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STORIES ABOUT SYMMETRY

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Abstract: In this note I would like to shine a light on topics where symmetry plays a role. Together we will explore how symmetry can be captured mathematically and how concepts involving symmetry help to understand viruses and self-organising materials, how they can be used to design better algorithms and how the classification of elementary symmetries leads to an exciting story about an unprecedented communal effort to prove a big mathematical theorem. I will give examples of open questions that still wait to be answered.

Keywords: symmetry, group theory, algorithms, finite simple groups, classification theorem

INTRODUCTION

What comes to your mind when you think about the word symmetry? Do you see pictures? Do you remember pieces of art, or of architecture? Do memories from school come up, maybe from geometry lessons? There are many famous examples of artwork where symmetry is displayed, for example in wall tilings, paintings or glass windows. An example of symmetry in Australian Aboriginal artwork is shown in Figure 1, in which totems of Indigenous culture and landscape have been incorporated into the painting. The artist has combined rough global symmetry with intricate symmetrical patterns.

M.C. Escher famously dedicated a lot of his work to symmetry, perspective and other mathematical concepts that he enjoyed exploring and analysing through his craft. The virtual Escher gallery (see https://mcescher.com/) exhibits numerous examples of this in his work. The artist Alex John Beck explores facial symmetry in his project ‘both sides of’ (see https://alexjohnbeck.com/projects/Both_Sides_Of), and he has kindly allowed me to show examples of his work. I came across this art project because of the following quote from *The Economist* (see Economist 2012):

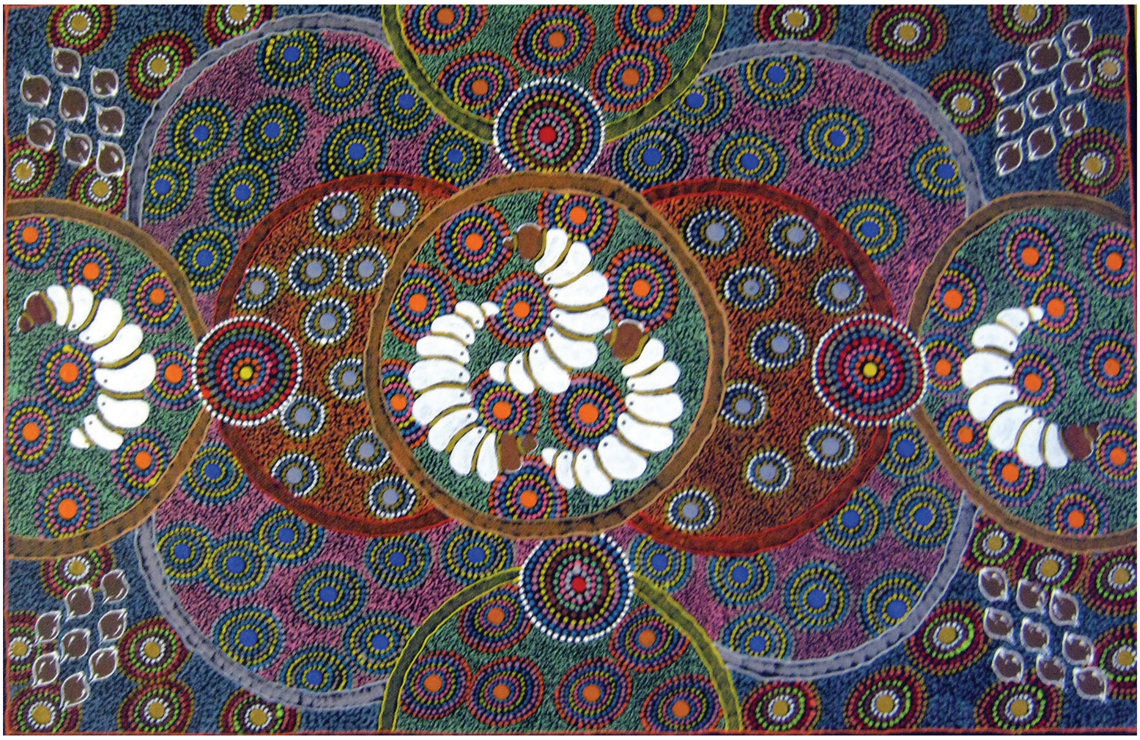

Figure 1: Painting by Australian Aboriginal artist Marie Young, c.2001 (46 x 30 cm). Maryvale Station Artifacts, Northern Territory, Australia. Photo from Bill Birch.

> ‘BEAUTY may be in the eye of the beholder, but a symmetrical face is usually a big help. … In theory, evolution provides a logical answer: unfit individuals are less likely than fitter folk to be able to maintain the symmetrical development of their bodies when exposed to stress and disease.’

Figure 2: Two examples from Alex John Beck’s project ‘Two sides of’.

This explains why psychologists are also interested in symmetry:

> ‘What is symmetry? Symmetry is the extent where one-half of the object is the same as the other half. Facial symmetry is the most commonly talked about wherein it is a specific measure of bodily symmetry.’ (quoted from https://ncpsychoanalysis.org/symmetrical-face/)

From a mathematical perspective, this description of symmetry is rather vague, and also narrow, because it only refers to mirror symmetry and neglects rotational symmetry. But, of course, every discipline describes what is intuitively relevant there, as we will see when we turn to physics and chemistry. In these subjects, symmetry has played a major role in the description and analysis of crystals, and an extensive historical overview is given in Burckhardt’s book on the symmetry of crystals (Burckhardt 1988). This book was inspired by an exhibition in 1986 titled ‘Symmetry in arts, in nature and in science’.

Concepts like regularity, repetition and symmetry also play a role when it comes to storing or transmitting information. For example, describing what a chessboard looks like is much easier than describing a completely chaotic floor tiling. It is exactly this principle, namely storing or communicating information of a highly regular structure effectively, that led to the virus models by Caspar, Klug, Crick, Watson and later Twarock (see Twarock, 2020 for a detailed and accessible overview). The basic idea often was to start with a sphere and then tile its surface with regular polygons, and these virus models made it possible to explain how very much information, in this case the reproduction information for the virus, could be coded and stored in such a tiny space, and with a tolerable error probability in the reproduction process. In computer algebra, a similar reduction principle is applied for many algorithms — we reduce the search and hence make problems accessible, or considerably reduce the run time, by detecting symmetry and then pruning parts of the search infrastructure that are superfluous — see for example Leon (1991) or Jefferson et al. (2021). This makes more problems accessible for us to solve, harder problems, and it contributes to the sustainability of doing computer-assisted research.

As a final example, we look at self-assembling materials, where we also often detect symmetry. The general example comes from columnar liquid crystals where the polyphilic molecules form 2D-honeycomb structures. (See Figure 3.)

An interesting special case occurs when the polyphilic molecules assemble in a quasiperiodic pattern. The following picture shows experimental electron density maps where we can see the development of the dodecagonal supertiles at the transition from triangular to square tiling patterns (A–C) with changing temperature. The lower part of the picture shows the models of the tiling patterns in the distinct honeycomb phases as follows: Black lines correspond to polyaromatic rods, blue dots correspond to glycerol groups, and the cells are filled by lateral alkyl chains (see Poppe et al. 2020).

In an ongoing project in the research training group ‘Beyond Amphiphilicity’ (https://beam.uni-halle.de/) we take this as motivation in order to understand the chemical perspective on symmetry with mathematical rigour, but also respecting the long history of using this concept in a variety of disciplines. Carsten Tschierske and I, together with our PhD students Virginia-Marie Fischer and Christian Anders, are working on a combination of mathematical analysis and chemical experiments. One of our aims is to close gaps in the existing descriptions, discuss and resolve ambiguities and contribute to a consistent description of well-known concepts that is both mathematically exact as well as accessible to non-mathematicians. In the future

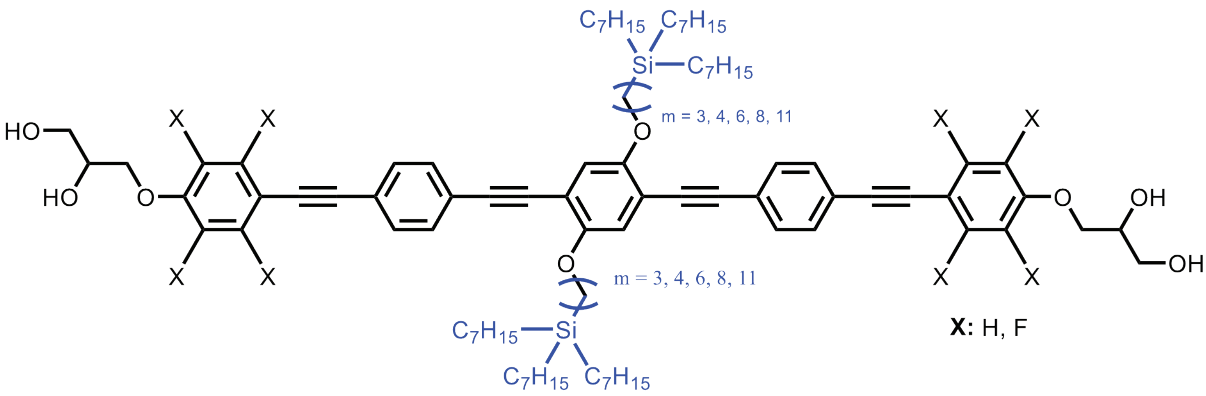

Figure 3: Example for our molecules with an indication of the different variants.

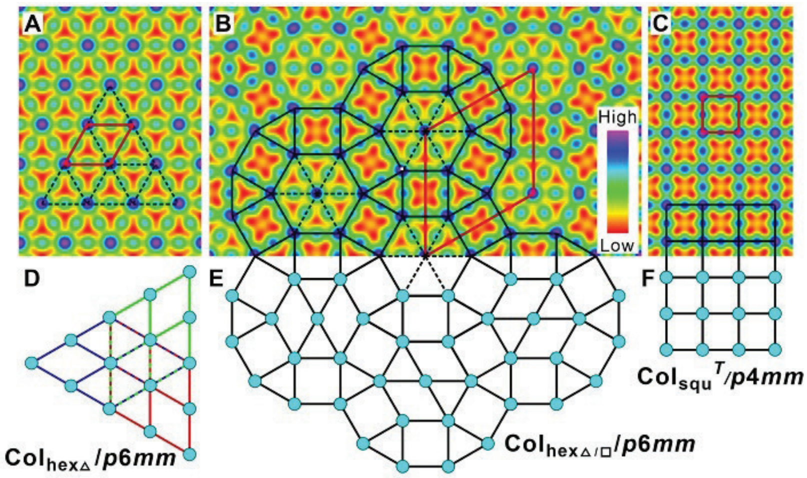

Figure 4: Change of symmetry — development of dodecagonal supertiles at the transition from triangular to square tiling patterns with rising temperature.

our work will hopefully make it possible to predict the behaviour of such self-organising materials when we change the molecular structure. But now it is time to turn to the mathematical language for symmetry. It goes back to Evariste Galois, in the 1830s, motivated by permutations of roots of polynomials.

MATHEMATICAL DESCRIPTION

Definition

A group is a pair (G, ∗), where G is a set and the following conditions are satisfied:

(1) ∗ is a binary operation on G, which means that if we take a and b from G, then a ∗ b is also an element from G.

(2) G and ∗ satisfy the associative law, which means that for all a, b and c from G, we have that (a ∗ b) ∗ c = a ∗ (b ∗ c).

(3) There is a neutral element n for ∗ in G, which means that n is an element such that for all g in G, it is true that g ∗ n = g.

(4) For all g in G there is an inverse element with respect to ∗ in G, i.e. an element h such that g ∗ h = n, where n is neutral as in (3).

Examples

- (Z,+) , the set of integers and addition, exhibits an example of a group where the binary operation satisfies the commutative law. This means that for all integers x and y, it is true that x+y= y+x. The neutral element in this group is 0, and for each integer z, its inverse is -z.
- Next we take the set of all permutations of a non-empty set S, together with the composition of maps. A permutation is a one-on-one map from S to itself, and by the composition of maps we just mean that we apply the maps one after the other. This also gives an example, and it can be seen as an early and very natural example of groups coming up in mathematics. The neutral element is the identity map (‘do nothing’), and each map is a permutation, i.e. one-on-one (or in mathematical terms bijective) and hence each map in S has an inverse map (‘undo’). This example sheds some light on why the notion of a group is so fundamental and has so many applications in various areas of mathematics and beyond.
- Another very natural example comes from regular polygons and their sets of symmetries. For instance, the set of rotations and reflections of a square and the compositions of maps gives rise to a group with eight elements.

These examples can already illustrate some of the subtleties around groups. In the first example, (Z,+), the set of integers and addition, it does not matter which way around we apply the binary operation. In contrast, for the second and third example, this is usually not the case.

As soon as the set S has three elements or more, or the regular polygon has three vertices or more, the group that we described above does not satisfy the commutative law with the given binary operation. While some permutations, or symmetries, commute, others do not.

If S={1,2,3}, for example, then let us denote by c the map that interchanges 1, 2 and 3 in a cycle. Let s denote the map that swaps 1 and 2 and leaves 3 alone. See what happens if you apply c first and then s, or the other way around!

If we take a square and denote its corners by 1, 2, 3 and 4, where the notation is used going round in a circle and counter-clockwise, and if we consider counter-clockwise rotations around the centre of this square, then any two such rotations commute. This means that it does not matter which way around we compose the rotations. But if we apply a rotation and a reflection, then it matters which way around we apply the maps. Most of the time! Again, I invite readers to try for themselves!

In fact, regular polygons and their symmetries are a good place to close the gap between our abstract definition of a group and the intuition of 'symmetry': If a geometrical object is given (e.g. a regular polygon), then a symmetry of this object is a bijective map from the object to itself that preserves distances. The set of symmetries, defined in this way, will always give rise to a group, with the composition of maps as binary operation. Depending on context, 'distance-preserving' might be replaced by 'continuous', or by 'compatible with an algebraic structure', or by other properties that are relevant to the area at hand.

Context of abstract group theory and its history

Groups have been abstractly studied for centuries, and the development of group theory has often been informed by (actual or potential) applications. The perspective of applications, in particular outside mathematics, comes with some challenges, because the various contexts lead to many different notations and often to ambiguities. From the perspective of an abstract mathematical theory, we need precise definitions, tools and methods in order to move the theory forward (i.e. results that are proven to be true, rather than just observations from examples), and we need questions and conjectures that move the theory forward and get researchers involved. It was Otto Hölder who, in 1892, asked a very natural and very hard question that turned out to be a driving force for the field for a long time to come: 'It would be of the greatest interest if it were possible to give an overview of the entire collection of finite simple groups.' (Solomon 2001)

Here, a group is said to be finite if and only if the number of elements in the underlying set is finite, and 'simple' means that the group cannot be split up into smaller groups by using normal subgroups and factor groups. Rather than going into technical details here, we rephrase Hölder's question in a nutshell: What are the 'building blocks of groups'?

If we want an analogy, then we might think of prime numbers and how they build up the integers. Given an integer, once we know all its prime divisors with multiplicities, we can easily obtain our original number by multiplying. However, the situation with finite groups is subtler. Even if we can split up a group into simple sections and determine their structure, then the knowledge of these simple building blocks is not enough in order to re-build the group. The same building blocks can be used to construct quite different groups — for example, the unique simple groups of size 2 and 3, respectively, can be combined in two ways, and one of the resulting groups is commutative while the other one is not.

Therefore, a much more appropriate analogy comes from chemistry: atoms and molecules.

We could then ask:

- What are the 'atoms' of group theory?
- How can they be combined to build up 'molecules'?
- Is there a 'periodic table' for group theory?

It took more than a hundred years, generations of mathematicians and a gigantic communal effort, but it was indeed possible to find all the 'atoms' of group theory, i.e. the finite simple groups, and we keep finding out more and more about their internal structure and their properties. They come in several infinite families, along with 26 so-called sporadic simple groups. For a better impression of the classification theorem and its background, I recommend overview articles and background literature by Gorenstein (e.g. Gorenstein 1985) and Solomon (e.g. Solomon 2001). The Classification of Finite Simple Groups continues to have an impact in mathematics and beyond, partly because many tentative results and strategies to prove big open conjectures were based on it being achieved. At the same time, it raises deep philosophical questions, because the original proof of the actual classification theorem is scattered across tens of thousands of pages in journal articles and books, it involves computer calculations, and some of the work was based on private communication or on the publication of articles that have been announced, but never happened. Michael Aschbacher, one of the key contributors to this classification theorem, raises some of these philosophical questions (see Aschbacher 2005). The importance and impact of the result calls for deep and extensive research into these philosophical questions as well as the history of the whole undertaking. Such a communal effort over several generations is unprecedented in mathematics, and there is much to learn from this great collective achievement:

1) How can we be convinced that the proof of the classification theorem is complete?
2) How did this communal work influence the perception of what a proof even is?
3) How did communication between colleagues change during this time, because of the work and because of technological advances?
4) How did the project influence work across sub-disciplines?
5) What was the role of conferences and special programs? Of networks and funding?
6) How did the rise of computers change the work in the classification program? How did the research community respond to computer-assisted proofs or constructions?

This is just a small selection of questions, and some of them turn out to be really big, difficult questions. In hindsight it becomes obvious why they are relevant — for example, the original proof was widely believed to be complete and correct, but then some gaps were found. Most of them were minor, but there was also a more substantial gap that was discovered and discussed not long after the announcement that the classification is complete, and it took more than 20 years and more than 1000 pages to fill this gap (see Aschbacher & Smith 2004). In order to increase confidence in the existing proofs and to understand the classification even better, several 'next generation proof projects' are ongoing. While the so-called GLS revision project is close to the original methods (nine books so far, see Gorenstein et al. 1994–2021), there are also new approaches with a different perspective, for example a strategy for a proof using fusion systems. Aschbacher gives an overview over this approach along with some background (see Aschbacher 2015).

As a finite group theorist, I am excited about the historical and philosophical perspective on the classification theorem and about working in this area. I began working on historical questions and using the Oberwolfach Digital Archive after meeting Volker Remmert, a science historian whose academic roots lie in group theory. His mini-workshop 'History of the Workshops in Oberwolfach, 1944 – ca. 1960' introduced me to a historical perspective on several of my research topics, and together we hope to shed new light on the history and philosophy of the Classification of Finite Simple Groups.

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Conflict of interest

The author declares no conflict of interest.

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