}\,, \quad (48)$$

with an analogous equation for E_0^-. The equation for E_1^+ follows from (48) on making the interchange $(0,1) \leftrightarrow (1,0)$ (when clearly $\delta \leftrightarrow -\delta$). We recall that in equation (48) the symmetric second-order responses (6) as well as third-order responses (7), which are symmetric on the two last indices, have been used. We applied the symmetrisation (7) instead of possible full symmetrisation of Σ^{eff} on all three indices k_1, k_2 and k_3 because, as a rule, $\Sigma^{\text{eff}} \propto F(\omega_2 + \omega_3)$, and the function $F(x)$ decreases sufficiently rapidly with increasing x that (because $|\omega_2 + \omega_3| \approx \Delta\omega \ll |\omega_1 + \omega_2| \approx 2\omega_{1,2}$)

$$\int \Sigma^{\text{eff}}_{kk_1k_2k_3} E_{k_1}^+ E_{k_2}^+ E_{k_3}^- d^{(3)} \gg \int \Sigma^{\text{eff}}_{kk_3k_2k_1} E_{k_3}^- E_{k_2}^+ E_{k_1}^+ d^{(3)}\,. \quad (49)$$

For electron plasma waves, this expression corresponds to the case when we can neglect electron nonlinearities [see the inequality (18)]. However, below in this section we also use the full symmetrisation (67).

In accordance with our assumptions, we have retained in the cubic-in-field terms the virtual fields at frequencies $\pm(\omega_0 + \delta\omega)$ and $\pm(\omega_1 - \delta\omega)$ only as linear factors. Thus the approximation considered is valid only if

$$\max\{|E^v_{\pm(\delta+0)}|,\ |E_{\pm(\delta-1)}|\} < |E_{0,1}|\,. \quad (50)$$

At the limit of validity of this inequality the fields at frequencies $\pm(\omega_0 + \delta\omega)$ and $\pm(\omega_1 - \delta\omega)$ clearly lose their virtual character; in this case another consideration is necessary. In addition to (50), the following inequality must be satisfied:

$$\min\{|E^v_{\pm(\delta+0)}|,\ |E_{\pm(\delta-1)}|\} > \max\{|E^v_{\pm 3\cdot 0(1)}|,\ |E_{\pm(2\cdot 0(1)+1(0)}|\}\,. \quad (51)$$

Moreover, the cubic terms containing virtual fields must dominate over terms of higher order in (real) fields. For modulational interactions these are fifth-order terms, whose explicit forms depend upon the medium under consideration. In a plasma estimates for Langmuir waves show that the fifth-order terms can practically always be neglected (Vladimirov and Tsytovich 1990, 1993*a*). In general, for our purposes it is sufficient to postulate a negligible contribution from higher nonlinearities (of course, it is necessary to check this supposition for any concrete situation).

For virtual fields at low and double frequencies we have from (3)

$$\varepsilon_\Delta E^v_\Delta = 2\int S(E^+_0 E^-_0 + E^+_1 E^-_1)d^{(2)}\,, \tag{52}$$

$$\varepsilon_\delta E^v_\delta = 2\int S(E^+_0 E^-_1 + E^v_{\delta+0}E^-_0 + E^v_{\delta-1}E^+_1)d^{(2)}\,, \tag{53}$$

$$\varepsilon_{2\delta} E^v_{2\delta} = 2\int S(E^v_{\delta+0}E^-_1 + E^v_{\delta-1}E^+_0)d^{(2)}\,, \tag{54}$$

$$\varepsilon_{2\cdot 0} E^v_{2\cdot 0} = \int S(E^+_0 E^+_0 + 2E^v_{\delta+0}E^+_1)d^{(2)}\,, \tag{55}$$

$$\varepsilon_{2\cdot 0-\delta} E^v_{2\cdot 0-\delta} = 2\int SE^+_0 E^+_1 d^{(2)}\,, \tag{56}$$

$$\varepsilon_{2\cdot 0+\delta} E^v_{2\cdot 0+\delta} = 2\int SE^+_0 E^v_{\delta+0} d^{(2)}\,. \tag{57}$$

Other virtual fields can easily be obtained from (52)–(57) on interchanging $(0,1) \leftrightarrow -(0,1)$ or $(0,1) \leftrightarrow \pm(1,0)$. For the virtual field at frequency $\omega_0 + \delta\omega$ we obtain

$$\begin{aligned}\varepsilon_{\delta+0} E^+_{\delta+0} = 2\int & S(E^+_0 E^v_\delta + E^+_1 E^v_{2\delta} + E^-_0 E^v_{2\cdot 0+\delta} \\ & + E^-_1 E^v_{2\cdot 0} + E^v_{\delta+0}E^v_\Delta + E^v_{\delta-1}E^v_{2\cdot 0-\delta})d^{(2)} \\ & + \int \Sigma[2E^+_0(E^+_1 E^v_{\delta-1} + E^-_0 E^v_{\delta+0} + E^+_0 E^-_1) \\ & + 2E^+_1(E^v_{\delta+0}E^-_1 + E^+_0 E^v_{\delta-1}) + 2E^v_{\delta-1}E^+_0 E^+_1 \\ & + 2E^v_{\delta+0}E^+_0 E^-_0 + E^-_1(E^+_0 E^+_0 + 2E^+_1 E^v_{\delta+0}) \\ & + 2E^v_{\delta+0}(E^+_0 E^-_0 + E^+_1 E^-_1]d^{(3)}\,.\end{aligned} \tag{58}$$

Other expressions are obtained from (58) by the standard interchange $(0,1) \leftrightarrow -(0,1)$ (for $E^v_{-\delta-0}$) or $(0,1) \leftrightarrow \pm(1,0)$ (for $E^v_{\pm(\delta-1)}$).

After substitution of the virtual fields (52)–(57) into (48), we find

$$\begin{aligned}\varepsilon_0 E^+_0 = \int & \Sigma^{\text{eff}}[E^+_0(E^+_0 E^-_0 + E^+_1 E^-_1) + E^+_1(E^+_0 E^-_1 + E^v_{\delta+0}E^-_0 + E^+_1 E^v_{\delta-1}) \\ & + E^-_0(E^+_0 E^+_0 + 2E^+_1 E^v_{\delta+0}) + E^-_1 E^+_0 E^+_1 + E^v_{\delta+0}E^+_1 E^-_0 + E^v_{\delta-1}E^+_1 E^+_1]d^{(3)}\,,\end{aligned} \tag{59}$$

where the effective third-order plasma responses Σ^{eff} are defined by (11) and (12). Furthermore, expressions like (59) for the fields E^-_0 and $E^\pm_1$ can easily be obtained by our standard interchanges.

For the virtual field at frequency $\omega_0 + \delta\omega$ we find [see equation (58)]

$$\begin{aligned}\tilde{\varepsilon}_{\delta+0} E^v_{\delta+0} - \int & \Sigma^{\text{eff}}(E^+_0 E^+_1 E^v_{\delta-1} + E^+_1 E^+_0 E^v_{\delta-1} + E_{\delta-1}E^+_0 E^+_1)d^{(3)} \\ & = \int \Sigma^{\text{eff}}(E^+_0 E^+_0 E^-_1 + E^-_1 E^+_0 E^+_0)d^{(3)}\,,\end{aligned} \tag{60}$$

where we have introduced the renormalised dielectric permittivity

$$\tilde{\varepsilon}_{\delta+0}E^v_{\delta+0} \equiv \varepsilon_{\delta+0}E^v_{\delta+0} - \int \Sigma^{\text{eff}}(E^+_0 E^v_{\delta+0}E^-_0 + E^v_{\delta+0}E^+_0E^-_0 + E_{\delta+0}E^+_1E^-_1 + E^+_1E^v_{\delta+0}E^-_1 + E^-_0E^+_0E^v_{\delta+0} + E^-_1E^+_1E^v_{\delta+0})d^{(3)}\,. \quad (61)$$

After the interchange $(0,1) \leftrightarrow -(1,0)$, we have from (61) an expression for the field $E^v_{\delta-1}$. It is easy to see that the virtual field at frequency $\omega_0 + \delta\omega$ is coupled only with the virtual field at frequency $-\omega_1 + \delta\omega$. Analogously, we find that the virtual field at frequency $-\omega_0 - \delta\omega$ is coupled only with the virtual field at frequency $\omega_1 - \delta\omega$. Thus (59) and (60) are the basic equations for our further investigation.

4.3 Zeroth Approximation

In the zeroth approximation we substitute monochromatic pump waves (28) into (59) and (60). We then have the following expression from (60) [and consequently we have all the equations that can be derived from (60) with the interchange $(0,1) \leftrightarrow -(0,1)$ and $(0,1) \leftrightarrow \pm(1,0)$]:

$$E^v_{\delta+0} = -\frac{\tilde{\Sigma}^{\text{eff}}_{0,0,-1} + \tilde{\Sigma}^{\text{eff}}_{-1,0,0}}{\tilde{\varepsilon}_{\delta+0}\Delta_{0,-1}}E^2_0E^*_1, \quad (62)$$

where

$$\tilde{\Sigma}^{\text{eff}}_{0,0,-1} \equiv \Sigma^{\text{eff}}{}_{0,0,-1} + \frac{\Sigma^{\text{eff}}_{-1,-1,0}}{\tilde{\varepsilon}_{\delta-1}}(\Sigma^{\text{eff}}_{0,1,\delta-1} + \Sigma^{\text{eff}}_{1,0,\delta-1} + \Sigma^{\text{eff}}_{\delta-1,0,1})|E_1|^2, \quad (63)$$

$$\tilde{\Sigma}^{\text{eff}}_{-1,0,0} \equiv \Sigma^{\text{eff}}{}_{-1,0,0} + \frac{\Sigma^{\text{eff}}_{0,-1,-1}}{\tilde{\varepsilon}_{\delta-1}}(\Sigma^{\text{eff}}_{0,1,\delta-1} + \Sigma^{\text{eff}}_{1,0,\delta-1} + \Sigma^{\text{eff}}_{\delta-1,0,1})|E_1|^2, \quad (64)$$

and

$$\Delta_{0,-1} \equiv 1 - \frac{1}{\tilde{\varepsilon}_{\delta+0}\tilde{\varepsilon}_{\delta-1}}(\Sigma^{\text{eff}}_{0,1,\delta-1} + \Sigma^{\text{eff}}_{1,0,\delta-1} + \Sigma^{\text{eff}}_{\delta-1,0,1}) \times (\Sigma^{\text{eff}}_{-0,-1,\delta+0} + \Sigma^{\text{eff}}_{-1,-0,\delta+0} + \Sigma^{\text{eff}}_{\delta+0,-0,-1})\,. \quad (65)$$

For the renormalised dielectric function we find from (61)

$$\tilde{\varepsilon}_{\delta+0} = \varepsilon_{\delta+0} + \Sigma^{\text{eff}}_{\delta+0,0,-0}|E_0|^2 + \Sigma^{\text{eff}}_{0,\delta+0,-0}|E_0|^2 + \Sigma^{\text{eff}}_{-0,0,\delta+0}|E_0|^2 + \Sigma^{\text{eff}}_{\delta+0,1,-1}|E_1|^2 + \Sigma^{\text{eff}}_{1,\delta+0,-1}|E_1|^2 + \Sigma^{\text{eff}}_{-1,1,\delta+0}|E_1|^2\,. \quad (66)$$

After the interchange $(0,1) \leftrightarrow -(1,0)$, we have an analogous expression for $\tilde{\varepsilon}_{\delta-1}$.

Below, we introduce the fully symmetric response (we recall that all Σ^{eff} are symmetric in the last pair of indices)

$$\Sigma^s_{i,j,l} \equiv \tfrac{1}{3}(\Sigma^{\text{eff}}_{i,j,l} + \Sigma^{\text{eff}}_{j,i,l})\,. \quad (67)$$

Then, to avoid repetition, we can write the expression for the virtual field $E^v_{-\delta+1}$ in the form:

$$E^v_{-\delta+1} = -\frac{3\tilde{\Sigma}^s_{1,1,0}}{\tilde{\varepsilon}_{-\delta+1}\Delta_{-0,1}} E_1^2 E_0^* , \tag{68}$$

where

$$\tilde{\Sigma}^s_{1,1,-0} \equiv \Sigma^s_{1,1,-0} + \frac{3\Sigma^s_{-0,-0,1}}{\tilde{\varepsilon}_{-\delta-0}} \Sigma^s_{0,1,-\delta-0}|E_0|^2 , \tag{69}$$

$$\Delta_{-0,1} \equiv 1 - \frac{9\Sigma^s_{0,1,-\delta-0}\Sigma^s_{-0,-1,-\delta+1}}{\tilde{\varepsilon}_{-\delta-0}\tilde{\varepsilon}_{-\delta+1}} |E_0|^2|E_1|^2 . \tag{70}$$

After substitution of the virtual fields (62) and (68) into the equations for the pump fields $E_{0,1}$, we find that in the zeroth approximation the nonlinear pump frequency shifts are described by the equations

$$\begin{aligned} &\varepsilon_0 + 3\Sigma^s_{0,0,-0}|E_0|^2 + 3\Sigma^s_{0,1,-1}|E_1|^2 \\ &\qquad = \frac{9\tilde{\Sigma}^s_{0,0,-1}\Sigma^s_{\delta+0,1,-0}}{\tilde{\varepsilon}_{\delta+0}\Delta_{0,-1}}|E_0|^2|E_1|^2 + \frac{9\tilde{\Sigma}^s_{-1,-1,0}\Sigma^s_{1,1,\delta-1}}{\tilde{\varepsilon}_{\delta-1}\Delta_{0,-1}}|E_1|^4 , \end{aligned} \tag{71}$$

$$\begin{aligned} &\varepsilon_1 + 3\Sigma^s_{1,1,-1}|E_1|^2 + 3\Sigma^s_{1,0,-0}|E_0|^2 \\ &\qquad = \frac{9\tilde{\Sigma}^s_{1,1,-0}\Sigma^s_{-\delta+1,0,-1}}{\tilde{\varepsilon}_{-\delta+1}\Delta_{-0,1}}|E_0|^2|E_1|^2 + \frac{9\tilde{\Sigma}^s_{-0,-0,1}\Sigma^s_{0,0,-\delta-0}}{\tilde{\varepsilon}_{-\delta-0}\Delta_{-0,1}}|E_0|^4 . \end{aligned} \tag{72}$$

We stress that a number of the above conditions must be satisfied in order to obtain a correct description, that given by (50) being the most important.

4.4 Equations for Modulational Perturbations

Now, we introduce the modulational perturbations (41). Therefore, on substituting (41) into (59) we obtain

$$\begin{aligned} &\left(\varepsilon_{\Delta+0} + 3\Sigma^s_{\Delta+0,0,-0}|E_0|^2 + 3\Sigma^s_{\Delta+0,1,-1}|E_1|^2\right)\delta E^+_{\Delta+0} \\ &\qquad - \left(3\Sigma^s_{0,0,\Delta-0}E_0^2 + 3\Sigma^s_{\delta+0,1,\Delta-0}E^v_{\delta+0}E_1\right)\delta E^-_{\Delta-0} \\ &\quad + \left(3\Sigma^s_{\Delta+1,0,-1}E_0E_1^* + 3\Sigma^s_{\Delta+1,\delta+0,-0}E^v_{\delta+0}E_0^* - 3\Sigma^s_{\Delta+1,1,\delta-1}E^v_{\delta-1}E_1\right)\delta E^+_{\Delta+1} \\ &\qquad -3\Sigma^s_{0,1,\Delta-1}E_0E_1\delta E^-_{\Delta-1} - 3\Sigma^s_{\Delta+\delta+0,1,-0}E_1E_0^*\delta E^v_{\Delta+\delta+0} \\ &\qquad\qquad -3\Sigma^s_{1,1,\Delta+\delta-1}E_1^2\delta E^v_{\Delta+\delta-1} = 0 . \end{aligned} \tag{73}$$

As before, from (73), with the interchanges $(0,1) \leftrightarrow -(0,1)$ and $(0,1) \leftrightarrow \pm(1,0)$ we find equations for the fields $\delta E^-_{\Delta-0}$ and $\delta E^\pm_{\Delta\pm1}$. All of these equations, however, also contain modulational perturbations from virtual waves. The expressions for

the perturbations from virtual waves follow from (60) and are given by

$$\begin{aligned}
&\tilde{\varepsilon}_{\Delta+\delta+0}\delta E^v_{\Delta+\delta+0} - 3\Sigma^s_{0,1,\Delta+\delta-1}E_0E_1\delta E^v_{\Delta+\delta-1} \\
&\qquad = -\left(3\Sigma^s_{\delta+0,\Delta+0,-0}E^v_{\delta+0}E^*_0 + 3\Sigma^s_{\Delta+0,0,-1}E_0E^*_1 - 3\Sigma^s_{\Delta+0,1,\delta-1}E^2_0\right)\delta E^+_{\Delta+0} \\
&\qquad - \left(3\Sigma^s_{\Delta+1,\delta+0,-1}E^v_{\delta+1}E^*_1 - 3\Sigma^s_{\Delta+1,0,\delta-1}E_0E^v_{\delta-1}\right)\delta E^+_{\Delta+1} \\
&+3\Sigma^s_{\delta+0,0,\Delta-0}E_0E^v_{\delta+0}\delta E^-_{\Delta-0} + \left(3\Sigma^s_{\delta+0,1,\Delta-1}E^v_{\delta+0}E_1 + 3\Sigma^s_{0,0,\Delta-1}E^2_0\right)\delta E^-_{\Delta-1}\,. \qquad (74)
\end{aligned}$$

Analogous equations can be obtained for the virtual fields $\delta E^v_{\Delta-\delta-0}$ and $\delta E^v_{\Delta\pm(\delta-1)}$ by our standard interchange.

Thus we have a modulational interaction matrix, whose dimension is 8×8. After some fairly simple linear transformations (bearing in mind, in particular, the simplifying facts that in the equations for $\delta E^+_{\Delta+0}$ and $\delta E^-_{\Delta-1}$ there are only modulational perturbations from the virtual fields $\delta E^v_{\Delta+\delta+0}$ and $\delta E^v_{\Delta+\delta-1}$, and that in the equations for $\delta E^-_{\Delta-0}$ and $\delta E^+_{\Delta+1}$ there are only perturbations from the virtual fields $\delta E^v_{\Delta-\delta-0}$ and $\delta E^v_{\Delta-\delta+1}$, respectively), we arrive at the conclusion that the determinant of the above matrix is equal to the determinant of a new 4×4 matrix containing 'renormalised' modulational perturbations of the real pump waves. However, it should be noted that the modulational perturbations of the virtual fields $\delta E^v_{\Delta\pm(\delta+0)}$ and $\delta E^v_{\Delta\pm(\delta-1)}$ are practically always present (apart from special cases.

Note that we present here a general description of the modulational interactions of two pump waves which is correct for every cubically nonlinear medium. Explicit expressions for the dispersion equations can now be obtained for particular models of a nonlinear medium by determining the linear dielectric function ε as well as the nonlinear second- and third-order responses S and Σ.

The above considerations, after some natural and simple generalisations, can be used to describe the interactions of transverse waves in a homogeneous and isotropic medium. In this case, instead of the longitudinal dielectric permittivity we should use the function $\varepsilon^t - \mathbf{k}^2c^2/\omega^2$, where ε^t is the transverse dielectric function, and instead of S and Σ, expressions like $e_iS_{ijl}e_je_l$, where e_i is the wave polarisation vector. A complication lies in taking into account both transverse and longitudinal virtual waves (even when considering only transverse pump waves). It should be noted that, in general, transverse virtual fields should be taken into consideration even when only longitudinal pump waves interact (Bel'kov and Tsytovich 1979; Kono *et al.* 1980). We have chosen here a model including only longitudinal perturbations for the sake of simplicity; besides, in a number of situations the effects connected with transverse virtual waves are small (Bel'kov and Tsytovich 1979).

The effects of modulational interactions of two pump waves are considerably more diverse than the effects of one-mode modulational instability. Therefore, in principle, situations are possible in which all satellites do not grow simultaneously (i.e. with the same rate), but some grow faster than others. This case will arise if the determinant of the 8×8 matrix can be written as a product of factors, each of which can in turn be treated as the determinant of a smaller matrix describing the interaction of a corresponding number of satellites (e.g. two or four).

Thus taking account of only the satellites from the pump waves is valid if (a) the instability rates of the satellites from the pump waves are greater than those of the satellites from virtual waves at frequencies $\pm(\omega+\delta\omega)$ and $\pm(\omega-\delta\omega)$, and (b) when simultaneous growth of all eight satellites is observed, but the instability dispersion equation (and consequently the instability rate) is not greately altered after reduction from the 8×8 matrix to a 4×4 matrix (without renormalisation of linear and nonlinear responses). In case (b) the satellites from the virtual fields are (for special reasons) smaller than the satellites from the pump waves at any given time although the rate of the instability is the same for all satellites. It should be clear that in the zeroth approximation the virtual waves $E^v_{\pm(\delta+0)}$ and $E^v_{\pm(\delta-1)}$ themselves can be absent, but their satellites may make a non-zero (and even rather large) contribution to the modulational interaction.

5. Rates of Two-pump Instability of Langmuir Waves

In this section, rates of the modulational instability of two pump Langmuir waves in homogeneous and isotropic plasmas are presented. Most attention we pay to perpendicular propagation of the pump waves. We demonstrate that even in this simpler analytic situation, the two-pump modulational interaction has qualitatively new features (and is considerably more complicated) compared with the modulational instability of one monochromatic pump.

5.1 Nonlinear Frequency Shift

Langmuir pump waves $E_{0,1}$ have the following dispersion:

$$\omega_{0,1}=\omega_{pe}^2+3\mathbf{k}_{0,1}^2 v_{Te}^2\,. \tag{75}$$

From this, we find (for $\mathbf{k}_{0,1}^2 r_{De}^2 \ll 1$)

$$\delta\omega=\tfrac{3}{2}\omega_{pe}[\delta\mathbf{k}\cdot(\mathbf{k}_0+\mathbf{k}_1)]r_{De}^2. \tag{76}$$

For the dielectric functions at frequencies $\pm(\omega_0+\delta\omega)$ and $\pm(\omega_1-\delta\omega)$ and taking into account (75) and (76), we find

$$\varepsilon_{\pm(\delta+0)}=\varepsilon_{\pm(\delta-1)}=-6\delta\mathbf{k}^2 r_{De}^2\equiv\varepsilon_\delta\ll 1\,. \tag{77}$$

Now, we consider two cases:

(a) If $\mathbf{k}_0\perp\mathbf{k}_1$ then [because of the angular dependence of the effective third-order response (32)] all virtual fields at frequencies $\pm(\omega_0+\delta\omega)$ and $\pm(\omega_1-\delta\omega)$ are zero. Thus when $\mathbf{k}_0\perp\mathbf{k}_1$, we have in the zeroth approximation equations (37)–(40). The static renormalisation of the plasma density leads to a static frequency shift which modifies (75)

$$\omega_{0,1}=\omega_{pe}^2\left[1-2\Sigma_0^{\text{eff}}\left(|E_0|^2+|E_1|^2\right)\right]+3\mathbf{k}_{0,1}^2 v_{Te}^2\,. \tag{78}$$

This frequency shift takes place in all plasma volume, and is the same for each pump wave.

(*b*) If $\mathbf{k}_0 \parallel \mathbf{k}_1$ then we have maximum values for the virtual fields $E^v_{\pm(\delta+0)}$ and $E^v_{\pm(\delta-1)}$. We should stress that in this case we can also renormalise the plasma density according to (40). After the renormalisation, we find

$$\varepsilon_0 + \Sigma^{\text{eff}}_{1,0,-1}|E_1|^2 + \frac{\tilde{\Sigma}^{\text{eff}}_{-1,-1,0}}{\tilde{\varepsilon}_{\delta-1}\Delta_{0,-1}}\Sigma^{\text{eff}}_{1,1,\delta-1}|E_1|^4 + \frac{\tilde{\Sigma}^{\text{eff}}_{0,0,-1}}{\tilde{\varepsilon}_{\delta+0}\Delta_{0,-1}}\Sigma^{\text{eff}}_{\delta+0,1,-0}|E_0|^2|E_1|^2 = 0\,, \tag{79}$$

where [compare with (61), (69) and (70)]

$$\tilde{\varepsilon}_{\delta+0} = \varepsilon_{\delta+0} + \Sigma^{\text{eff}}_{0,\delta-0,-0}|E_0|^2 + \Sigma^{\text{eff}}_{1,\delta+0,-1}|E_1|^2\,, \tag{80}$$

$$\tilde{\Sigma}^{\text{eff}}_{0,0,-1} = \Sigma^{\text{eff}}_{0,0,-1} + \frac{\Sigma^s_{0,1,\delta-1}}{\tilde{\varepsilon}_{\delta-1}}\Sigma^{\text{eff}}_{-1,-1,0}|E_1|^2\,, \tag{81}$$

$$\Delta_{0,-1} = 1 - \frac{\Sigma^s_{-0,-1,\delta+0}\Sigma^s_{0,1,\delta-1}}{\tilde{\varepsilon}_{\delta+0}\tilde{\varepsilon}_{\delta-1}}|E_0|^2|E_1|^2\,. \tag{82}$$

An equation for the pump wave E_1 can be obtained from (79)–(82) on interchanging $(0,1) \leftrightarrow (1,0)$.

Now we determine when inequality (50) is satisfied and also when the contributions from virtual fields at triple frequencies, from electron nonlinearities as well as from fifth-order terms are small. For simplicity we suppose $|E_0| = |E_1|$. Then, from (32) and (77), we have

$$\varepsilon_0 + 2\Sigma_\delta|E_0|^2 D_0 = 0\,, \tag{83}$$

where

$$D_0 = \frac{\varepsilon_\delta + 2\Sigma_\delta|E_0|^2(1-D_0)}{\varepsilon_\delta + 2\Sigma_\delta|E_0|^2(4-D_0)}\,, \qquad \Sigma_\delta = \frac{1}{8\pi n T_e}\,\frac{\delta\mathbf{k}^2 v_s^2}{\delta\mathbf{k}^2 v_s^2 - \delta\omega^2}\,. \tag{84, 85}$$

We can see that in the zeroth approximation the nonlinear frequency shift for the wave E_0 is determined by equation (83) in which a form factor (84) appears. The presence of this form factor in both numerator and denominator of the r.h.s. of (84) is connected with the appearance of a nonlinear frequency shift [following from (83)] in calculating $\varepsilon_{\delta+0}$ and $\varepsilon_{\delta-1}$ [these dielectric functions are contained in the expressions for $E^v_{\delta+0}$ and $E^v_{\delta-1}$, see (62)].

For the virtual fields we have

$$E^v_{\delta+0} = -\frac{2\Sigma_\delta|E_0|^2 E_0}{\varepsilon_\delta + 2\Sigma_\delta|E_0|^2(4-D_0)}\,, \qquad E^v_{\delta-1} = -\left(E^v_{\delta+0}\right)^*\,. \tag{86, 87}$$

Using equations (86) and (87), we can easily conclude that (50) is satisfied if

$$\left|\frac{\Sigma_\delta|E_0|^2}{\varepsilon_\delta}\right| < \min\left\{1,\ \frac{1}{2(5-D_0)}\right\}\,. \tag{88}$$

If D_0 is close to unity, (84) gives

$$D_0 \approx \frac{1}{1 + 6\Sigma_\delta |E_0|^2/\varepsilon_\delta}\,. \tag{89}$$

Condition (51) for small fields at triple frequencies leads to (we suppose $|\mathbf{k}_0| \approx |\mathbf{k}_1|$)

$$\left|\frac{\Sigma_\delta |E_0|^2}{\varepsilon_\delta}\right| \gg \mathbf{k}_0^2 r_{De}^2 \Sigma_0 |E_0|^2 \tag{90}$$

[we remind the reader that Σ_0 is defined by (39)]. Condition (18) which compares a small contribution to Σ [which is defined by (32)] from electron nonlinearities with the contributions of the virtual fields $E^v_{\pm(\delta+0)}$ and $E^v_{\pm(\delta-1)}$ is (Vladimirov and Tsytovich 1993*b*)

$$|\Sigma_\delta |E_0|^2/\varepsilon_\delta| \gg \mathbf{k}_0^2 r_{De}^2. \tag{91}$$

The fifth-order effects can be estimated as (Khakimov and Tsytovich 1976)

$$\max\Sigma^{(5)} \approx \left[\frac{\delta \mathbf{k}^2 v_s^2}{4\pi n T_e(\delta\omega^2 - \delta\mathbf{k}^2 v_s^2)}\right]^2. \tag{92}$$

Thus we can neglect them if

$$|\Sigma_\delta^2/\varepsilon_\delta| \gg \max\Sigma^{(5)} \approx \Sigma_\delta^2\,. \tag{93}$$

This inequality is automatically satisfied.

5.2 Modulational Perturbations

In investigating modulational interactions we proceed to the most interesting case of perpendicular propagation of the pump waves (which is analogous to the case of orthogonal polarisation for two pump electromagnetic waves). We demonstrate that even if the interference terms (i.e. the virtual fields $E^v_{\pm(\delta+0)}$ and $E^v_{\pm(\delta-1)}$) are zero, their satellites can effect the modulational interaction (and consequently must be taken into consideration).

First we consider modulations with $\Delta\mathbf{k} \perp \mathbf{k}_0 \perp \mathbf{k}_1$. In this situation the virtual fields $E^v_{\pm(\delta+0)}$ and $E^v_{\pm(\delta-1)}$ are zero, as are their satellites $\delta E^v_{\Delta\pm(\delta+0)}$ and $\delta E^v_{\Delta\pm(\delta-1)}$. Finally, we have a system of four equations containing only modulational perturbations from the pump waves $E_{0,1}$ (and therefore a 4×4 modulational matrix without any renormalisation). Hence there are no restrictions on the pump amplitudes and frequencies, and the results are very similar to those for one monochromatic Langmuir pump (Vladimirov and Tsytovich 1990).

Now let us consider modulations with $\Delta\mathbf{k} \parallel \mathbf{k}_0 \perp \mathbf{k}_1$. In this situation we have non-zero modulational perturbations of the zero virtual fields $E^v_{\pm(\delta+0)}$ and $E^v_{\pm(\delta-1)}$. Using (74) (and analogous expressions) to substitute into (73) and

analogous expressions, we have the 4×4 modulational matrix

$$T = \begin{pmatrix} \varepsilon_{\Delta+0} + \Sigma_\Delta |E_0|^2 & -\Sigma_\Delta E_0^2 & \Sigma_\Delta E_0 \tilde{E}_1^* & -\Sigma_\Delta E_0 \tilde{E}_1 \\ -\Sigma_\Delta (E_0^*)^2 & \varepsilon_{\Delta-0} + \Sigma_\Delta |E_0|^2 & -\Sigma_\Delta E_0^* \tilde{E}_1^* & \Sigma_\Delta E_0^* \tilde{E}_1 \\ \Sigma_\Delta E_0^* \tilde{E}_1 & -\Sigma_\Delta E_0 \tilde{E}_1 & \bar{\varepsilon}_{\Delta+1} + \Sigma_\Delta |\tilde{E}_1|^2 & -\Sigma_\Delta \tilde{E}_1^2 \\ -\Sigma_\Delta E_0^* \tilde{E}_1^* & \Sigma_\Delta E_0 \tilde{E}_1^* & -\Sigma_\Delta (\tilde{E}_1^*)^2 & \bar{\varepsilon}_{\Delta-1} + \Sigma_\Delta |\tilde{E}_1|^2 \end{pmatrix}, \tag{94}$$

where we have put

$$\Sigma_\Delta \equiv -\frac{1}{4\pi n T_e} \frac{\Delta \mathbf{k}^2 v_s^2}{\Delta\omega^2 - \Delta \mathbf{k}^2 v_s^2}, \qquad |\tilde{E}_1|^2 \equiv \frac{\mathbf{k}_1^2}{\Delta \mathbf{k}^2 + \mathbf{k}_1^2} |E_1|^2. \tag{95, 96}$$

We note that the matrix (94) posesses the symmetry properties, namely the element T_{ik} of this matrix is equal to T_{ki}^*. This is associated with the reality of the effective third-order nonlinear response $\Sigma_{1,2,3}^{\text{eff}}$ in a homogeneous and isotropic medium. For the case of anisotropic and/or inhomogeneous plasma the effective third-order response can contain the imaginary part (see e.g. Tsytovich *et al.* 1992; Popel *et al.* 1994*a*) and the above statement about the symmetry properties of the matrix (94) does not remain valid.

From (96) we can see that spectrum of the pump E_1 is 'smoothed' in the k representation. Furthermore, in (94) we have the renormalised dielectric function

$$\bar{\varepsilon}_{\Delta+1} \equiv \varepsilon_{\Delta+1} + \Sigma_{0,\Delta+1,-0}^{\text{eff}} |E_0|^2 - \frac{\tilde{\Sigma}_{1,\Delta+1,-0}}{\tilde{\varepsilon}_{\Delta-\delta+1} \Delta_{-0,1}^{\text{mod}}} \Sigma_{0,\Delta-\delta+1,1}^{\text{eff}} |E_0|^2 |E_1|^2 - \frac{\tilde{\Sigma}_{-0,-0,\Delta+1}}{\tilde{\varepsilon}_{\Delta-\delta-0} \Delta_{-0,1}^{\text{mod}}} \Sigma_{0,0,\Delta-\delta-0}^{\text{eff}} |E_0|^2 |E_1|^2, \tag{97}$$

with an analogous expression for $\bar{\varepsilon}_{\Delta-1}$ on interchanging $(0,1) \leftrightarrow -(0,1)$. The third and fourth terms on the r.h.s. of (97) are due to the modulational perturbations from the (zero in our special case $\mathbf{k}_0 \perp \mathbf{k}_1$) virtual fields $E_{\pm(\delta+0)}^v$ and $E_{\pm(\delta-1)}^v$. In (97) we also have

$$\tilde{\Sigma}_{1,\Delta+1,-0} \equiv \Sigma_{1,\Delta+1,-0}^{\text{eff}} + \frac{\Sigma_{0,1,\Delta-\delta-0}^s}{\tilde{\varepsilon}_{\Delta-\delta-0}} \Sigma_{-0,-0,\Delta+1}^{\text{eff}} |E_0|^2, \tag{98}$$

$$\tilde{\Sigma}_{-0,-0,\Delta+1} \equiv \Sigma_{-0,-0,\Delta+1}^{\text{eff}} + \frac{\Sigma_{-0,-1,\Delta-\delta+1}^s}{\tilde{\varepsilon}_{\Delta-\delta+1}} \Sigma_{1,\Delta+1,-0}^{\text{eff}} |E_1|^2, \tag{99}$$

$$\tilde{\varepsilon}_{\Delta-\delta-0} \equiv \varepsilon_{\Delta-\delta-0} + \Sigma_{-0,\Delta-\delta-0,0} |E_0|^2 + \Sigma_{-1,\Delta-\delta-0,1} |E_1|^2, \tag{100}$$

$$\Delta_{-0,1}^{\text{mod}} \equiv 1 - \frac{\Sigma_{-0,-1,\Delta-\delta+1}^s \Sigma_{0,1,\Delta-\delta-0}^s}{\tilde{\varepsilon}_{\Delta-\delta-0} \tilde{\varepsilon}_{\Delta-\delta+1}} |E_0|^2 |E_1|^2 \tag{101}$$

[other renormalised linear and nonlinear plasma responses are obtained from (98)–(101) by the standard interchanges $(0,1) \leftrightarrow -(0,1)$ and $(0,1) \leftrightarrow \pm(1,0)$].

In calculating (94) we have renormalised the plasma density according to (40). This allows us to omit in, for example, $\varepsilon_{\Delta+0}$ the term $\Sigma_{\Delta+0,0,-0}|E_0|^2$ (note that an analogous procedure has been performed in all other linear dielectric functions).

From equating the determinant of the matrix (94) to zero, the following dispersion equation is obtained:

$$1 = -\Sigma_\Delta \left[|E_0|^2 \left(\frac{1}{\varepsilon_{\Delta+0}} + \frac{1}{\varepsilon_{\Delta-0}} \right) + |\tilde{E}_1|^2 \left(\frac{1}{\bar{\varepsilon}_{\Delta+1}} + \frac{1}{\bar{\varepsilon}_{\Delta-1}} \right) \right]. \tag{102}$$

An analogous expression has been investigated by Vladimirov and Tsytovich (1990). The present analysis allows us to establish limits on the above approximation (Vladimirov and Tsytovich 1993*b*).

First we note that in the situation considered any restrictions on the pump amplitude such as (88) are still absent. Furthermore, let us ascertain when it is possible to neglect the contributions from modulational perturbations $\delta E^v_{\Delta\pm(\delta+0)}$ and $\delta E^v_{\Delta\pm(\delta-1)}$ in (97) [i.e. when the third and fourth terms in the r.h.s. of (97) can be dropped]. The most important question in this context is that of when $\Delta^{\rm mod}_{\mp 0,\pm 1}$ can be taken to be zero.

In the limit when $|\Delta\omega| \ll |\delta\omega|$ and $|\Delta\mathbf{k}| \ll |\mathbf{k}_{0,1}|$ from the equation $\Delta^{\rm mod} = 0$ we find (where, for simplicity, we put $|E_0| = |E_1|$ and thus $\Delta^{\rm mod}_{-0,1} = \Delta^{\rm mod}_{0,-1} \equiv \Delta^{\rm mod}$)

$$4\,\frac{\Delta\omega^2}{\omega_{pe}^2} = \left[6\left(\mathbf{k}_0^2 + \mathbf{k}_1^2\right) r_{De}^2 - 2\Sigma_\delta |E_0|^2\right]^2 - \frac{25\mathbf{k}_0^2\mathbf{k}_1^2\left(2\Sigma_\delta|E_0|^2\right)^2}{\left(\mathbf{k}_1^2 + 4\mathbf{k}_0^2\right)\left(4\mathbf{k}_1^2 + \mathbf{k}_0^2\right)}. \tag{103}$$

This equation has imaginary solutions when (note that in the case considered $\mathbf{k}_0 \perp \mathbf{k}_1$ we have $\mathbf{k}_0^2 + \mathbf{k}_1^2 = \delta\mathbf{k}^2$)

$$\frac{6\delta\mathbf{k}^2 r_{De}^2}{1 + 5k_0k_1/(k_1^2 + 4k_0^2)^{\frac{1}{2}}(4k_1^2 + k_0^2)^{\frac{1}{2}}} < 2\Sigma_\delta|E_0|^2$$

$$< \frac{6\delta\mathbf{k}^2 r_{De}^2}{1 - 5k_0k_1/(k_1^2 + 4k_0^2)^{\frac{1}{2}}(4k_1^2 + k_0^2)^{\frac{1}{2}}} \tag{104}$$

(the expression in the denominator of the r.h.s. is non-negative, and is equal to zero when $k_0 = k_1$, $k_j \equiv |\mathbf{k}_j|$).

In the limit when $|\Delta\omega| \ll |\delta\omega|$ and $|\Delta\mathbf{k}| \gg k_1$ (if $k_1 \ll k_0$), instead of (104), we have

$$\frac{3(\Delta\mathbf{k}^2 + 2k_0^2) r_{De}^2}{1 + 3k_0|\Delta\mathbf{k}|/(\Delta\mathbf{k}^2 + 4k_0^2)^{\frac{1}{2}}(4k_0^2 + \Delta\mathbf{k}^2)^{\frac{1}{2}}} < 2\Sigma_\delta|E_0|^2$$

$$< \frac{3(\Delta\mathbf{k}^2 + 2k_0^2) r_{De}^2}{1 - 3k_0|\Delta\mathbf{k}|/(\Delta\mathbf{k}^2 + 4k_0^2)^{\frac{1}{2}}(4k_0^2 + \Delta\mathbf{k}^2)^{\frac{1}{2}}}. \tag{105}$$

Thus if $\Delta^{\rm mod} = 0$, modulational amplification of satellites $\delta E^v_{\Delta\pm(\delta+0)}$ and $\delta E^v_{\Delta\pm(\delta-1)}$ can be realised. Here, we note that satellites $\delta E_{\Delta\pm 1}$ can also be amplified, but

their intensity is considerably less: $\max\delta E_{\Delta\pm1} \ll \min\{|\delta E^v_{\Delta\pm(\delta+0)}|,\ |\delta E^v_{\Delta\pm(\delta-1)}|\}$. However, (104) and (105) are necessary but not sufficient conditions for this modulational amplification to occur. In addition the instability rate [which is a solution of (103)] must satisfy the equation

$$1 = -\Sigma_\Delta |E_0|^2 \left(\frac{1}{\varepsilon_{\Delta+0}} + \frac{1}{\varepsilon_{\Delta-1}} \right) \tag{106}$$

connected with the satellites $\delta E_{\Delta\pm0}$.

If this pump level does not satisfy (104) or (105), then all satellites increase simultaneously with a rate determined by (102); in this case we can neglect the contribution from modulational perturbations of the virtual fields $\delta E^v_{\Delta\pm(\delta+0)}$ and $\delta E^v_{\Delta\pm(\delta-1)}$ if

$$2\Sigma_\delta |E_0|^2 \ll \frac{6\delta \mathbf{k}^2 r_{De}^2}{1 + 5k_0k_1/(k_1^2+4k_0^2)^{\frac{1}{2}}(4k_1^2+k_0^2)^{\frac{1}{2}}} \tag{107}$$

when $|\Delta\mathbf{k}| \ll k_1$, and if

$$2\Sigma_\delta |E_0|^2 \ll \frac{3(\Delta\mathbf{k}^2 + 2k_0^2) r_{De}^2}{1 + 3k_0|\Delta\mathbf{k}|/(\Delta\mathbf{k}^2+4k_0^2)^{\frac{1}{2}}(4k_0^2+\Delta\mathbf{k}^2)^{\frac{1}{2}}} \tag{108}$$

when $|\Delta\mathbf{k}| \gg k_1$. Thus we have established (following Vladimirov and Tsytovich 1993) the limits of the approximation that was used by Vladimirov and Tsytovich (1990) to investigate equation (102) (when all terms connected with modulational perturbations from the virtual fields $\delta E^v_{\Delta\pm(\delta+0)}$ and $\delta E^v_{\Delta\pm(\delta-1)}$ have been omitted).

5.3 Instability Rates

Here, we present expressions for the modulational instability rates obtained by Vladimirov and Tsytovich (1993*b*). For a low pump level satisfying (107) and (108) the maximum rates are realised in the supersonic limit $|\Delta\omega| \gg |\Delta\mathbf{k}| v_s$. Note here that, because of our condition $|\delta\omega| > |\Delta\omega|$, the above limit is realised when $|\Delta\mathbf{k}| \ll k_1$ or $k_1 \ll |\Delta\mathbf{k}| \ll k_0$. In these cases we find the instability rate γ^{mod} to be proportional to $|\Delta\mathbf{k}|$; its maximum value is achieved when

$$|\Delta\mathbf{k}| \approx \frac{1}{r_{De}} \left[\frac{|E_0|^2}{4\pi n T_e} \frac{m_e}{m_i} \left(1 + \frac{\bar{r}_{De}^2}{r_{De}^2} \right) \right]^{\frac{1}{4}} \tag{109}$$

if

$$\frac{|E_0|^2}{4\pi n T_e} \gg \frac{m_e}{m_i}, \tag{110}$$

and when

$$|\Delta\mathbf{k}| \approx \frac{1}{r_{De}} \left[\frac{|E_0|^2}{4\pi n T_e} \left(1 + \frac{\bar{r}_{De}^2}{r_{De}^2} \right) \right]^{\frac{1}{2}} \tag{111}$$

if

$$\frac{|E_0|^2}{4\pi n T_e} \ll \frac{m_e}{m_i}\,. \tag{112}$$

Of course, the inequality (110) holds simultaneously with (107) or (108). Furthermore, in (109) and (111) we have

$$\bar{r}_{De}^2 \equiv r_{De}^2 - \frac{1}{k_1^2 + \Delta\mathbf{k}^2}\,\frac{|E_0|^2}{4\pi n T_e}\,. \tag{113}$$

Note that $r_{De}^2 + \bar{r}_{De}^2 > 0$ for (107), as for (108). Finally, we have for the maximum instability rate

$$\gamma_{\max} \approx \omega_{pe}\left[6\,\frac{|E_0|^2}{4\pi n T_e}\,\frac{m_e}{m_i}\left(1 + \frac{\bar{r}_{De}^2}{r_{De}^2}\right)\right]^{\frac{1}{2}} \tag{114}$$

for the pump level (110), and

$$\gamma_{\max} \approx \omega_{pe} 3\,\frac{|E_0|^2}{4\pi n T_e}\left(1 + \frac{\bar{r}_{De}^2}{r_{De}^2}\right) \tag{115}$$

for the pump level (112). For the rates (114) and (115) to be valid all inequalities adopted in the approximation considered here must be satisfied. From the conditions $|\delta\omega| > \gamma$ and $|\Delta\mathbf{k}| \ll k_1$ or $k_1 \ll |\Delta\mathbf{k}| \ll k_0$, the inequality $\gamma \ll k_0^2 r_{De}^2$ follows. By combining this inequality with the pump levels (107) or (108), we arrive at the conclusion that the above condition is satisfied if $k_0^2 r_{De}^2 \gg m_e/m_i$ [for (110)] and is compatible with the other restrictions for (112). The inequality $k_0^2 r_{De}^2 \gg m_e/m_i$ means that $|\delta\omega| \gg |\delta\mathbf{k}| v_s$, i.e. decays $l \to l' + s$ are possible; we then have to take $\gamma\tau_{\rm dec} \gg 1$, where $\tau_{\rm dec}$ is defined by (26). Finally, we find that the decays can be ignored if

$$1 \ll k_0^2 r_{De}^2\,\frac{m_e}{m_i}\,. \tag{116}$$

This inequality is clearly not satisfied. Thus a situation with maximum instability rate (114) in the case considered cannot be realised because of violation of the adopted conditions.

Now, we consider the case of a high pump level

$$2\Sigma_\delta |E_0|^2 \gg \frac{6\delta\mathbf{k}^2 r_{De}^2}{1 - 5k_0k_1/(k_1^2 + 4k_0^2)^{\frac{1}{2}}(4k_1^2 + k_0^2)^{\frac{1}{2}}} \tag{117}$$

when $|\Delta\mathbf{k}| \ll k_1$ [and clearly $k_0 \neq k_1$, i.e. when the denominator of the r.h.s. of

(117) is not equal (or even close) to zero], or

$$2\Sigma_\delta|E_0|^2 \gg \frac{3(\Delta\mathbf{k}^2+2k_0^2)r_{De}^2}{1-3k_0|\Delta\mathbf{k}|/(\Delta\mathbf{k}^2+4k_0^2)^{\frac{1}{2}}(4k_0^2+\Delta\mathbf{k}^2)^{\frac{1}{2}}} \tag{118}$$

when $|\Delta\mathbf{k}| \gg k_1$, and $|\Delta\mathbf{k}| \neq k^+ \approx 2k_0$ when the denominator of the r.h.s. of (118) is not equal (or even close) to zero. For these cases, we again find a maximum instability rate (115) [and the rate (114) may not be realised for the same reasons as above]. For the 'renormalised' Debye length we have

$$\bar{r}_{De}^2 = r_{De}^2\left[1+\frac{\delta\mathbf{k}^2(4k_0^4+2k_0^2k_1^2+4k_1^4)}{2(\Delta\mathbf{k}^2+k_1^2)(k_0^2-k_1^2)}\right] \tag{119}$$

if $|\Delta\mathbf{k}| \ll k_1$, and

$$\bar{r}_{De}^2 = r_{De}^2\left[1+\frac{(\Delta\mathbf{k}^2+2k_0^2)(\Delta\mathbf{k}^4+2\Delta\mathbf{k}^2k_0^2+4k_0^4)}{(\Delta\mathbf{k}^2+k_0^2)(\Delta\mathbf{k}^2-2k_0^2)}\right] \tag{120}$$

if $|\Delta\mathbf{k}| \gg k_1$.

The most interesting situation, from our point of view, is realised under condition (104) or (105). A solution of the dispersion equation (103) is

$$\gamma \approx \omega_{pe}\,\frac{5k_0k_1}{(k_1^2+4k_0^2)^{\frac{1}{2}}(4k_1^2+k_0^2)^{\frac{1}{2}}}\Sigma_\delta|E_0|^2 \tag{121}$$

for $|\Delta\mathbf{k}| \ll k_1$, or

$$\gamma \approx \omega_{pe}\,\frac{3|\Delta\mathbf{k}|k_0}{(\Delta\mathbf{k}^2+4k_0^2)^{\frac{1}{2}}(\Delta\mathbf{k}^2+k_0^2)^{\frac{1}{2}}}\Sigma_\delta|E_0|^2 \tag{122}$$

for $|\Delta\mathbf{k}| \gg k_1$. These solutions can satisfy (106) if

$$|\Delta\mathbf{k}|r_{De} \approx |\delta\mathbf{k}|r_{De}\left(\frac{|E_0|^2}{4\pi nT_e}\right)^{\frac{3}{4}}\left(\frac{m_e}{m_i}\right)^{\frac{1}{4}}\frac{5k_0k_1}{(k_1^2+4k_0^2)^{\frac{1}{2}}(4k_1^2+k_0^2)^{\frac{1}{2}}} \tag{123}$$

for $|\Delta\mathbf{k}| \ll k_1$, or

$$|\Delta\mathbf{k}|r_{De} \approx \left(|\delta\mathbf{k}|^2+k_0^2\right)^{\frac{1}{2}}r_{De}\left(\frac{|E_0|^2}{4\pi nT_e}\right)^{\frac{3}{4}}\left(\frac{m_e}{m_i}\right)^{\frac{1}{4}}\frac{3|\Delta\mathbf{k}|k_0}{(\Delta\mathbf{k}^2+4k_0^2)^{\frac{1}{2}}(\Delta\mathbf{k}^2+k_0^2)^{\frac{1}{2}}} \tag{124}$$

for $|\Delta\mathbf{k}| \gg k_1$. By analysing (123) and (124) we can conclude that they satisfy all necessary conditions if $|\Delta\mathbf{k}| \ll k_1 < k_{cr}$ [for (123)] or $k_1 \ll |\Delta\mathbf{k}| \ll k_0 < k_{cr}$ [for (124)], where $k_{cr} = (m_e/m_i)^{\frac{1}{2}}/3r_{De}$. Thus the above situation can be realised.

Let us stress that the absence of an instability threshold for the above situation has been demonstrated by Vladimirov and Tsytovich (1990). Our present consideration (see also Vladimirov and Tsytovich 1993*b*) confirms this [see expressions (107) and (108) for a low pump level].

Thus the theory considered of modulational interactions of two pump waves allows (within its limits of applicability) the exact solution of the problem. The most important results are as follows.

(1) An infinite system of equations describing modulational interactions of two modes can be reduced to a system of eight equations. These equations describe the interactions of four satellites from the pump waves and also four satellites from virtual waves $\delta E^v_{\Delta\pm(\delta+0)}$ and $\delta E^v_{\Delta\pm(\delta-1)}$. The inequality (88) establishes limits within which the theory is valid for the case of Langmuir waves.

(2) In the particular situation where the virtual fields $\delta E^v_{\Delta\pm(\delta+0)}$ and $\delta E^v_{\Delta\pm(\delta-1)}$ are absent (for Langmuir waves this situation is realised when $\mathbf{k}_0 \perp \mathbf{k}_1$, i.e. the pump waves propagate in perpendicular directions) their modulational perturbations are still present. We can neglect these perturbations only for a sufficiently low pump level [see inequalities (107) and (108)]. When conditions (107) and (108) are satisfied, we have a system of four equations (when only satellites from pump waves are taken into account). The maximum instability rate in this case (115) is less than the maximum rate of one-mode modulational instability (Rudakov and Tsytovich 1978) because of the additional factor $1+\bar{r}^2_{De}/r^2_{De} < 1$, where $\bar{r}^2_{De}$ is defined by (113). We should stress here that an analogous decrease in the instability rate also occurs for turbulent fields (see Section 7 below).

(3) For larger pump levels, when (107) and (108) are not satisfied, the satellites from the virtual waves $\delta E^v_{\Delta\pm(\delta+0)}$ and $\delta E^v_{\Delta\pm(\delta-1)}$ significantly affect the modulational interaction. In this case a situation is possible in which satellites from some pump waves can be suppressed; for example, for (104) and (105) the satellites from the pump E_1 are suppressed. For high pump levels (117) and (118) the maximum rates are greater than the one-mode instability rate by a factor of $1+\bar{r}^2_{De}/r^2_{De} > 1$, where $\bar{r}^2_{De}$ is defined by (119) or (120), and can be of the order of the instability rate of one mode with intensity E_0+E_1.

6. Interaction of Broad Wave Packets

Here, we consider random pump fields. To model the third-order-in-field properties of the plasma, we adopt the Zakharov model, where the effective plasma response is given by equation (32). That is, we neglect all high-order effects such as electron (including interaction through virtual fields at double-harmonic frequency), relativistic, etc. nonlinearities. We demonstrate that the modulational interaction is described by a coupled set of integral equations which generalise an infinite system of coupled equations describing the modulational interaction of two monochromatic pumps in the general case.

6.1 General Equations

The nonlinear equation for random fields can be obtained by averaging (10) over a statistical ensemble (for details see e.g. Tsytovich 1977). We find

$$\varepsilon\delta E^+ = \int \Sigma^{\text{eff}}(\delta E_1^+\delta E_2^+\delta E_3^- - \delta E_1^+\langle\delta E_2^+\delta E_3^-\rangle - \langle\delta E_1^+\delta E_2^+\delta E_3^-\rangle)d^{(3)}\,, \qquad (125)$$

where δE is the random field, and angle brackets denote the average quantity. The terms of (125) which contain the average values are obtained by expansion

of the plasma distribution function f in the random and regular components. Note that in general the total field is $E + \delta E$, where E is its regular component; however, in (125) the regular component of the field E is neglected. For the fields of the waves propagating in a plasma, the non-zero regular fields can in principle be determined by external conditions. Also, the random fields can excite regular virtual fields. An example is the following interaction:

$$\varepsilon E = 2 \int S \langle \delta E_1^+ \delta E_2^- \rangle d^{(2)} , \tag{126}$$

which takes place, in particular, when the nonlinear interaction of regular fields is neglected.

It should be noted that for homogeneous and isotropic turbulence [when $\langle \delta E_1^+ \delta E_2^- \rangle \sim \delta(k_1 + k_2)$], the field E in (126) is proportional to $\delta(k)$. Also, the second-order response S is proportional to $\mathbf{k}$, which can easily be seen from its concrete expression (Rudakov and Tsytovich 1978). Therefore, we have $E = 0$. However, if turbulent perturbations are inhomogeneous then $E \neq 0$. Thus, as the modulational instability grows, the low-frequency and large-scale regular fields develop. This effect, obtained by Tsytovich (1970), is the characteristic feature of the process of turbulent self-organisation in which the nonlinear interaction of regular fields is essential.

Furthermore, we assume that the nonlinear spectrum of weak turbulence $\delta E^{(0)}$ is a solution of (125), taking into account turbulent sources described by Q_k. We have

$$\begin{aligned} \varepsilon \delta E^{+(0)} = \int \Sigma^{\text{eff}} (\delta E_1^{+(0)} \delta E_2^{+(0)} \delta E_3^{-(0)} - \delta E_1^{+(0)} \langle \delta E_2^{+(0)} \delta E_3^{-(0)} \rangle \\ - \langle \delta E_1^{+(0)} \delta E_2^{+(0)} \delta E_3^{-(0)} \rangle) d^{(3)} + Q_k . \end{aligned} \tag{127}$$

Note that the source Q_k can, in particular, eliminate wave phase correlations sufficiently rapidly that the nonlinear interaction has not sufficient time to react to this process. Below, we assume that the source Q_k is not perturbed by the modulational interaction. Thus, we have

$$\delta E^{\pm} = \delta E^{\pm(0)} + \delta' E^{\pm} . \tag{128}$$

From (126) we then find (for $E^{(0)} = 0$)

$$\varepsilon E = 2 \int S (\langle \delta' E_1^+ \delta E_2^{-(0)} \rangle + \langle \delta E_1^{+(0)} \delta' E_2^- \rangle) d^{(2)} \tag{129}$$

Therefore, from (127) we finally obtain

$$\begin{aligned} \varepsilon \delta' E^+ = \int \Sigma^{\text{eff}} (\delta' E_1^+ \delta E_2^{+(0)} \delta E_3^{-(0)} + \delta E_1^{+(0)} \delta' E_2^+ \delta E_3^{-(0)} + \delta E_1^{+(0)} \delta E_2^{+(0)} \delta' E_3^- \\ - \delta' E_1^+ \langle \delta E_2^{+(0)} \delta E_3^{-(0)} \rangle - \delta E_1^{+(0)} \langle \delta' E_2^+ \delta E_3^{-(0)} \rangle - \delta E_1^{+(0)} \langle \delta E_2^{+(0)} \delta' E_3^- \rangle \\ - \langle \delta' E_1^+ \delta E_2^{+(0)} \delta E_3^{-(0)} \rangle - \langle \delta E_1^{+(0)} \delta' E_2^+ \delta E_3^{-(0)} \rangle - \langle \delta E_1^{+(0)} \delta E_2^{+(0)} \delta' E_3^- \rangle) d^{(3)} . \end{aligned} \tag{130}$$

We have taken into account in (130) only linear (with respect to field perturbations) terms, as well as the assumption that the source Q_k has not been perturbed in the process of the interaction. The analogous equation can be written for $\delta' E^-$ from (130) using the interchange $(+) \leftrightarrow (-)$. These two coupled equations are the basis for our investigation of the modulational instability.

6.2 Regular and Random Fields

Since the regular fields (129) are excited as a result of the modulational instability, it is necessary to discuss how they should be accounted for in equation (130) or in the more general equation (125).

In kinetic theory, the nonlinearity is determined by the term

$$(\mathbf{E} + \delta\mathbf{E}) \cdot \frac{\partial f}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \cdot (\mathbf{E} + \delta\mathbf{E}) f \tag{131}$$

(we note that here $\mathbf{E}$ is the regular component of the electric field). Thus we analyse the expression $(\mathbf{E} + \delta\mathbf{E})f$ where both regular and random components of the electric field and the distribution function f should be considered. Below, we denote the regular component of the distribution function as Φ. The kinetic equation for the regular component contains nonlinear terms such as

$$E\Phi + \langle \delta E \delta f \rangle \,, \tag{132}$$

which can be obtained by averaging the basic kinetic equation. The kinetic equation for the random distribution has the nonlinear term

$$E\delta f + \delta E \Phi + \delta E \delta f - \langle \delta E \delta f \rangle \,. \tag{133}$$

Furthermore, we expand δf and Φ in terms proportional to powers of the fields E and δE up to third order. The last term in (132) can be treated either on the basis of perturbation theory or not. The issue is that in the first approximation (when only the linear term $\delta f \sim \delta E$ is taken into account) this term describes quasi-linear effects and can be included in the distribution function Φ which becomes weakly inhomogeneous and weakly nonstationary. Otherwise, if the second term in (132) is treated on the basis of perturbation theory, then the quasi-linear interaction is considered as a perturbation. If such an interaction takes place in the absence of resonances of waves with plasma particles, then the quasi-linear effects (in the case of inhomogeneous random fields) results in the ponderomotive force.

Below, we consider the quasi-linear interaction as a perturbation. Then the cubic nonlinear equations for the random and regular fields have the

following form:

$$\varepsilon\delta E^+ = \int \Sigma^{\text{eff}}(\delta E_1^+\delta E_2^+\delta E_3^- - \langle\delta E_1^+\delta E_2^+\delta E_3^-\rangle)d^{(3)}$$
$$+\int \Sigma^{\text{eff}}(\delta E_1^+E_2^+E_3^- + E_1^+E_2^+\delta E_3^- + E_1^+\delta E_2^+E_3^-)d^{(3)}$$
$$+\int \Sigma^{\text{eff}}[E_1^+(\delta E_2^+\delta E_3^- - \langle\delta E_2^+\delta E_3^-\rangle) + \delta E_1^+E_2^+\delta E_3^-$$
$$-\langle\delta E_1^+E_2^+\delta E_3^-\rangle + \delta E_1^+\delta E_2^+E_3^- - \langle\delta E_1^+\delta E_2^+\rangle E_3^-]d^{(3)}\,, \quad (134)$$

$$\varepsilon E^+ = \int \Sigma^{\text{eff}}(E_1^+E_2^+E_3^- + \langle\delta E_1^+\delta E_2^+\delta E_3^-\rangle)d^{(3)}$$
$$+\int \Sigma^{\text{eff}}(E_1^+\langle\delta E_2^+\delta E_3^-\rangle + \langle\delta E_1^+E_2^+\delta E_3^-\rangle + \langle\delta E_1^+\delta E_2^+\rangle E_3^-)d^{(3)}\,. \quad (135)$$

Contrary to (125), the first integral on the r.h.s. of (134) has no term $\delta E_1^+\langle\delta E_2^+\delta E_3^-\rangle$ which is present if the quasi-linear interaction is not considered as a perturbation. In our derivation of equations (134) and (135) we have assumed that the virtual field has both regular

$$\varepsilon E^v = 2\int S(E_1^+E_2^- + \langle\delta E_1^+\delta E_2^-\rangle)d^{(2)}\,, \quad (136)$$

and random component

$$\varepsilon\delta E^v = 2\int S(E_1^+\delta E_2^- + \delta E_1^+E_2^- + \delta E_1^+\delta E_2^- - \langle\delta E_1^+\delta E_2^-\rangle)d^{(2)} \quad (137)$$

(we recall that we account for only virtual fields at 'zero' frequency). Expression (136) can be transformed to (126) if the nonlinear interaction is neglected. We assume that at the initial moment the regular fields are absent, and the problem is to investigate their further development. Here, we restrict our attention to the linear development of the modulational instability. Then the regular fields are proportional to the random field squared, see (126).

Thus, neglecting terms which contain higher than first powers of the regular field, we find that:

(a) Equation (136) coincides with (126).

(b) Because of the second integral on the r.h.s. of (135), which contains $\langle\delta E_1^+\delta E_2^+\delta E_3^-\rangle$, the regular fields arise also in the cubic approximation (in the random electric fields). However, in contrast to (126), they are high-frequency. In the limit of weak turbulence, the average of three random fields is equal to zero, and in the first approximation the term $\langle\delta E_1^+\delta E_2^+\delta E_3^-\rangle$ can be neglected.

(c) We can neglect the first two integrals of equation (135) if we assume that the correlator with two positive frequency random fields is equal to zero and that the random fields (in the first approximation) are stationary and homogeneous, i.e.

$$\langle\,\delta E_k^+\delta E_{k_1}^-\,\rangle = -|E^{+(0)}|_k^2\delta(k+k_1)\,. \quad (138)$$

Therefore, we find

$$(\varepsilon_k + \varepsilon_0^N)E_k^+ = 0\,, \tag{139}$$

where

$$\varepsilon_0^N = \int \Sigma^{\text{eff}}_{k,k_1,-k_1}|E|^2_{k_1}dk_1\,. \tag{140}$$

Here, $\Sigma^{\text{eff}}_{k_1,k_2,k_3} = \Sigma^{\text{eff}}_{kk_1k_2k_3}$, and the minus sign on the r.h.s. of (138) appears because the absolute value of the electric field vector $\mathbf{E}_k = (\mathbf{k}/k)E_k$ is included in (138), while the vectors $\mathbf{k}$ and $\mathbf{k}_1$ are opposite. We see that the high-frequency regular field does not couple with the low-frequency perturbations. Consequently, if the regular field is equal to zero at the initial moment, it will be zero at all other moments.

(d) The low-frequency regular field (126) is excited.

Now we see that we can use equation (130) for further analysis. Taking into account that the modulational instability can be expressed in terms of the regular fields excited and that equation (129) is valid, we can multiply equation (130) by $\delta E_{k'}^{-(0)}$ and average the expression obtained. We find

$$\varepsilon_k\langle\delta' E_k^+\delta E_{k'}^{-(0)}\rangle = \int \Sigma^{\text{eff}}(\langle\delta' E_1^+\delta E_{k'}^{-(0)}\delta E_2^{+(0)}\delta E_3^{-(0)}\rangle$$
$$+\langle\delta E_1^{+(0)}\delta' E_2^+\delta E_{k'}^{-(0)}\delta E_3^{-(0)}\rangle + \langle\delta E_1^{+(0)}\delta E_2^{+(0)}\delta E_{k'}^{-(0)}\delta' E_3^-\rangle)d^{(3)}\,. \tag{141}$$

The analogous equation can be obtained for $\langle\delta' E_k^-\delta E_{k'}^{+(0)}\rangle$, namely

$$\varepsilon_k\langle\delta' E_k^-\delta E_{k'}^{+(0)}\rangle = \int \Sigma^{\text{eff}}(\langle\delta' E_1^-\delta E_{k'}^{+(0)}\delta E_2^{-(0)}\delta E_3^{+(0)}\rangle$$
$$+\langle\delta E_1^{-(0)}\delta' E_2^-\delta E_{k'}^{+(0)}\delta E_3^{+(0)}\rangle + \langle\delta E_1^{-(0)}\delta E_2^{-(0)}\delta E_{k'}^{+(0)}\delta' E_3^+\rangle)d^{(3)}\,. \tag{142}$$

The appearance of regular fields in the development of the modulational instability of random fields is important from the physical point of view. In particular, the non-zero ponderomotive force (which is the force of high-frequency pressure) arises for the random fields

$$F_x \sim \frac{\partial}{\partial x}\langle|E|^2\rangle \neq 0\,. \tag{143}$$

This force creates regular perturbations of the concentration δn which, in turn, lead to generation of the regular low-frequency electric fields.

6.3 Perturbations of Correlation Functions

For weak turbulent fields $E^{\pm(0)}$ we have the approximate relationship

$$\langle \delta E_1 \delta E_2 \delta E_3 \delta E_4 \rangle \approx \langle \delta E_1 \delta E_2 \rangle \langle \delta E_3 \delta E_4 \rangle + \langle \delta E_1 \delta E_3 \rangle \langle \delta E_2 \delta E_4 \rangle + \langle \delta E_1 \delta E_4 \rangle \langle \delta E_2 \delta E_3 \rangle . \tag{144}$$

The modulational perturbations are low-frequency, so they weakly shift the frequencies of the initial pump wave packet (from $\approx \omega_{pe}$ for Langmuir waves). This means that correlations are equal to zero for the components of the same frequency sign (even in the presence of modulational perturbations). Thus we have

$$\langle \delta' E^+ \delta E^+ \rangle = \langle \delta' E^- \delta E^- \rangle = 0 . \tag{145}$$

Furthermore, although modulational perturbations are correlated, their correlation with the initial random field cannot be strong, because the random character of the pump field is determined independently by the initial conditions of the random pumping. We will assume that relationship (144) is valid, in particular, in the case when only *one* of the fields is the field of the modulational perturbations $\delta' E$ [for the case of two fields equation (144) can be violated, since in this case the fields can be strongly correlated]. We note that in the linear approximation only linear perturbations $\delta' E$ are included in (141) and (142).

Now, we introduce the functions

$$G^{\pm}_{k,k'} = \langle \delta' E^{\pm}_{\pm k+k'} E^{\mp(0)}_{\mp k} \rangle . \tag{146}$$

Then we obtain from (141)

$$\begin{aligned} \varepsilon_{k+k'} G^{+}_{k,k'} = & \\ & - \int \Sigma^{\text{eff}}_{k+k',k_1,-k_1} G^{+}_{k,k_1} |E^{+(0)}|^2_{k_1} dk_1 - \int \Sigma^{\text{eff}}_{k_1,k+k',-k_1} G^{+}_{k,k_1} |E^{+(0)}|^2_{k_1} dk_1 \\ & - |E^{+(0)}|^2_k \int G^{+}_{k_1,k'} (\Sigma^{\text{eff}}_{k_1+k',k,-k_1} + \Sigma^{\text{eff}}_{k,k_1+k',-k_1}) dk_1 \\ & - |E^{+(0)}|^2_k \int G^{-}_{k_1,k'} (\Sigma^{\text{eff}}_{k,k_1,k'-k_1} + \Sigma^{\text{eff}}_{k_1,k,k'-k_1}) dk_1 , \end{aligned} \tag{147}$$

$$\begin{aligned} \varepsilon_{-k+k'} G^{-}_{k,k'} = & \\ & - \int \Sigma^{\text{eff}}_{-k+k',-k_1,k_1} G^{-}_{k,k_1} |E^{+(0)}|^2_{k_1} dk_1 - \int \Sigma^{\text{eff}}_{-k_1,-k+k',k_1} G^{-}_{k,k_1} |E^{+(0)}|^2_{k_1} dk_1 \\ & - |E^{+(0)}|^2_k \int G^{-}_{k_1,k'} (\Sigma^{\text{eff}}_{-k_1+k',-k,k_1} + \Sigma^{\text{eff}}_{-k,-k_1+k',k_1}) dk_1 \\ & - |E^{+(0)}|^2_k \int G^{+}_{k_1,k'} (\Sigma^{\text{eff}}_{-k,-k_1,k'+k_1} + \Sigma^{\text{eff}}_{-k_1,-k,k'+k_1}) dk_1 . \end{aligned} \tag{148}$$

In the derivation of these equations, we have used that

$$\langle \delta E_{k_1}^{+(0)} \delta E_{k_2}^{-(0)} \rangle = -|E^{+(0)}|_{k_1}^2 \delta(k_1 + k_2) \,. \tag{149}$$

The correlator $|E^{+(0)}|_{k_1}^2$ contains (in the limits of weak turbulence theory) only $\delta(\omega - \omega_{\mathbf{k}})$, but not $\delta(\omega + \omega_{\mathbf{k}})$. The total correlator $|E^{(0)}|_k^2$ is given by

$$|E^{(0)}|_k^2 = |E^{+(0)}|_k^2 + |E^{+(0)}|_{-k}^2 \,. \tag{150}$$

The first terms on the r.h.s. of equations (147) and (148) depend on the way of switching on the random pumping. Indeed, we have [compare with (32)]

$$\Sigma^{\text{eff}}_{k+k',k_1,-k_1'} \approx -\frac{e^2}{m_e^2 \omega_{pe}^2 v_{Te}^2} \cdot \frac{\mathbf{k}_1 \cdot \mathbf{k}_1'}{|\mathbf{k}_1 \parallel \mathbf{k}_1'|} \frac{(\mathbf{k}_1 - \mathbf{k}_1')^2 v_s^2}{(\omega_1 - \omega_1')^2 - (\mathbf{k}_1 - \mathbf{k}_1')^2 v_s^2} \,. \tag{151}$$

This expression is equal to zero if at the beginning $\mathbf{k}_1 - \mathbf{k}_1'$ approaches zero, and later $\omega_1 - \omega_1'$ tends to zero. If at the beginning $\omega_1 - \omega_1'$ approaches zero and later $\mathbf{k}_1 - \mathbf{k}_1'$ tends to zero, then the expression (151) is a non-zero constant which can be removed by renormalisation of the initial plasma density $n_0 \to n_0 + \delta n_0$. Thus, below we do not take into account the first terms on the r.h.s. of (147) and (148). The second terms on the r.h.s. of (147) and (148) describe the nonlinear dielectric permittivity

$$\varepsilon_k^N = \int \Sigma^{\text{eff}}_{k_1,k,-k_1} |E^{+(0)}|_{k_1}^2 dk_1 \,. \tag{152}$$

Furthermore, if $\mathbf{k}_2 + \mathbf{k}_3$ satisfies the relationships

$$|\mathbf{k}_2 + \mathbf{k}_3| v_{Ti} \ll \omega_2 + \omega_3 \ll |\mathbf{k}_2 + \mathbf{k}_3| v_{Te} \,, \tag{153}$$

then we have expression (32) for the effective third-order plasma response. If

$$\omega_2 + \omega_3 \ll |\mathbf{k}_2 + \mathbf{k}_3| v_{Ti} \,, \tag{154}$$

then

$$\Sigma^{\text{eff}}_{k_1,k_2,k_3} \approx -\frac{1}{4\pi n_0 (T_e + T_i)} \frac{(\mathbf{k} \cdot \mathbf{k}_1)(\mathbf{k}_2 \cdot \mathbf{k}_3)}{|\mathbf{k} \parallel \mathbf{k}_1 \parallel \mathbf{k}_2 \parallel \mathbf{k}_3|} \,. \tag{155}$$

Equation (147) contains the following types of expressions:

$$\Sigma^{\text{eff}}_{k,k_1+k',-k_1} \approx -\frac{1}{4\pi n_0 T_e} \frac{(\mathbf{k}_1 + \mathbf{k}') \cdot \mathbf{k}_1}{|\mathbf{k}_1 + \mathbf{k}' \parallel \mathbf{k}_1|} \frac{|\mathbf{k}'|^2 v_s^2}{\omega'^2 - |\mathbf{k}'|^2 v_s^2} \tag{156}$$

if $|\mathbf{k}'| v_{Ti} \ll \omega' \ll |\mathbf{k}'| v_{Te}$, and

$$\Sigma^{\text{eff}}_{k,k_1+k',-k_1} \approx \frac{1}{4\pi n_0 (T_e + T_i)} \frac{(\mathbf{k}_1 + \mathbf{k}') \cdot \mathbf{k}_1}{|\mathbf{k}_1 + \mathbf{k}' \parallel \mathbf{k}_1|} \tag{157}$$

if $\omega' \ll |\mathbf{k}'|v_{Ti}$. Also, we have

$$\Sigma^{\text{eff}}_{k_1+k',k,-k_1} \approx -\frac{1}{4\pi n_0 T_e}\,\frac{(\mathbf{k}+\mathbf{k}')\cdot(\mathbf{k}_1+\mathbf{k}')}{|\mathbf{k}+\mathbf{k}'\parallel\mathbf{k}_1+\mathbf{k}'|}\,\frac{(\mathbf{k}\cdot\mathbf{k}_1)}{|\mathbf{k}\parallel\mathbf{k}_1|}\,\frac{(\mathbf{k}-\mathbf{k}_1)^2 v_s^2}{(\omega-\omega_1)^2-(\mathbf{k}-\mathbf{k}_1)^2 v_s^2} \tag{158}$$

if $|\mathbf{k}-\mathbf{k}_1|v_{Ti} \ll |\omega-\omega_1| \ll |\mathbf{k}-\mathbf{k}_1|v_{Te}$, and

$$\Sigma^{\text{eff}}_{k_1+k',k,-k_1} \approx \frac{1}{4\pi n_0(T_e+T_i)}\,\frac{(\mathbf{k}+\mathbf{k}')\cdot(\mathbf{k}_1+\mathbf{k}')}{|\mathbf{k}+\mathbf{k}'\parallel\mathbf{k}_1+\mathbf{k}'|}\,\frac{(\mathbf{k}\cdot\mathbf{k}_1)}{|\mathbf{k}\parallel\mathbf{k}_1|} \tag{159}$$

if $|\omega-\omega_1| \ll |\mathbf{k}-\mathbf{k}_1|v_{Ti}$.

Thus we see that $\Sigma^{\text{eff}}_{k,k_1+k',-k_1}$ depends on the wavevectors ($\mathbf{k}'$) and frequencies (ω') of modulational perturbations, while $\Sigma^{\text{eff}}_{k_1+k',k,-k_1}$ depends on the difference of wavevectors and frequencies of the pump Langmuir waves.

Furthermore, we introduce the notation

$$\alpha_{k'} = \begin{cases} \dfrac{1}{4\pi n_0 T_e}\,\dfrac{|\mathbf{k}'|^2 v_s^2}{\omega'^2-|\mathbf{k}'|^2 v_s^2}\,, & \text{if } |\mathbf{k}'|v_{Ti} \ll \omega' \ll |\mathbf{k}'|v_{Te}\,, \\[2ex] -\dfrac{1}{4\pi n_0(T_e+T_i)}\,, & \text{if } \omega' \ll |\mathbf{k}'|v_{Ti}\,. \end{cases} \tag{160}$$

Then we obtain

$$\tilde{\varepsilon}_k = \varepsilon_k + \varepsilon_k^N\,, \qquad \varepsilon^N_{\pm k+k'} = \int \frac{(\mathbf{k}\cdot\mathbf{k}_1)^2}{|\mathbf{k}|^2|\mathbf{k}_1|^2}\alpha_{\pm k\mp k_1+k'}|E^{+(0)}|^2_{k_1}dk_1\,. \tag{161, 162}$$

Thus, we finally find (Popel *et al.* 1994*b*)

$$\begin{aligned}(\varepsilon_{k+k'}+\varepsilon^N_{k+k'})G^+_{k,k'} = |E^{+(0)}|^2_k \Bigg\{ &\int \frac{[(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}][(\mathbf{k}_1+\mathbf{k}')\cdot\mathbf{k}_1]}{|\mathbf{k}\parallel\mathbf{k}_1\parallel\mathbf{k}+\mathbf{k}'\parallel\mathbf{k}_1+\mathbf{k}'|}\alpha_{k'}G^+_{k_1,k'}dk_1 \\ &+\int \frac{[(\mathbf{k}+\mathbf{k}')\cdot(\mathbf{k}_1+\mathbf{k}')][\mathbf{k}\cdot\mathbf{k}_1]}{|\mathbf{k}\parallel\mathbf{k}_1\parallel\mathbf{k}+\mathbf{k}'\parallel\mathbf{k}_1+\mathbf{k}'|}\alpha_{k-k_1}G^+_{k_1,k'}dk_1 \\ &+\int \frac{[(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}][(\mathbf{k}_1-\mathbf{k}')\cdot\mathbf{k}_1]}{|\mathbf{k}\parallel\mathbf{k}_1\parallel\mathbf{k}+\mathbf{k}'\parallel\mathbf{k}_1-\mathbf{k}'|}\alpha_{k'}G^-_{k_1,k'}dk_1 \\ &+\int \frac{[(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}_1][(\mathbf{k}_1-\mathbf{k}')\cdot\mathbf{k}]}{|\mathbf{k}\parallel\mathbf{k}_1\parallel\mathbf{k}+\mathbf{k}'\parallel\mathbf{k}_1-\mathbf{k}'|}\alpha_{k-k_1+k'}G^-_{k_1,k'}dk_1 \Bigg\}\,, \end{aligned} \tag{163}$$

$$(\varepsilon_{-k+k'} + \varepsilon^N_{-k+k'})G^-_{k,k'} = |E^{+(0)}|^2_k \left\{ \int \frac{[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{k}][(\mathbf{k}_1-\mathbf{k}')\cdot\mathbf{k}_1]}{|\mathbf{k}\parallel\mathbf{k}_1\parallel\mathbf{k}-\mathbf{k}'\parallel\mathbf{k}_1-\mathbf{k}'|}\alpha_{k'}G^-_{k_1,k'}dk_1 \right.$$

$$+\int \frac{[(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{k}_1-\mathbf{k}')][\mathbf{k}\cdot\mathbf{k}_1]}{|\mathbf{k}\parallel\mathbf{k}_1\parallel\mathbf{k}-\mathbf{k}'\parallel\mathbf{k}_1-\mathbf{k}'|}\alpha_{k-k_1}G^-_{k_1,k'}dk_1$$

$$+\int \frac{[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{k}][(\mathbf{k}_1+\mathbf{k}')\cdot\mathbf{k}_1]}{|\mathbf{k}\parallel\mathbf{k}_1\parallel\mathbf{k}-\mathbf{k}'\parallel\mathbf{k}_1+\mathbf{k}'|}\alpha_{k'}G^+_{k_1,k'}dk_1$$

$$\left. +\int \frac{[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{k}_1][(\mathbf{k}_1+\mathbf{k}')\cdot\mathbf{k}]}{|\mathbf{k}\parallel\mathbf{k}_1\parallel\mathbf{k}-\mathbf{k}'\parallel\mathbf{k}_1+\mathbf{k}'|}\alpha_{k-k_1-k'}G^+_{k_1,k'}dk_1 \right\}. \quad (164)$$

This system of integral equations is analogous to the system of coupled equations describing modulational interaction of many monochromatic modes. We stress that due to inhomogeneity and isotropism of the unperturbed plasma state, this system again has symmetry properties analogous to those of the matrix for modulational instability of two monochromatic pumps (94).

7. Rates of Instability of Broad Wave Packets

In this section, the rates of the modulational instability of broad packets of random Langmuir waves are presented. We consider the one-dimensional situation when all modulational perturbations are in the direction of waves in the packet, as well as the case of isotropic turbulence and modulational instability. Thresholds of the instability of the broad wave packets are discussed.

7.1 One-dimensional Equations

First, we assume that the direction of propagation of the pump packet and of the modulational perturbations are the same. In this case equations (163) and (164) reduce to, respectively,

$$(\varepsilon_{k+k'} + \varepsilon^N_{k+k'})G^+_{k,k'} = |E^{+(0)}|^2_k \int (\alpha_{k'} + \alpha_{k-k_1})G^+_{k_1,k'}dk_1$$
$$+ |E^{+(0)}|^2_k \int (\alpha_{k'} + \alpha_{k-k_1+k'})G^-_{k_1,k'}dk_1\,, \quad (165)$$

$$(\varepsilon_{-k+k'} + \varepsilon^N_{-k+k'})G^-_{k,k'} = |E^{+(0)}|^2_k \int (\alpha_{k'} + \alpha_{k-k_1})G^-_{k_1,k'}dk_1$$
$$+ |E^{+(0)}|^2_k \int (\alpha_{k'} + \alpha_{k-k_1-k'})G^+_{k_1,k'}dk_1\,. \quad (166)$$

Consider some particular solutions of equations (165) and (166). If all pump waves have sufficiently small wavevectors, and their frequency difference is small, then inequality (159) is fulfilled, and we have

$$\alpha_{k-k_1} \approx -\frac{1}{4\pi n_0(T_e + T_i)} = -\alpha_0\,. \quad (167)$$

We will also consider small growth rates which satisfy the inequality $\omega' \ll |\mathbf{k}'|v_{Ti}$. In this case we have

$$\alpha_{k'} \approx -\alpha_0 \,. \tag{168}$$

Furthermore, if we also assume that $\alpha_{k-k_1-k'} \approx -\alpha_0$, then

$$G^+_{k,k'} = -2\,\frac{\alpha_0|E^{+(0)}|^2_k}{\varepsilon_{k+k'} + \varepsilon^N_{k+k'}} \int (G^+_{k_1,k'} + G^-_{k_1,k'})dk_1 \,, \tag{169}$$

$$G^-_{k,k'} = -\,\frac{2\alpha_0|E^{+(0)}|^2_k}{\varepsilon_{-k+k'} + \varepsilon^N_{-k+k'}} \int (G^+_{k_1,k'} + G^-_{k_1,k'})dk_1 \,. \tag{170}$$

To find the dispersion relation for the modulational instability, we integrate these equations over k to find

$$1 + 2\alpha_0 \int |E^{+(0)}|^2_k \left(\frac{1}{\varepsilon_{k+k'} + \varepsilon^N_{k+k'}} + \frac{1}{\varepsilon_{-k+k'} + \varepsilon^N_{-k+k'}} \right) dk = 0 \,. \tag{171}$$

Equation (171) is similar to the usual dispersion equation for the modulational instability of one monochromatic mode, the only differences are that summation over all harmonics is carried out and the nonlinear dielectric permittivity is taken into account in (171). Note that the conditions used to derive this equation correspond to those for which the Zakharov system of equations is not applicable.

In another limit of large growth rates, when $\omega' \gg |\mathbf{k}'|v_s$ but $\alpha_{k-k'} \approx -\alpha_0$, we have $(\alpha_{k'} \ll \alpha_{k-k_1},\ \alpha_{k-k_1-k'} \approx \alpha_{k'})$

$$\begin{aligned}(\varepsilon_{k+k'} + \varepsilon^N_{k+k'})G^+_{k,k'} = &-\alpha_0|E^{+(0)}|^2_k \int G^+_{k_1,k'}dk_1 \\ &+ 2\alpha_0 \frac{|\mathbf{k}'|^2 v_s^2}{\omega'^2}\,\frac{T_e + T_i}{T_e}|E^{+(0)}|^2_k \int G^-_{k_1,k'}dk_1 \,,\end{aligned} \tag{172}$$

$$\begin{aligned}(\varepsilon_{-k+k'} + \varepsilon^N_{-k+k'})G^-_{k,k'} = &-\alpha_0|E^{+(0)}|^2_k \int G^-_{k_1,k'}dk_1 \\ &+ 2\alpha_0 \frac{|\mathbf{k}'|^2 v_s^2}{\omega'^2}\,\frac{T_e + T_i}{T_e}|E^{+(0)}|^2_k \int G^+_{k_1,k'}dk_1 \,.\end{aligned} \tag{173}$$

Furthermore, if we introduce the notation

$$g^\pm_{k'} = \int G^\pm_{k,k'}dk \,, \qquad S^\pm_{k'} = \int \frac{\alpha_0|E^{+(0)}|^2_k}{\varepsilon_{\pm k+k'} + \varepsilon^N_{\pm k+k'}}dk \,, \tag{174}$$

we find

$$g^+_{k'} = -S^+_{k'} g^+_{k'} + 2\,\frac{|\mathbf{k}'|^2 v_s^2}{\omega'^2}\,\frac{T_e + T_i}{T_e} S^+_{k'} g^-_{k'}\,, \tag{175}$$

$$g^-_{k'} = -S^-_{k'} g^-_{k'} + 2\,\frac{|\mathbf{k}'|^2 v_s^2}{\omega'^2}\,\frac{T_e + T_i}{T_e} S^-_{k'} g^+_{k'}\,. \tag{176}$$

With help of these expressions, we obtain the dispersion equation

$$(1 + S^+_{k'})(1 + S^-_{k'}) = 4\,\frac{|\mathbf{k}'|^4 v_s^4}{\omega'^4}\,\frac{T_e + T_i}{T_e} S^+_{k'} S^-_{k'}\,. \tag{177}$$

Equation (171) written in the notation of (175) and (176) has the form

$$1 + 2(S^+_{k'} + S^-_{k'}) = 0\,. \tag{178}$$

Finally, we can consider the case, when the following inequalities are valid:

$$(\omega - \omega_1)^2 \gg (\mathbf{k} - \mathbf{k}_1)^2 v_s^2, \quad \omega'^2 \gg |\mathbf{k}'|^2 v_s^2\,, \qquad \frac{|\mathbf{k}'|^2 v_s^2}{\omega'^2} \gg \frac{(\mathbf{k} - \mathbf{k}_1)^2 v_s^2}{(\omega - \omega_1)^2}\,. \tag{179}$$

In this case we have $\alpha_{k'} \ll \alpha_0$ and $\alpha_{k-k_1} \ll \alpha_0$; however, $\alpha_{k'} \gg \alpha_{k-k_1}$. If we assume that $\omega' \gg |\omega - \omega_1|$ and $|\mathbf{k}'| \gg |\mathbf{k}| - |\mathbf{k}_1|$, then we have $\alpha_{k-k_1-k'} \gg \alpha_{k'}$. Finally, equations (165) and (166) can be reduced to

$$(\varepsilon_{k+k'} + \varepsilon^N_{k+k'}) G^+_{k,k'} = \frac{|E^{+(0)}|^2_k}{2\pi n_0 T_e}\,\frac{|\mathbf{k}'|^2 v_s^2}{\omega'^2} \int (G^+_{k_1,k'} + G^-_{k_1,k'})\,dk_1, \tag{180}$$

$$(\varepsilon_{-k+k'} + \varepsilon^N_{-k+k'}) G^-_{k,k'} = \frac{|E^{+(0)}|^2_k}{2\pi n_0 T_e}\,\frac{|\mathbf{k}'|^2 v_s^2}{\omega'^2} \int (G^+_{k_1,k'} + G^-_{k_1,k'})\,dk_1. \tag{181}$$

From these equations we obtain the dispersion equation

$$1 = \int \frac{|E^{+(0)}|^2_k}{2\pi n_0 T_e}\,\frac{|\mathbf{k}'|^2 v_s^2}{\omega'^2} \left(\frac{1}{\varepsilon_{k+k'} + \varepsilon^N_{k+k'}} + \frac{1}{\varepsilon_{-k+k'} + \varepsilon^N_{-k+k'}} \right) dk\,. \tag{182}$$

The above dispersion equations are the basis for calculation of instability rates in the one-dimensional situation.

7.2 Rates in the One-dimensional Case

To calculate the instability rates, we have to consider the integrals $S^{\pm}_{k'}$. The dielectric function (taking into account the nonlinear frequency shift) is given by

$$\varepsilon_{k+k'} \approx 1 - \frac{\omega_{pe}^2}{(\omega+\omega')^2} - \frac{3(\mathbf{k}+\mathbf{k}')^2 v_{Te}^2}{\omega_{pe}^2}\,, \tag{183}$$

$$\varepsilon^N_{k+k'} = -\alpha_0 \int |E^{+(0)}|^2_{k_1} dk_1\,. \tag{184}$$

If the wavevectors of Langmuir oscillations are small ($|\mathbf{k}| \ll |\mathbf{k}'|$), and their frequencies correspondingly renormalised to include the nonlinear shift, then

$$\omega = \omega_{pe} + \frac{\alpha_0}{2}\int |E^{+(0)}|^2_{k_1} dk_1\,, \qquad \varepsilon_{k+k'} + \varepsilon^N_{k+k'} \approx \frac{2\omega'}{\omega_{pe}} - \frac{3|\mathbf{k}'|^2 v_{Te}^2}{\omega_{pe}^2}\,. \tag{185, 186}$$

Hence, we obtain

$$S^+_{k'} + S^-_{k'} = \alpha_0 \int dk |E^{+(0)}|^2_k \left\{ \left[\frac{2\omega'}{\omega_{pe}} - \frac{3|\mathbf{k}'|^2 v_{Te}^2}{\omega_{pe}^2}\right]^{-1} + \left[-\frac{2\omega'}{\omega_{pe}} - \frac{3|\mathbf{k}'|^2 v_{Te}^2}{\omega_{pe}^2}\right]^{-1} \right\}. \tag{187}$$

In the case

$$\omega' \gg \frac{|\mathbf{k}'|^2 v_{Te}^2}{\omega_{pe}} \tag{188}$$

we find

$$S^+_{k'} + S^-_{k'} = \alpha_0 \int dk |E^{+(0)}|^2_k \frac{3|\mathbf{k}'|^2 v_{Te}^2}{4\omega'^2}\,. \tag{189}$$

For further purposes, it is convenient to introduce

$$\tilde{v}^2 = \frac{3}{8\pi}\int dk \frac{|E^{+(0)}|^2_k}{n_0 m_e}\,, \tag{190}$$

which is the squared amplitude of electron motion in the field of turbulent oscillations. We find

$$\omega'^2 = -2|\mathbf{k}'|^2 \tilde{v}^2 \frac{T_e}{T_e+T_i}\,, \qquad \gamma \approx |\mathbf{k}'|\tilde{v}\left(\frac{2T_e}{T_e+T_i}\right)^{\frac{1}{2}} \tag{191, 192}$$

Two conditions $\omega' \ll |\mathbf{k}'| v_{Ti}$ and (188) imply that

$$\tilde{v} \ll v_{Ti}\,, \qquad |\mathbf{k}'| \ll \frac{\omega_{pe}}{v_{Te}}\frac{\tilde{v}}{v_{Te}}. \tag{193}$$

That is, we obtain

$$\gamma_{\max} \approx \omega_{pe} \left(\frac{2T_e}{T_e + T_i} \right)^{\frac{1}{2}} \frac{\tilde{v}^2}{v_{Te}^2} = \omega_{pe} \frac{3}{8\pi} \left(\frac{2T_e}{T_e + T_i} \right)^{\frac{1}{2}} \int dk \, \frac{|E^{+(0)}|_k^2}{n_0 T_e} . \tag{194}$$

If inequality (188) is not fulfilled then we have

$$S_{k'}^{+} + S_{k'}^{-} = \alpha_0 \frac{3|\mathbf{k}'|^2 v_{Te}^2}{\omega'^2 - (3|\mathbf{k}'|^2 v_{Te}^2/\omega_{pe})^2} \int dk |E^{+(0)}|_k^2 , \tag{195}$$

$$\omega'^2 = -4|\mathbf{k}'|^2 \tilde{v}^2 \frac{T_e}{T_e + T_i} + \left(\frac{3|\mathbf{k}'|^2 v_{Te}^2}{\omega_{pe}} \right)^2 . \tag{196}$$

The maximum wave number $|\mathbf{k}'|$, which corresponds to development of the modulational instability, is given by

$$|\mathbf{k}'|_{\max} = \frac{2\omega_{pe}}{3v_{Te}} \frac{\tilde{v}}{v_{Te}} \left(\frac{T_e}{T_e + T_i} \right)^{\frac{1}{2}} . \tag{197}$$

We note that the rates determined by expressions (194) and (196) are coincident with those of monochromatic waves of the same energy. We emphasise that here we consider the case $\omega' \ll |\mathbf{k}'| v_{Ti}$ and $|\omega - \omega_1| \ll |\mathbf{k} - \mathbf{k}'| v_{Ti}$. As it will be shown in the next section, this case corresponds to the inequality $\gamma_{\max} \gg \delta\omega$, and consequently in this case we can consider the wave spectrum as narrow.

In the case of a broad wave spectrum when $\omega' \gg |\mathbf{k}'| v_s$ and $\omega' \gg |\mathbf{k}'|^2 v_{Te}^2/\omega_{pe}$ [when equation (177) is valid], the r.h.s. of (177) can be neglected. Then the equations

$$S_k^{+} = -1, \qquad S_k^{-} = -1 \tag{198}$$

have the solutions

$$\omega' = \pm \frac{3|\mathbf{k}'|^2 v_{Te}^2}{2\omega_{pe}} \mp \frac{\omega_{pe}}{2} \int dk \, \frac{|E^{+(0)}|_k^2}{4\pi n_0 (T_e + T_1)} , \tag{199}$$

which are stable. We note that the case $\omega' \gg |\mathbf{k}'| v_s$ corresponds to the maximum rates of the instability of the monochromatic pump. Thus we see that the modulational instability of the broad wave spectrum is significantly suppressed compared with that of a monochromatic pump wave of the same energy.

Finally, we find from the dispersion equation (182) the growth rates analogous to those for a monochromatic pump wave. Note that the nonlinear frequency

shift depends on k. Therefore we have [as a consequence of (179), (180) and (181)]

$$\varepsilon^N_{k+k'} \approx \frac{|\mathbf{k}'|^2 v_s^2}{\omega'^2} \int dk_1 \frac{|E^{+(0)}|^2_{k_1}}{4\pi n_0 T_e}, \tag{200}$$

$$\varepsilon^N_k \approx \int dk_1 \frac{(\mathbf{k}-\mathbf{k}_1)^2 v_s^2}{(\omega-\omega_1)^2} \frac{|E^{+(0)}|^2_{k_1}}{4\pi n_0 T_e}. \tag{201}$$

Hence, the nonlinear frequency shift of Langmuir turbulent oscillations depends on k. As a consequence of the last inequality in (179), $\varepsilon^N_{k\pm k'}$ dominates in the denominators of (182). So (182) is transformed to

$$1 = \int dk \frac{|E^{+(0)}|^2_k}{2\pi n_0 T_e} \frac{|\mathbf{k}'|^2 v_s^2}{\omega'^2} \left\{ \left[\frac{2\omega'}{\omega_{pe}} - \frac{3|\mathbf{k}'|^2 v_{Te}^2}{\omega_{pe}^2} + \frac{|\mathbf{k}'|^2 v_s^2}{\omega'^2} \int dk_1 \frac{|E^{+(0)}|^2_{k_1}}{4\pi n_0 T_e} \right]^{-1} \right.$$
$$\left. + \left[-\frac{2\omega'}{\omega_{pe}} - \frac{3|\mathbf{k}'|^2 v_{Te}^2}{\omega_{pe}^2} + \frac{|\mathbf{k}'|^2 v_s^2}{\omega'^2} \int dk_1 \frac{|E^{+(0)}|^2_{k_1}}{4\pi n_0 T_e} \right]^{-1} \right\} \tag{202}$$

Thus we see that if

$$\omega' \gg \omega_{pe} \frac{m_e}{m_i} \int dk_1 \frac{|E^{+(0)}|^2_{k_1}}{4\pi n_0 T_e}, \tag{203}$$

we can find the growth rate analogous to that for the case of the monochromatic pump wave. In this case the maximum instability rate is given by

$$\gamma_{\max} \approx \omega_{pi} \left(dk_1 \frac{|E^{+(0)}|^2_{k_1}}{4\pi n_0 T_e} \right)^{\frac{1}{2}} \tag{204}$$

When the inequality opposite to (203) is fulfilled, we have

$$\omega'^3 \approx \omega_{pe} |\mathbf{k}'|^2 v_s^2 \int dk_1 \frac{|E^{+(0)}|^2_{k_1}}{4\pi n_0 T_e}. \tag{205}$$

Thus we obtain

$$1 = -\left[\int dk_1 \frac{|E^{+(0)}|^2_{k_1}}{2\pi n_0 T_e} \right]^2 \frac{|\mathbf{k}'|^4 v_s^4}{\omega'^4} \omega_{pe}^2 \left\{ 4\omega'^2 - \left[\frac{|\mathbf{k}'|^2 v_s^2}{\omega'^2} \int dk_1 \frac{|E^{+(0)}|^2_{k_1}}{4\pi n_0 T_e} \right]^2 \omega_{pe}^2 \right\}^{-1} \tag{206}$$

Finally, we find

$$\sim \approx (|\mathbf{k}'|v_s)^{\frac{2}{3}}\omega_{pe}^{\frac{1}{3}}\left[\int dk_1 \frac{|E^{+(0)}|_{k_1}^2}{2\pi n_0 T_e}\right]^{\frac{1}{3}}, \tag{207}$$

which takes place if

$$|\mathbf{k}'|v_s \ll \omega_{pe}\left(\frac{m_e}{m_i}\right)^{\frac{3}{2}}\int dk_1 \frac{|E^{+(0)}|_{k_1}^2}{2\pi n_0 T_e}. \tag{208}$$

Thus we have the following upper limit for the instability rate:

$$\gamma \leq \omega_{pe}\frac{m_e}{m_i}\int dk_1 \frac{|E^{+(0)}|_{k_1}^2}{2\pi n_0 T_e}. \tag{209}$$

Below, we consider the case of small growth rates to find conditions where the modulational instability can develop.

7.3 Conditions of the One-dimensional Instability

For broad wave spectra when the maximum rate of the instability does not exceed the width of the spectrum $\delta\omega$, the condition for modulational instability can be determined from the energy principle (Gailitis 1964, 1965) and has the form

$$\int \frac{dk}{2\pi}\frac{\omega_{pe}^2|E^{+(0)}|_k^2}{|\mathbf{k}|^2 v_{Te}^2} > 12n_0(T_e + T_i). \tag{210}$$

This inequality defines the threshold for modulational instability.

Here, on the basis of the above equations we consider the case $\omega' \ll |\mathbf{k}'|v_{Ti}$ and $|\omega - \omega_1| \ll |\mathbf{k} - \mathbf{k}_1|v_{Ti}$, when the dispersion equation (171) is valid. We show that for a turbulent spectrum when $\gamma_{\max}$ exceeds $\delta\omega$, the instability is possible practically for any levels of turbulence and plasma parameters. The case considered corresponds to the above condition $\gamma_{\max} \gg \delta\omega$. Indeed, ω and ω_1, $\mathbf{k}$ and $\mathbf{k}_1$ characterise the spectrum of the waves, while ω' and $\mathbf{k}'$ refer to the modulational perturbations. As we will demonstrate, the instability develops for any levels of turbulence when $|\mathbf{k}'|$ is close to the spectral width δk.

We can obtain from $|\omega - \omega_1| \ll |\mathbf{k} - \mathbf{k}_1|v_{Ti}$ the condition $|\omega - \omega_1| \ll |\mathbf{k}'|v_{Ti}$, which is satisfied when $|\mathbf{k}'| \sim \delta k$. But $|\mathbf{k}'|v_{Ti}$ determines the maximum growth rate $\gamma_{\max}$ in the situation considered. So we have $|\omega - \omega_1| \sim \delta\omega \ll \gamma_{\max}$.

We also assume that

$$|E^{+(0)}|_k^2 = \begin{cases} |E^{+(0)}|^2\delta(\omega - \omega_{\mathbf{k}}), & \text{if } k_{\min} < |\mathbf{k}| < k_{\max}, \\ 0, & \text{if } |\mathbf{k}| \leq k_{\min} \text{ or } |\mathbf{k}| \geq k_{\max}, \end{cases} \tag{211}$$

where $|E^{+(0)}|^2 \approx$ const. In this case (171) can be rewritten in the form

$$1+\frac{\alpha_0\omega_{pe}^2}{3v_{Te}^2|\mathbf{k}'|\delta k}\ln\left[\left|\frac{6k_{\max}|\mathbf{k}'|v_{Te}^2}{\omega_{pe}^2}-\frac{3|\mathbf{k}'|^2v_{Te}^2}{\omega_{pe}^2}-\frac{2\omega'}{\omega_{pe}}\right|\right.$$

$$\times\left|\frac{6k_{\min}|\mathbf{k}'|v_{Te}^2}{\omega_{pe}^2}+\frac{3|\mathbf{k}'|^2v_{Te}^2}{\omega_{pe}^2}-\frac{2\omega'}{\omega_{pe}}\right|\left|\frac{6k_{\min}|\mathbf{k}'|v_{Te}^2}{\omega_{pe}^2}-\frac{3|\mathbf{k}'|^2v_{Te}^2}{\omega_{pe}^2}-\frac{2\omega'}{\omega_{pe}}\right|^{-1}$$

$$\left.\times\left|\frac{6k_{\max}|\mathbf{k}'|v_{Te}^2}{\omega_{pe}^2}+\frac{3|\mathbf{k}'|^2v_{Te}^2}{\omega_{pe}^2}-\frac{2\omega'}{\omega_{pe}}\right|^{-1}\right]\int dk|E^{+(0)}|_k^2=0\,, \quad (212)$$

where $\delta k = k_{\max} - k_{\min}$. In order for this equation to have solutions, the term containing ln(...) should be real. This allows us to find the real part of the modulation frequency $\mathrm{Re}\omega'$ (we recall that $\omega' = \mathrm{Re}\omega' + i\mathrm{Im}\omega'$), assuming that $\mathrm{Im}\omega' \neq 0$ and all functions in ln(...) (with the exception of ω') are real. We have

$$\mathrm{Re}\omega' = \tfrac{3}{2}\frac{|\mathbf{k}'|(k_{\min}+k_{\max})v_{Te}^2}{\omega_{pe}}. \quad (213)$$

Expression (213) satisfies the inequality $\omega' \ll |\mathbf{k}'|v_{Ti}$ for the wave numbers

$$k_{\min}+k_{\max} \ll \frac{v_{Ti}}{v_{Te}}\frac{\omega_{pe}}{v_{Te}}. \quad (214)$$

We note that equation (196) has either exactly real or exactly imaginary solutions. But it is well to bear in mind that this equation has been derived under the condition $|\mathbf{k}'| \gg |\mathbf{k}|$, and all terms in the dielectric permittivity of order $|\mathbf{k} \parallel \mathbf{k}'|v_{Te}^2/\omega_{pe}^2$ have been neglected compared with those of order $|\mathbf{k}'|^2v_{Te}^2/\omega_{pe}^2$. Thus the terms having order (213) are neglected in (196). Furthermore, in the case when dispersion equation (171) has an exact real solution, the condition $\mathrm{Im}\omega' \neq 0$ is not satisfied, and this solution should be determined (for the case $|\mathbf{k}'| \gg |\mathbf{k}|$) from equation (196). Here we are interested in the conditions of the modulational instability development. Hence, we can consider (without loss of generality) the case $\mathrm{Im}\omega' \neq 0$.

Taking into account (213), we rewrite equation (212) in the form

$$1+\frac{\alpha_0\omega_{pe}^2}{3v_{Te}^2|\mathbf{k}'|\delta k}\ln\left\{\left[\frac{|\mathbf{k}'|^2(\delta k-|\mathbf{k}'|)^2v_{Te}^4}{\omega_{pe}^4}+\frac{4(\mathrm{Im}\omega')^2}{\omega_{pe}^2}\right]\right.$$

$$\left.\times\left[\frac{|\mathbf{k}'|^2(\delta k+|\mathbf{k}'|)^2v_{Te}^4}{\omega_{pe}^4}+\frac{4(\mathrm{Im}\omega')^2}{\omega_{pe}^2}\right]\right\}\int dk|E^{+(0)}|_k^2=0\,. \quad (215)$$

Let us consider the case $\mathrm{Im}\omega'$ where $(\mathrm{Im}\omega')^2 \to 0$. It corresponds to the transition from a stable to unstable regime: the condition for the development of the instability can be obtained from the requirement that there are values $(\mathrm{Im}\omega')^2 > 0$ satisfying equation (215). Assume that $\delta k \neq |\mathbf{k}'|$, and $(\mathrm{Im}\omega')^2$ is sufficiently small, so that we can expand the l.h.s. of (215) in the small parameters

$$\frac{4(\mathrm{Im}\omega')^2\omega_{pe}^2}{|\mathbf{k}'|^2(\delta k \pm |\mathbf{k}'|)^2 v_{Te}^4}\,. \tag{216}$$

Taking into account the first terms of this expansion we find

$$1 + \frac{16\alpha_0\omega_{pe}^4}{3v_{Te}^6|\mathbf{k}'|^2(\delta k^2 - |\mathbf{k}'|^2)^2}(\mathrm{Im}\omega')^2 \int dk|E^{+(0)}|_k^2$$

$$+\frac{2\alpha_0\omega_{pe}^2}{3v_{Te}^2|\mathbf{k}'|\delta k}\ln\left|\frac{\delta k - |\mathbf{k}'|}{\delta k + |\mathbf{k}'|}\right| \int dk|E^{+(0)}|_k^2 = 0\,. \tag{217}$$

We see that unstable solutions of the dispersion equation exist for

$$\frac{2\alpha_0\omega_{pe}^2}{3v_{Te}^2|\mathbf{k}'|\delta k}\ln\left|\frac{\delta k + |\mathbf{k}'|}{\delta k - |\mathbf{k}'|}\right| \int dk|E^{+(0)}|_k^2 > 1\,. \tag{218}$$

We note that for any values of α_0, $|E^{+(0)}|^2$, ω_{pe}^2 and v_{Te}^2 there are the wave numbers $|\mathbf{k}'|$ which satisfy (218).

If $\delta k = |\mathbf{k}'|$ then we find from (215)

$$1 + \frac{\alpha_0\omega_{pe}^2}{3v_{Te}^2(\delta k)^2}\ln\left[\frac{(\mathrm{Im}\omega')^2\omega_{pe}^2}{\delta k^4 v_{Te}^4}\right] \int dk|E^{+(0)}|_k^2 = 0\,, \tag{219}$$

which always has solutions $(\mathrm{Im}\omega')^2 > 0$ $[(\mathrm{Im}\omega')^2 \to 0]$,

$$(\mathrm{Im}\omega')^2 = \omega_{pe}^2\,\frac{\delta k^4 v_{Te}^4}{\omega_{pe}^4}\exp\left(-3\delta k^2 v_{Te}^2/\alpha_0\omega_{pe}^2 \int dk|E^{+(0)}|_k^2\right) \tag{220}$$

Thus, if the one-dimensional approximation is valid, and the spectrum of Langmuir waves is concentrated in the region of sufficiently small wavevectors $\mathbf{k}$

$$|\mathbf{k}| \ll \frac{v_{Ti}}{v_{Te}}\,\frac{\omega_{pe}}{v_{Te}} \tag{221}$$

[see (214)], then the modulational instability is possible practically for any levels of turbulence and plasma parameters.

Thus we see that the modulational instability of turbulent spectra can, in principle, develop for any levels of turbulence and plasma parameters if this spectrum is sufficiently narrow, so that the maximum growth rate of the modulational instability exceeds the spectral width $\delta\omega$.

7.4 Correlation Functions in the Isotropic Case

Let us again assume that $|\mathbf{k}'| \gg |\mathbf{k}|, |\mathbf{k}_1|$. Then equations (163) and (164) can be written in the form

$$(\varepsilon_{k'} + \varepsilon_{k'}^N) G_{k,k'}^+ = |E^{+(0)}|_k^2 \left\{ \int \frac{(\mathbf{k}\cdot\mathbf{k}')(\mathbf{k}'\cdot\mathbf{k}_1)}{|\mathbf{k} \parallel \mathbf{k}_1||\mathbf{k}'|^2} \alpha_{k'} G_{k_1,k'}^+ dk_1 \right.$$

$$+ \int \frac{(\mathbf{k}\cdot\mathbf{k}_1)}{|\mathbf{k} \parallel \mathbf{k}_1|} \alpha_{k-k_1} G_{k_1,k'}^+ dk_1 - \int \frac{(\mathbf{k}\cdot\mathbf{k}')(\mathbf{k}'\cdot\mathbf{k}_1)}{|\mathbf{k} \parallel \mathbf{k}_1 \parallel \mathbf{k}'|^2} \alpha_{k'} G_{k_1,k'}^- dk_1$$

$$\left. - \int \frac{(\mathbf{k}'\cdot\mathbf{k}_1)(\mathbf{k}\cdot\mathbf{k}')}{|\mathbf{k} \parallel \mathbf{k}_1 \parallel \mathbf{k}'|^2} \alpha_{k'} G_{k_1,k'}^- dk_1 \right\}, \tag{222}$$

$$(\varepsilon_{k'} + \varepsilon_{k'}^N) G_{k,k'}^- = |E^{+(0)}|_k^2 \left\{ \int \frac{(\mathbf{k}\cdot\mathbf{k}')(\mathbf{k}'\cdot\mathbf{k}_1)}{|\mathbf{k} \parallel \mathbf{k}_1||\mathbf{k}'|^2} \alpha_{k'} G_{k_1,k'}^- dk_1 \right.$$

$$+ \int \frac{(\mathbf{k}\cdot\mathbf{k}_1)}{|\mathbf{k} \parallel \mathbf{k}_1|} \alpha_{k-k_1} G_{k_1,k'}^- dk_1 - \int \frac{(\mathbf{k}\cdot\mathbf{k}')(\mathbf{k}'\cdot\mathbf{k}_1)}{|\mathbf{k} \parallel \mathbf{k}_1 \parallel \mathbf{k}'|^2} \alpha_{k'} G_{k_1,k'}^+ dk_1$$

$$\left. - \int \frac{(\mathbf{k}'\cdot\mathbf{k}_1)(\mathbf{k}\cdot\mathbf{k}')}{|\mathbf{k} \parallel \mathbf{k}_1 \parallel \mathbf{k}'|^2} \alpha_{k'} G_{k_1,k'}^+ dk_1 \right\}. \tag{223}$$

In the case $\omega' \gg |\mathbf{k}'|v_s$ and $|\omega - \omega_1| \ll |\mathbf{k} - \mathbf{k}_1|v_s$ we have $\alpha_k' \ll \alpha_0$ and $\alpha_{k-k_1} \approx \alpha_0$. Now, we introduce the vectors

$$\mathbf{G}_{k'}^{\pm} = \int \frac{\mathbf{k}_1}{|\mathbf{k}_1|} G_{k_1,k'}^{\pm} dk_1 \,, \tag{224}$$

and find

$$\mathbf{G}_{k'}^{\pm} = \int dk \, \frac{|E^{+(0)}|_k^2}{(\varepsilon_{k'} + \varepsilon_{k'}^N)} \, \frac{\mathbf{k}(\mathbf{k}\cdot\mathbf{G}_{k'}^{\pm})}{|\mathbf{k}|^2} \tag{225}$$

For the case of isotropic turbulence we find that the modulational instability is absent.

In the case $\alpha_{k-k_1} \ll \alpha_{k'} \ll \alpha_0$ we obtain for the functions

$$G'^{\pm}_{k'} = \int \frac{\mathbf{k}_1\cdot\mathbf{k}'}{|\mathbf{k}_1 \parallel \mathbf{k}'|} G_{k_1,k'}^{\pm} dk_1 \tag{226}$$

the following equations:

$$G'^{+}_{k'} = \int dk \, \frac{\alpha_{k'}(\mathbf{k}\cdot\mathbf{k}')^2|E^{+(0)}|^2_k}{|\mathbf{k}|^2|\mathbf{k}'|^2(\varepsilon_{k+k'} + \varepsilon^N_{k+k'})}(G'^{+}_{k'} - 2G'^{-}_{k'}) \,, \tag{227}$$

$$G'^{-}_{k'} = \int dk \, \frac{\alpha_{k'}(\mathbf{k}\cdot\mathbf{k}')^2|E^{+(0)}|^2_k}{|\mathbf{k}|^2|\mathbf{k}'|^2(\varepsilon_{-k+k'} + \varepsilon^N_{-k+k'})}(G'^{-}_{k'} - 2G'^{+}_{k'}) \,. \tag{228}$$

Using these equations we obtain the dispersion equation of the modulational instability

$$\left(1 - \alpha_{k'} \int dk \, \frac{(\mathbf{k}\cdot\mathbf{k}')^2|E^{+(0)}|^2_k}{|\mathbf{k}|^2|\mathbf{k}'|^2(\varepsilon_{k+k'} + \varepsilon^N_{k+k'})}\right)\left(1 - \alpha_{k'} \int dk \, \frac{(\mathbf{k}\cdot\mathbf{k}')^2|E^{+(0)}|^2_k}{|\mathbf{k}|^2|\mathbf{k}'|^2(\varepsilon_{-k+k'} + \varepsilon^N_{-k+k'})}\right)$$
$$= 4\alpha^2_{\mathbf{k}'} \int dk \, \frac{(\mathbf{k}\cdot\mathbf{k}')^2|E^{+(0)}|^2_k}{|\mathbf{k}|^2|\mathbf{k}'|^2(\varepsilon_{k+k'} + \varepsilon^N_{k+k'})} \int dk \, \frac{(\mathbf{k}\cdot\mathbf{k}')^2|E^{+(0)}|^2_k}{|\mathbf{k}|^2|\mathbf{k}'|^2(\varepsilon_{-k+k'} + \varepsilon^N_{-k+k'})} \,. \tag{229}$$

This equation can be rewritten as

$$1 = \alpha_{k'} \int dk \, \frac{(\mathbf{k}\cdot\mathbf{k}')^2|E^{+(0)}|^2_k}{|\mathbf{k}|^2|\mathbf{k}'|^2}\left(\frac{1}{(\varepsilon_{k+k'} + \varepsilon^N_{k+k'})} + \frac{1}{(\varepsilon_{-k+k'} + \varepsilon^N_{-k+k'})}\right)$$
$$+ 3\alpha^2_{\mathbf{k}'} \int dk \, \frac{(\mathbf{k}\cdot\mathbf{k}')^2|E^{+(0)}|^2_k}{|\mathbf{k}|^2|\mathbf{k}'|^2(\varepsilon_{k+k'} + \varepsilon^N_{k+k'})} \int dk \, \frac{(\mathbf{k}\cdot\mathbf{k}')^2|E^{+(0)}|^2_k}{|\mathbf{k}|^2|\mathbf{k}'|^2(\varepsilon_{-k+k'} + \varepsilon^N_{-k+k'})} \,. \tag{230}$$

The last equation is basic for our investigation of the modulational instability in the isotropic case.

7.5 Instability Rates in the Isotropic Case

Now we investigate the instability described by equation (230). If $\omega' \gg |\mathbf{k}'|^2 v^2_{Te}/\omega_{pe}$ then we have

$$1 = \tfrac{3}{2}\,\frac{|\mathbf{k}'|^2 v^2_{Te}}{\omega'^2}\,\frac{|\mathbf{k}'|^2 v^2_s}{\omega'^2} \int dk \, \frac{(\mathbf{k}\cdot\mathbf{k}')^2|E^{+(0)}|^2_k}{|\mathbf{k}|^2|\mathbf{k}'|^2}$$
$$- \tfrac{3}{4}\left(\frac{|\mathbf{k}'|^2 v^2_s}{\omega'^2}\right)^2 \frac{\omega^2_{pe}}{\omega'^2}\left[\int dk \, \frac{(\mathbf{k}\cdot\mathbf{k}')^2|E^{+(0)}|^2_k}{|\mathbf{k}|^2|\mathbf{k}'|^2}\right]^2 . \tag{231}$$

Taking into account the isotropic character of the spectrum, i.e. $|E^{+(0)}|^2_k = |E^{+(0)}|^2(\omega, |\mathbf{k}|)$, we find from (231)

$$1 = \frac{|\mathbf{k}'|^2 v^2_{Te}}{\omega'^2}\,\frac{|\mathbf{k}'|^2 v^2_s}{\omega'^2} \int dk |E^{+(0)}|^2_k - \tfrac{1}{3}\left(\frac{|\mathbf{k}'|^2 v^2_s}{\omega'^2}\right)^2 \frac{\omega^2_{pe}}{\omega'^2}\left[\int dk |E^{+(0)}|^2_k\right]^2 . \tag{232}$$

When the inequality

$$\omega' \gg \omega_{pe} \frac{m_e}{m_i} \int dk |E^{+(0)}|_k^2 \tag{233}$$

is fulfilled, we can neglect the last term on the r.h.s. of (232). The maximum growth rate in this case is given by

$$\gamma_{\max} \sim \omega_{pe} \left(\frac{m_e}{m_i} \int dk |E^{+(0)}|_k^2 \right)^{\frac{1}{2}} . \tag{234}$$

Note that condition (233) is valid for the maximum rate (234).

If the inequality opposite to (233) takes place, i.e.

$$\omega' \ll \omega_{pe} \frac{m_e}{m_i} \int dk |E^{+(0)}|_k^2 , \tag{235}$$

then the first term on the r.h.s. of (232) is negligible, and we obtain

$$\gamma \sim \omega_{pe}^{\frac{1}{3}} (|\mathbf{k}'| v_s)^{\frac{2}{3}} \left(\int dk |E^{+(0)}|_k^2 \right)^{\frac{2}{3}} . \tag{236}$$

Furthermore, using the condition

$$\gamma \gg \frac{|\mathbf{k}'|^2 v_{Te}^2}{\omega_{pe}} = \frac{|\mathbf{k}'|^2 v_s^2}{\omega_{pe}^2} \frac{m_i}{m_e} , \tag{237}$$

we find that $|\mathbf{k}'| v_s$ should not exceed the frequency

$$\sim \omega_{pe} \left(\frac{m_e}{m_i} \right)^{\frac{3}{4}} \left(\int dk |E^{+(0)}|_k^2 \right)^{\frac{1}{2}} . \tag{238}$$

However, for this frequency we have

$$\gamma \sim \omega_{pi} \int dk |E^{+(0)}|_k^2 , \tag{239}$$

which is in contradiction to the assumption (235). Thus, in this case the maximum rate of the instability and the maximum value of $|\mathbf{k}'| v_s$ are given by

$$\gamma_{\max} \sim \omega_{pe} \frac{m_e}{m_i} \int dk |E^{+(0)}|_k^2 , \tag{240}$$

$$|\mathbf{k}'|_{\max} v_s \sim \omega_{pe} \left(\frac{m_e}{m_i} \right)^{\frac{3}{2}} \left(\int dk |E^{+(0)}|_k^2 \right)^{\frac{1}{2}} . \tag{241}$$

Thus the theory of modulational interactions of broad wave packets allows us to make the following conclusions:

(1) The instability is described by integral equations for perturbations of the wave field correlation functions. These equations play the same role as the set of coupled equations for the fields of modulational perturbations in the case of a single monochromatic pump wave (or an infinite system of equations describing the modulational interactions of two modes).

(2) The instability of the broad wave spectrum is significantly suppressed compared with that of a monochromatic pump wave of the same energy.

(3) The presence of the threshold (210) for the modulational instability is possible only for sufficiently broad spectra, when the width of the spectra $\delta\omega$ exceeds the maximum rate of the modulational instability. For narrow spectra (when the spectral width is less than the maximum rate of the modulational instability), the instability develops for any level of turbulence and, consequently, its threshold is absent.

8. Conclusions

We have reviewed the development of modulational interactions of two monochromatic pump waves and broad packets of random waves. The modulational interactions of even two monochromatic pumps differ qualitatively from the modulational instability of one monochromatic pump. In particular, in the case of two pump modes interference terms [see (23)] appear which significantly effect the development of the instability. These terms are due to nonlinear coupling of two pump waves which produces low-frequency density perturbations at the beat frequency. Their appearance leads to a non-stationary distribution even in the zeroth approximation. In equations for modulated perturbations, coupling between modulational fields and the low-frequency density variations leads to an infinite system of coupled equations [see (42)] which only for some specific cases can be reduced to a finite one. This is possible, for example, for the case where the frequency difference of two monochromatic pumps is large compared with the instability growth rate. In this case the system of equations describing the modulational interactions of two modes can be reduced to a system of eight equations [see (73) and (74)]. These equations describe the interactions of four satellites from the pump waves and also four satellites from virtual waves. The limits of validity of such an approximation for the case of Langmuir waves are determined by the inequality (88). In the particular situation where the two pump Langmuir waves propagate in perpendicular directions and for a sufficiently low pump level [see conditions (107) and (108)], the modulational instability of these pump waves can be described by a system of four equations (when only satellites from the pump waves are taken into account). It is well to bear in mind that even in these specific cases the instability is strongly affected by coupling of different modes (as well as their modulational perturbations) excited by the two pumps. All these effects lead in general to suppression of the instability development [see e.g. the maximum rate (115) of the two-pump instability which is less than the maximum rate of the modulational instability of one mode with the intensity equal to the sum of intensities of the two pumps]. However, the

physics of the interactions is richer, and interesting resonance effects may appear (e.g. when some wave satellites are amplified while the others suppressed etc.)

The modulational instability of packets of random waves is described by a set of integral equations (163) and (164) which is the generalisation of the equations which appear in the description of the modulational instability of two monochromatic pump waves. The solvability condition of these equations results in the dispersion relation of the modulational instability. The investigation of the modulational instability on the basis of the proposed formalism has shown that the instability of the broad wave spectrum (when the spectral width exceeds the maximum growth rate of the modulational instability) is also significantly suppressed compared with that of a monochromatic pump wave of the same energy. In this case the presence of the threshold of the modulational instability (210) is possible. For narrow spectra (when the spectral width is less than the maximum instability rate) the instability develops for any levels of turbulence and its character is similar to that of the monochromatic pump.

Thus we can conclude that interference terms play an important role in the development of the modulational instability of many pumps leading in particular to suppression of the instability.

To summarise, all main features of the development of modulational instabilities of two monochromatic pumps and a packet of random Langmuir waves (comparing them with the features of the modulational instability of one monochromatic pump) can be presented in the form shown in Table 1.

The modulational interactions of Langmuir waves play an important role in the description of numerous phenomena in the lower part of the Earth's ionosphere, in active geophysical experiments, in laser plasmas, etc. Consideration of the modulational processes is also important for a proper description of the transition from a weak-turbulent to strong-turbulent state.

Excitation of strong turbulence is different in the cases of a monochromatic pump and a random pump. Here, we have presented the effects of the modulational interaction of two monochromatic pumps and a packet of random modes for the case where the unperturbed plasma is homogeneous, unmagnetised, and collisionless. We can expect that the modulational instabilities of many pumps, as well as of broad wave spectra, will be important for many other real plasmas (e.g. for plasmas in the presence of the external magnetic field, for inhomogeneous plasmas, for collision-dominated plasmas, etc.). We note that the modulational interactions of many pumps in such plasmas have not yet been studied.

Investigations of the modulational interactions in plasmas in the presence of an external magnetic field and in inhomogeneous plasmas are more complicated than that presented here for the case of Langmuir waves. This is associated with a wealth of collective effects in such plasmas, as well as with the more complicated description of the modulational interactions of each of the possible collective modes in the presence of plasma anisotropy and/or inhomogeneity. However, the nonlinear formalism presented can be applied for the description of modulational instability in such situations.

Of course, it is impossible to predict exactly the character of the modulational instabilities of broad wave spectra for each of all possible modes [even in the case of the monochromatic pump the development of the modulational instability of different plasma modes can vary largely: compare, e.g., the character of the

Table 1. Main features of the development of modulational instabilities of two monochromatic pumps and a packet of random Langmuir waves

	One monochromatic pump	Two monochromatic pumps	Broad spectrum of random waves ($\gamma_{\max} \ll \delta\omega$)	Narrow spectrum of random waves ($\gamma_{\max} \gg \delta\omega$)
Description	By set of two equations coupling high- and low-frequency satellites $\omega_0 \pm \Delta\omega$ of the pump mode	By infinite set of equations coupling high- and low-frequency satellites not only from the pump modes, but also from 'beat' fields $\omega_{0,1} \pm n\delta\omega$	By set of integral equations coupling satellites from each mode in the spectrum taking into account their stochastic properties	By set of integral equations from each mode in the spectrum taking into account their correlations
Behaviour	Maximum possible instability rates	In general, suppression of the instability in comparison with the case of monochromatic pump	Significant suppression of the instability in comparison with the case of monochromatic pump	Small suppression takes place, but the character basically similar to that of monochromatic pump
Thresholds	No instability thresholds	Absent for specific considered cases; however, further investigations necessary to compare with typical situations (one-dimensional and isotropic) considered for packets of random waves	Thresholds present	Thresholds absent

modulational instability of the lower-hybrid waves (Tsytovich *et al.* 1992) and that of the oscillations excited as a result of the lower-hybrid drift instability (Popel *et al.* 1994*a*)]. Nevertheless, one can assume that for the oscillations in these plasmas which have properties close to those of Langmuir waves (e.g. the lower-hybrid waves in plasmas in the presence of an external magnetic field), the character of the modulational instability of broad wave spectra can acquire some properties similar to those mentioned above (namely, suppression of the modulational instability of broad wave spectra in comparison with the instability of a monochromatic pump, and the appearance of the instability thresholds).

Furthermore, one can suppose that the character of the modulational interactions of broad wave spectra in collision-dominated plasmas differs significantly from that described in the present review, because the modulational instability in such plasmas is mostly determined by collisional effects (the differential Joule heating nonlinearity, see the recent review by Vladimirov and Popel 1994), in contrast to the situation in the collisionless plasmas (where the ponderomotive force nonlinearity dominates). Thus, consideration of the modulational instabilities of broad wave spectra is a problem of significant interest and would be useful for the interpretation of different phenomena in various space, astrophysical, and laboratory plasmas.

And, finally, we note that many effects observed in other nonlinear media (e.g. self-focusing phenomena in nonlinear optics) are similar to modulational interactions. These effects can be described by a set of equations analogous to the Zakharov equations. Therefore, development of these instabilities can have the same features; in particular, thresholds can occur for multi-mode self-focusing instabilities. We stress that the general formalism presented in this survey can be applied, with some minor modifications, to study these phenomena.

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