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Structure of a Sequence of Two Zone Polytropic Stellar Models with Indices 0 and 1

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Abstract

The structure and physical properties of a sequence of two zone polytropic stellar models, based on E-type composite analytical solutions of the Lane–Emden equation for indices $n = 0$ and 1, have been determined. The resulting models are characterized by an inner zone of constant density, corresponding to the $n = 0$ component, and a relatively small envelope with a steep polytropic density gradient and index $n = 1$. At the surface of these near uniform density composite configurations the physically significant condition $\rho = 0$ is always satisfied, a condition which is not fulfilled by the complete polytrope of index $n = 0$. In each model the surface corresponds to the first zero of the associated composite function, which in turn is given by one of the higher order zeros of the E solution of the Lane–Emden equation for $n = 1$. The radial pulsational properties of these low central condensation type models have also been investigated with the relative amplitudes of radial displacement and density variation given as a function of radial distance for the first six modes of some selected models. Overall, these results establish that the radial displacement amplitudes are only significant in the $n = 1$ outer layer and accordingly the pulsations can be classified as essentially surface phenomena.

1. Introduction

The three classical E-type analytical solutions of the Lane–Emden equation

$$\xi\frac{d^2\Theta}{d\xi^2}+2\frac{d\Theta}{d\xi}+\xi\Theta^n = 0\,; \qquad 0 \leqslant \xi \leqslant \xi_1,$$

for indices $n = 0$, 1 and 5, which establish complete polytropes with constant index throughout, are well known. For astrophysical applications these three solutions present a rather restricted range of models; the $n = 0$ polytrope with $\Theta_0(\xi) = 1-\frac{1}{6}\xi^2$ has constant density and does not comply with the normal condition of $\rho = 0$ at the surface, while the $n = 5$ model based on the solution $\Theta_5(\xi) = (1+\frac{1}{3}\xi^2)^{-\frac{1}{2}}$ has infinite radius and central condensation. When $n = 1$, the Lane–Emden equation is linear and accordingly has a simple solution $\Theta_1(\xi) = (\sin\xi)/\xi$ but, unlike the other two analytical solutions, it cannot be transformed following the homology theorem which does not hold specifically when $n = 1$.

In terms of the central condensation these three models possess polytropic density distributions which can be considered as extremes, with $\rho_c/\bar{\rho} = 1, 1\cdot84, \infty$ for $n = 0, 1, 5$ respectively. Alternatively, simple composite analytical solutions, which

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generate physically consistent polytropes, have many attractive features for applications and also extend the choice of possible density distributions. Recently, a finite radius composite analytical E-type solution for the polytrope $n = 5$ has been given (Murphy 1980, 1981), where the model has an extensive low density outer zone with $\zeta_1 = 62{\cdot}2665$ and $\rho_c/\bar{\rho} = 35\,326{\cdot}88$. In the present paper a sequence of two zone

Table 1. Numerical values of parameters for the composite model sequence

Parameter or feature	Model $m = 1$	$m = 2$	$m = 3$	$m = 4$
A_m	0·40702	0·26456	0·19683	0·15691
B_m	5·80232	14·04316	25·56271	40·36885
$\Theta_m(0)$	6·03627	14·28735	25·81175	40·61719
ζ_{Jm}	5·76345	9·09501	12·32294	15·51460
$\Theta_m(\zeta_{Jm})$	0·5	0·5	0·5	0·5
Fraction of R for const. ρ	92%	96·5%	98%	99%
First zero at $\zeta_1 =$	2π	3π	4π	5π
$\rho_c^{-1}\,d\rho/d\zeta$ at ζ_1	−1·84694	−2·98005	−4·06843	−5·13992

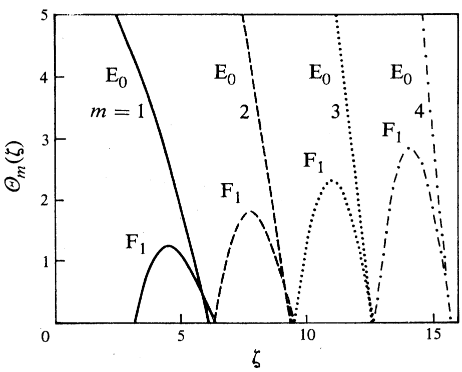


Fig. 1. Transformed $n = 0$ and $n = 1$ components of the composite solutions $\Theta_m(\zeta)$ for the first four models $m = 1, 2, 3, 4$. At the joining point, $\Theta_m(\zeta_{Jm}) = 0{\cdot}5$ for all values of m and the first zero occurs at $\zeta_1 = (m+1)\pi$.

polytropic models with a central zone of index $n = 0$ and an outer zone of index $n = 1$ is developed with the condition $\rho = 0$ holding at the surface in all cases. These near uniform density models, which have a steep density gradient near the surface, are based on the following E-type sequence of solutions of the Lane–Emden equation (Murphy 1982):

$$\Theta_m(\zeta) = \Theta_{m,0}(\zeta) = \frac{\zeta_{Jm}^2(2\tan\zeta_{Jm} - \zeta_{Jm})}{6(\tan\zeta_{Jm} - \zeta_{Jm})} - \frac{\zeta^2}{6}; \qquad 0 \leqslant \zeta \leqslant \zeta_{Jm}, \tag{1a}$$

$$= \Theta_{m,1}(\zeta) = \frac{\zeta_{Jm}\sin\zeta}{2\zeta\sin\zeta_{Jm}}; \qquad \zeta_{Jm} \leqslant \zeta \leqslant (m+1)\pi, \tag{1b}$$

for $m = 1, 2, 3, \ldots$, where ζ_{Jm} is the joining point for the two polytropic zones of model m. The homological scaling factor for the $n = 0$ component of each model, which is incorporated in the solution (1), is given by

$$A_m = \frac{1}{\zeta_{Jm}}\left(\frac{6(\tan\zeta_{Jm} - \zeta_{Jm})}{2\tan\zeta_{Jm} - \zeta_{Jm}}\right)^{\frac{1}{2}},$$

and the corresponding linear scaling of the $n = 1$ component is achieved by the factor

$$B_m = \zeta_{Jm}/2\,|\sin\zeta_{Jm}|\,.$$

Further numerical values which define the model sequence are given in Table 1 and the joining of the transformed $n = 0$ and $n = 1$ components of the composite solutions $\Theta_m(\zeta)$ are illustrated in Fig. 1 for the first four models.

Although numerical composite solutions, which are non-singular at the centre and which consist of an E-type central zone with index n_1 and an F-type outer zone with index n_2, such that $n_1 < n_2 < 5$, have been considered and tabulated in detail by Milne (1930, 1932), building on the general inward integrations of the collapsed type for the Lane–Emden equation given by Fairclough (1930, 1932*a*, 1932*b*), there appears to be no published account of any E-type analytic composite solutions given in the rather extensive literature on both the mathematical analysis of the Lane–Emden equation and the associated composite polytropic structure (Russell 1931; Eddington 1931). The present results, representing a hitherto missing facet in the historical development of polytropic stellar models, illustrate in simple analytical form the important principles of continuity of the physical variables at the zone interface for any composite stellar model. In both the numerical and analytical approach to constructing composite polytropic stellar models it is expedient to satisfy these requirements by matching the two solutions of the Lane–Emden equation with different indices in the U–$(n+1)V$ plane, using the techniques described in detail by Milne (1932) and Chandrasekhar (1939).

2. Model Characteristics

The analytical formulation for $\Theta_m(\zeta)$, given by equations (1), now allows the model characteristics to be specifically determined and, in turn, expressions for the density, pressure and temperature variation and the mass distribution throughout the model can be incorporated into other astrophysical investigations analytically, when the polytropic structure defined is considered physically appropriate. Further, the result established for the density complies with the usual zero surface requirement for stellar models, and in terms of potential applications involving this model sequence represents a significant feature particularly when large scale numerical simulation is contemplated. The atmospheric nature of the index $n = 1$ envelope and the variability of this layer depth with m provide a relevant set of models when, for example, rotational and tidal distortions as functions of density are considered on an analytical basis. In the next section the amplitudes of radial pulsation for these models are shown to be significant only in this envelope layer with overtone modes making only a small contribution to the total displacement.

With standard notation (Chandrasekhar 1939) the total mass of each model, based on the composite solutions (1), can be determined from the expression

$$M_m = 4\pi\alpha_0^3\rho_c\int_0^{\zeta_{Jm}}\zeta^2\,\mathrm{d}\zeta + 4\pi\alpha_1^3\lambda_1\int_{\zeta_{Jm}}^{(m+1)\pi}\zeta^2\Theta_m\,\mathrm{d}\zeta\,. \tag{2}$$

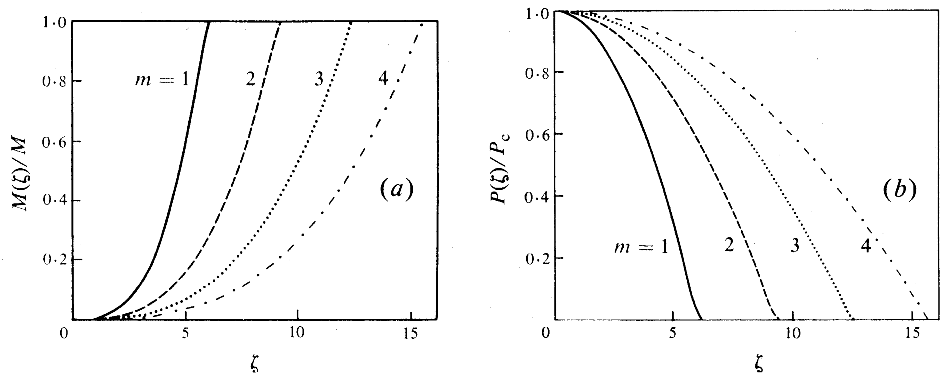
Now we have $\rho = \rho_c =$ constant over the zone $0 \leqslant \zeta \leqslant \zeta_{Jm}$, and at the polytropic interface ζ_{Jm}

$$\rho = \lambda_1\,\Theta_m(\zeta_{Jm}) = \tfrac{1}{2}\lambda_1\,.$$

Hence for continuity we require $\lambda_1 = 2\rho_c$ together with $\alpha_1 = \alpha_0 = R/(m+1)\pi$. The mass of model m in the sequence is then given explicitly by

$$M_m = 4\pi\alpha_0^3\,\rho_c\left(\frac{\zeta_{Jm}^3}{3} + \frac{\zeta_{Jm}}{\sin\zeta_{Jm}}\{\zeta_{Jm}\cos\zeta_{Jm} - \sin\zeta_{Jm} + (-1)^m(m+1)\pi\}\right). \tag{3}$$

The relative mass function $M_m(\zeta)/M$, for $0 \leqslant \zeta \leqslant \zeta_1$, is given in Fig. 2$a$ for the four models labelled $m = 1, 2, 3, 4$.



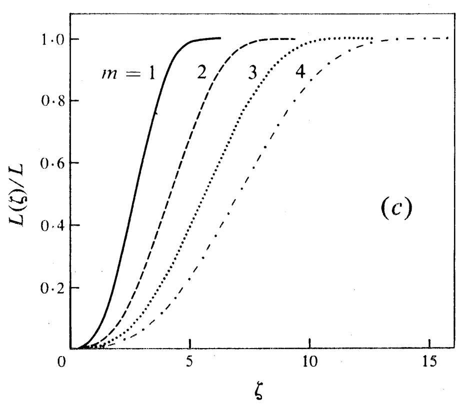


Fig. 2. Distribution of
(a) the relative mass $M(\zeta)/M$,
(b) the relative pressure $P(\zeta)/P_c$,
(c) the relative luminosity $L(\zeta)/L$,
as a function of
the radial variable ζ
for models $m = 1, 2, 3, 4$.

The distribution of the relative density $\rho(\zeta)/\rho_c$ for each composite model is defined by

$$\rho(\zeta)/\rho_c = 1; \qquad \text{for} \quad 0 \leqslant \zeta \leqslant \zeta_{Jm}, \tag{4a}$$

$$= (\zeta_{Jm}\sin\zeta)/(\zeta\sin\zeta_{Jm}); \qquad \text{for} \quad \zeta_{Jm} \leqslant \zeta \leqslant (m+1)\pi = \zeta_1, \tag{4b}$$

and clearly ρ is zero at the surface of these polytropes. Fig. 2b illustrates the computed pressure $P(\zeta)$ as a function of the central pressure P_c. Here for the inner $n = 0$ zone we have

$$P(\zeta)/P_c = 1 - (\tan\zeta_{Jm} - \zeta_{Jm})\zeta^2/\zeta_{Jm}^2(2\tan\zeta_{Jm} - \zeta_{Jm}); \qquad 0 \leqslant \zeta \leqslant \zeta_{Jm}, \tag{5}$$

while for the $n = 1$ zone the pressure is given by

$$P = K_1 \rho^{1+1/n} = 4K_1 \rho_c^2 \Theta_m^2,$$

where

$$K_1 = P_c/2\Theta_m(0)\rho_c^2, \tag{6}$$

and hence

$$P(\zeta)/P_c = (\zeta_{Jm}^2 \sin^2\zeta)/\{2\Theta_m(0)\zeta^2 \sin^2\zeta_{Jm}\}; \qquad \text{for} \quad \zeta_{Jm} \leqslant \zeta \leqslant (m+1)\pi. \tag{7}$$

Any solution of the Lane–Emden equation determines the distribution of the relative temperature $T(\zeta)/T_c$ directly, and for the $n = 0$ zone of each model it is given by the same expression (5) for $P(\zeta)/P_c$, but in the outer zone it is governed by

$$T(\zeta)/T_c = (\zeta_{Jm}\sin\zeta)/\{2\Theta_m(0)\zeta\sin\zeta_{Jm}\}. \tag{8}$$

Moreover, the established form (6) for K_1 is consistent with the equation of state being satisfied at the centre of the model.

The total luminosity of these polytropes can be evaluated from

$$L_m = \frac{4\pi\varepsilon_0 \rho_0^2 T_c^\nu \alpha_0^3}{\Theta_m^\nu(0)} \left(\int_0^{\zeta_{Jm}} \Theta_m^\nu(\zeta)\zeta^2 \, d\zeta + \frac{\zeta_{Jm}^{2+\nu}}{2^\nu(\sin\zeta_{Jm})^{2+\nu}} \int_{\zeta_{Jm}}^{(m+1)\pi} \frac{(\sin\zeta)^{2+\nu}}{\zeta^\nu} \, d\zeta \right), \tag{9}$$

when the nuclear energy generation is based on a power law of the form

$$\varepsilon = \varepsilon_0 \rho T^\nu.$$

For $\nu = 4$, the relative luminosity $L(\zeta)/L$ as a function of the polytropic radius ζ for four models corresponding to $m = 1, 2, 3, 4$ is shown in Fig. 2*c*. In fact for these near constant density models the energy producing core extends over a large fraction of the radius of each model. Specifically, for $m = 1$, we have $L(\zeta)/L > 0{\cdot}99$ only when $\zeta > 5{\cdot}0$; this represents 80% of the total radius which in turn contains only 56% of the total mass. For $m = 4$ the values are nearly the same at 82% and 58% respectively. Furthermore, because $\zeta_{J1} = 5{\cdot}76$, the second integral in equation (9), which gives the nuclear energy contribution from the $n = 1$ zone, is clearly negligible. This in confirmed in numerical terms by the ratios of the contributions from the two zones

$$10\,689{\cdot}227 : 0{\cdot}157 \quad \text{for } m = 1; \qquad 382\,503\,677{\cdot}314 : 0{\cdot}416 \quad \text{for } m = 4.$$

In contrast the outer layers contribute significantly to the gravitational potential energy when $m = 1$ and 2, but overall their contribution to the total is a decreasing function of m. By definition the negative potential energy for a spherically symmetric distribution of matter is

$$-\Omega = 4\pi G \int_0^R r\rho M(r) \, dr,$$

and for the case of a complete polytrope of constant index n in hydrostatic equilibrium it is

$$-\Omega = \frac{3}{5-n}\frac{M^2 G}{R}. \tag{10}$$

For the $n = 0$ and $n = 1$ composite model family the potential energy is

$$-\Omega_m = \frac{M^2G}{R}\left\{(m+1)\pi \Big/ \left(\frac{\zeta_{Jm}^3}{3} + \frac{\zeta_{Jm}}{\sin\zeta_{Jm}}\{\zeta_{Jm}\cos\zeta_{Jm} - \sin\zeta_{Jm} + (-1)^m(m+1)\pi\}\right)^2\right\}$$

$$\times\left\{\frac{\zeta_{Jm}^5}{15} + \frac{\zeta_{Jm}^2}{\sin\zeta_{Jm}}\int_{\zeta_{Jm}}^{(m+1)\pi}\sin\zeta\left(\frac{\zeta_{Jm}^3}{3} + \frac{1}{\sin\zeta_{Jm}}(\sin\zeta - \zeta\cos\zeta\right.\right.$$

$$\left.\left. + \zeta_{Jm}\cos\zeta_{Jm} - \sin\zeta_{Jm})\right)\mathrm{d}\zeta\right\}. \qquad (11)$$

Upon numerical evaluation we get

$$-\Omega = (M^2G/R)(0\cdot50103 + 0\cdot12371)$$

for the $m = 1$ model, representing a $19\cdot8\%$ contribution from the outer layer, while the (total) numerical factors for $m = 2, 3, 4$ are $0\cdot61049, 0\cdot60581, 0\cdot60369$ respectively, with the $n = 1$ outer layer contributing only $3\cdot06\%$ towards the total for the $m = 4$ value. It is evident that for higher values of m in the model sequence the limiting value of $0\cdot6$ is approached, which corresponds to $n = 0$ in the simple expression (10) for $-\Omega$ and is consistent with the smaller fraction of the total radius occupied by the $n = 1$ component in the composite model.

Table 2. Relative values of physical variables at interface of the two polytropic zones

Ratio	Model $m = 1$	$m = 2$	$m = 3$	$m = 4$
$M(\zeta_{Jm})/M$	0·884	0·952	0·979	0·990
$P(\zeta_{Jm})/P_c$	0·092	0·033	0·017	0·008
$\rho(\zeta_{Jm})/\rho_c$	1·000	1·000	1·000	1·000
$T(\zeta_{Jm})/T_c$	0·092	0·033	0·017	0·008
$-\Omega(\zeta_{Jm})/(-\Omega)$	0·802	0·915	0·952	0·970

The relative contribution of the inner zone to the total potential energy, and to several other relevant physical variables, is summarized in Table 2 for the four models $m = 1, 2, 3, 4$.

3. Radial Pulsations

A numerical investigation, incorporating the structure of the zero surface density sequence of polytropic stellar models developed above, has been undertaken to determine some of the salient radial pulsational features of stars with very low central condensation. Physically these results will reflect the oscillatory characteristics of stars with near uniform density, which in turn have a large percentage of their mass contained within the outer half radius. Specifically, for all the models considered we have the mass function $M(\zeta) < 0\cdot12\,M$ over the range $0 \leqslant \zeta \leqslant \frac{1}{2}(m+1)\pi$. Two questions of particular significance are, firstly, whether or not the relative radial oscillation displacements are only significant throughout the $n = 1$ surface layers, which have a steep density gradient and can be considered as representative of an atmospheric layer, and, secondly, the nature of the associated role of the higher modes in contributing to the radial velocity curves.

In terms of the transformed Emden variable ζ, the differential equations governing the relative amplitude of displacement η for small radial adiabatic oscillations of these composite polytropes are

$$\frac{d^2\eta}{d\zeta^2}+\frac{1}{\zeta}\left(4-\frac{2(n+1)(\tan\zeta_{Jm}-\zeta_{Jm})\zeta^2}{\zeta_{Jm}^2(2\tan\zeta_{Jm}-\zeta_{Jm})-(\tan\zeta_{Jm}-\zeta_{Jm})\zeta^2}\right)\frac{d\eta}{d\zeta}$$
$$+\frac{6(\tan\zeta_{Jm}-\zeta_{Jm})}{\zeta_{Jm}^2(2\tan\zeta_{Jm}-\zeta_{Jm})-(\tan\zeta_{Jm}-\zeta_{Jm})\zeta^2}\{\omega_k^2-\tfrac{1}{3}(n+1)\alpha^*\}\eta=0, \qquad (12)$$

for $0<\zeta\leqslant\zeta_{Jm}$, where $n=0$, and

$$\frac{d^2\eta}{d\zeta^2}+\frac{1}{\zeta}\left(4+\frac{2(\zeta\cos\zeta-\sin\zeta)}{\sin\zeta}\right)\frac{d\eta}{d\zeta}$$
$$+\left(\frac{\omega_k^2\zeta_{Jm}\sin\zeta_{Jm}(2\tan\zeta_{Jm}-\zeta_{Jm})\zeta}{3(\tan\zeta_{Jm}-\zeta_{Jm})\sin\zeta}+\frac{(n+1)\alpha^*(\zeta\cos\zeta-\sin\zeta)}{\zeta^2\sin\zeta}\right)\eta=0, \qquad (13)$$

for $\zeta_{Jm}\leqslant\zeta<(m+1)\pi$, where $n=1$.

The solutions to this set of differential equations are initialized about $\zeta=0$ by the series expansion

$$\eta=1+\tfrac{1}{30}A_m^2(\alpha^*-3\omega_k^2)\zeta^2+\tfrac{1}{2520}A_m^4(7+\alpha^*-3\omega_k^2)(\alpha^*-3\omega_k^2)\zeta^4+\dots, \qquad (14)$$

and ω_k^2 is determined subject to the boundary condition

$$\lim_{\zeta\to\zeta_R}\left\{-\gamma\left(\zeta\frac{d\eta}{d\zeta}+3\eta\right)+\eta\left(4-\frac{2(m+1)^2\pi^2\sin\zeta_{Jm}\omega_k^2}{\zeta_{Jm}\cos(m+1)\pi}\right)\right\}=0, \qquad (15)$$

being satisfied at the surface ζ_R, for each mode of oscillation. Here $\omega_k^2=\sigma_k^2\alpha_0^2\rho_c/\gamma P_c$ and $\alpha^*=3-4/\gamma$, with $\gamma=\frac{5}{3}$, σ_k being the corresponding frequency. The established values of the characteristic frequencies ω_k for the first six modes of the first four models in the composite sequence are given in Table 3.

Table 3. Characteristic frequencies for radial pulsations

Mode	ω_k			
k	model 1	model 2	model 3	model 4
0	0·30759	0·29594	0·28496	0·28491
1	1·07891	1·00658	0·95895	0·92968
2	1·53258	1·55347	1·58641	1·68403
3	2·15048	2·08564	2·03920	2·02887
4	2·52622	2·60263	2·71692	2·83292
5	3·15450	3·09475	3·05080	3·16310

The computed relative amplitudes η of the displacements in the fundamental oscillatory mode and the first five overtones as functions of the distance $x=\zeta/\zeta_R$ from the centre are given in Figs 3*a* and 3*b* for the composite polytropic models

$m = 1$ and 4 respectively. In both cases an extended scale for $x > 0 \cdot 90$ has been utilized to illustrate the number of nodes occurring for each mode in the $n = 1$ variable density zone. In fact, for model 1 in Fig. 3*a*, 60% of the nodal points fall within this region, which in turn occupies only 8·3% of the total radius, while the respective values for model 4 in Fig. 3*b* are 40% and 1·2%. On a comparative basis Schwarzschild's (1941) results for the complete polytrope $n = 3$ show that only two nodal points lie within the outer 10% of the radius when five modes are considered,

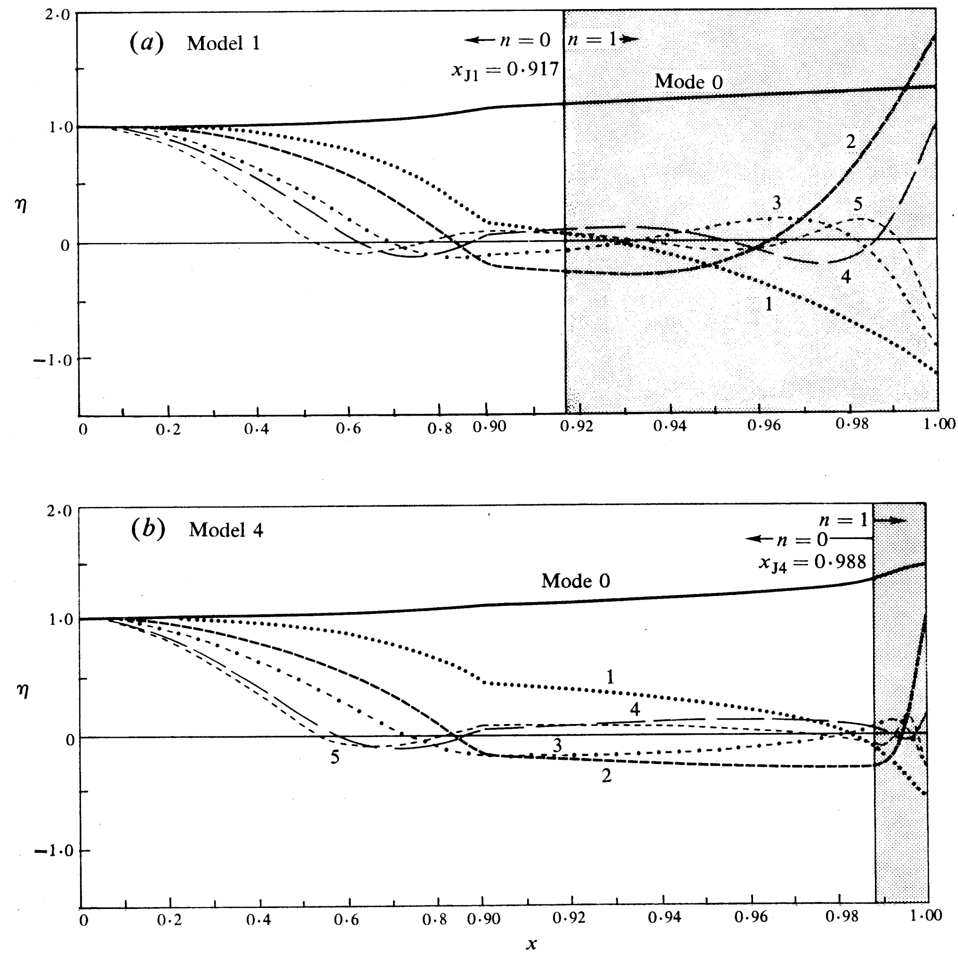


Fig. 3. Relative amplitudes of radial oscillation η for the first six modes of (*a*) model $m = 1$ and (*b*) model $m = 4$, as a function of the relative radius $x = \zeta/\zeta_R$. The polytropic zone interface is at (*a*) $x = 0 \cdot 917$ and (*b*) $x = 0 \cdot 988$ respectively. At $\zeta = 0$, η is scaled to 1.

and for an early main sequence star of $10 M_{\odot}$ (Van der Borght and Murphy 1966) only two nodal points of the first six modes lie within the equivalent region. Consequently a sharp density gradient in the outer regions has the direct effect of concentrating the nodal points (which in mathematical terms are the zeros of the eigenfunctions corresponding to the computed eigenvalues) of all the modes near the surface. Overall, the pulsation displacement is characterized by small relative amplitudes in both the $n = 0$ and $n = 1$ zones for all modes, except in the low density layer in the immediate vicinity of the surface. These two features taken in conjunction establish the atmospheric nature of radial pulsations.

Moreover, with the possible exception of the second overtone of model 1, the higher modes are not significant in terms of their collective displacement at the surface. On the basis of these results one could anticipate that the fundamental mode of pulsation is the dominant one for stars of low central condensation and the resulting observed features, such as the radial velocity and light curves, are of sinusoidal character.

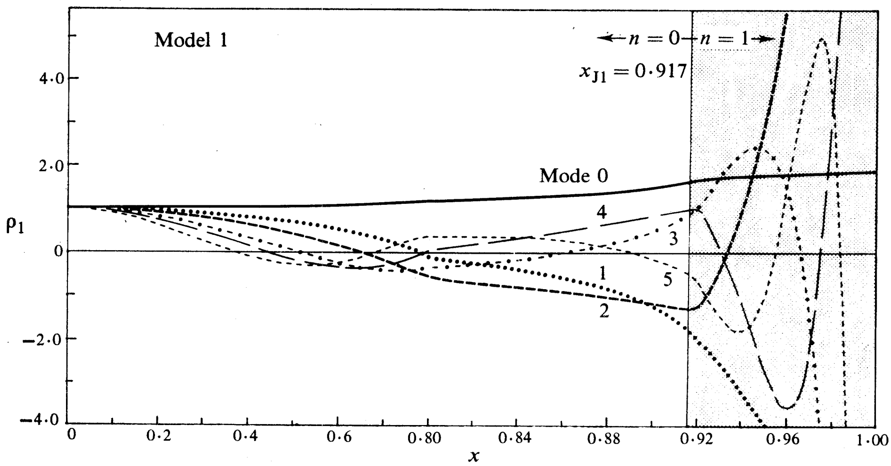


Fig. 4. Relative amplitudes of the density variation ρ_1 for the first six density modes of model $m = 1$, as a function of the relative radius $x = \zeta/\zeta_R$. At $\zeta = 0$, ρ_1 is scaled to 1.

A brief examination of the density variations associated with the pulsation modes reveals that, in terms of the relative amplitude ρ_1, they are considerably greater than the corresponding displacement amplitudes, particularly in the $n = 1$ zone component of the models. The values of ρ_1 as a function of the radial distance ζ, which can be derived from the expression

$$\rho_1 = -\gamma(\zeta\, \mathrm{d}\eta/\mathrm{d}\zeta + 3\eta), \tag{16}$$

are illustrated in Fig. 4 for the first six modes of model $m = 1$. These density modes exhibit the same accumulation of zeros in the (shaded) $n = 1$ zone and the role of the derivative term in equation (16) is clearly defined in this context. In numerical terms the values of $|\rho_1|$ at the relative radius $x = 1$ for model $m = 1$ range from $1{\cdot}79$ for the fundamental mode to $32{\cdot}70$ for the fifth overtone when $|\rho_1|$ is scaled to 1 at $x = 0$. The higher modes for other models in this series show even greater fluctuations in $|\rho_1|$ throughout the reduced $n = 1$ zones. Finally, an investigation into the dependence of the pulsational stability of these modes on the steepening density gradient associated with the outer $n = 1$ zones for the various models in the sequence for increasing m would be informative. A comparison of the results could then be made with those obtained by Van der Borght (1969), where the influence of a hydrogen envelope on the vibrational stability of helium stars was examined.

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