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Pulsar Magnetospheres: Some Fundamental Considerations

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Abstract

The equations governing a pulsar magnetosphere under quasi-static conditions are presented in a vector form from which the theory can be developed in a systematic manner. In particular, integrals applicable in the cylindrical and axisymmetric cases are obtained direct from these vector equations.

1. Introduction

The canonical model of a pulsar as a rotating magnetized neutron star has been developed for quasi-static (QS) conditions, under which the pulsar is assumed to rotate with uniform angular velocity $\Omega \boldsymbol{k}$ and all quantities that determine the electromagnetic field and the plasma flow depend jointly on the time t and the azimuth ϕ, referred to the axis of rotation $\boldsymbol{k}$, via the variable $\phi - \Omega t$.

For the general case, where the magnetic axis of the pulsar is not aligned with the axis of rotation, the model proves most tractable mathematically when the physically unrealistic 'cylindrical' constraint that all quantities are independent of the axial coordinate z in the direction of $\boldsymbol{k}$ is introduced. Then it has been found convenient (Mestel, Wright and Westfold 1976; hereinafter referred to as MWW) to express the field components in terms of a corotating scalar potential Φ, the axial component A of a vector potential, and a stream function χ for the plasma current perpendicular to the axis.

For the axisymmetric case, where the magnetic axis is along the axis of rotation, the conditions are fully static. It has likewise been found convenient (Mestel, Phillips and Wang 1979; hereinafter referred to as MPW) to express the field components in terms of a scalar potential ψ (the same as Φ in MWW), a 'magnetic Stokes stream function' P, and a Stokes stream function S for the poloidal plasma current.

The plasma current is generally determined by the magnetohydrodynamical equations of motion for the separate electron and ion species. Again, these are most tractable (Mestel 1973) where it can be assumed that the inertial terms are negligible, yielding the force-free condition familiar in classical plasma theory. It is evident that this condition cannot be maintained out to distances where the inertial terms would assume relativistic values. Indeed, the equations then lead to the physically unacceptable conclusion (cf. Mestel 1973; MWW) that there is no flow of electromagnetic energy across the light cylinder.

In a recent series of papers Burman and Mestel (1978, 1979) and Burman (1980) have shown how, by exploiting Endean's (1972*a*, 1972*b*) integrals of the plasma equations of motion, the inertial terms can readily be incorporated into the theory.

The treatment of the problem of the pulsar magnetosphere has generally proceeded from a consideration of the governing equations after these have been expressed in terms of cylindrical or spherical polar coordinates. While this procedure may seem to be an inevitable consequence of the QS specification, it has at the same time tended to obscure the essential morphology of the governing equations as a generalization of the corresponding equations for a static plasma. As a consequence the problem has not been attacked in the systematic manner that would follow from the recognition of this morphology.

The purpose of the present paper is to exhibit the theory developed so far in a coordinate-free vector form that makes it evident that the relativistic QS governing equations are extensions of the 'classical' nonrelativistic static equations, and thereby to gain further insights into the cylindrical and axisymmetric cases. This is effected by developing suitable vector forms for the QS operator $\partial/\partial t$.

The resulting electromagnetic equations suggest the form of appropriate scalar and vector potentials with an accompanying gauge condition. These are fruitfully exploited in the cylindrical case, in which the field is formally decoupled into two modes dependent on (I) the axial component of the vector potential $\boldsymbol{A}$ (whose source is the axial component of the plasma current density $\boldsymbol{J}$) and (II) the perpendicular component of $\boldsymbol{A}$ and the corotating scalar potential Φ (whose sources are the perpendicular component of the corotating current density $\boldsymbol{J}-\rho_{\mathrm{e}}\Omega\boldsymbol{k}\times\boldsymbol{x}/c$ and the free-charge density ρ_{e}).

Again, in the axisymmetric case it is possible formally to decouple the field into two modes dependent on (I) the toroidal component of $\boldsymbol{A}$ (whose source is the toroidal component of $\boldsymbol{J}$) and (II) the poloidal component of $\boldsymbol{A}$ and the rest-frame scalar potential φ (whose sources are the poloidal component of $\boldsymbol{J}$ and ρ_{e}).

In the magnetosphere the current and charge densities $\boldsymbol{J}$ and ρ_{e} themselves depend on the field vectors via the magnetohydrodynamical equations of motion of the plasma species. Close to the pulsar surface the force-free equations apply, and in the cylindrical case these provide a useful integral (Mestel 1973; MWW) which must, however, in the neighbourhood of the light cylinder give place to the equivalent of a momentum integral obtained by MWW and Burman and Mestel (1979) for the unrestricted relativistic equations.

Finally we show that, under the conditions prescribed by Burman and Mestel (1978, 1979), the vector form of the equations of motion leads direct to generalizations of the force-free integrals that obtain for the cylindrical and axisymmetric cases. These and the corresponding equations for the potentials suffice to determine these restricted problems of the pulsar magnetosphere.

2. Field Equations in QS Form

We begin with the fundamental equations for the electromagnetic field vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ due to distributions of current and charge of densities $\boldsymbol{J}$ and ρ_{e} in free space, expressed in the rationalized symmetric general (RSG) form†

$$\text{Faraday–Neumann} \qquad \frac{\partial}{\partial \boldsymbol{x}}\times\boldsymbol{E}+\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}=0, \tag{1}$$

† To obtain corresponding unrationalized gaussian forms we replace $\boldsymbol{E}$, $\boldsymbol{B}$ by $\boldsymbol{E}/(4\pi)^{\frac{1}{2}}$, $\boldsymbol{B}/(4\pi)^{\frac{1}{2}}$ and $\boldsymbol{J}$, ρ_{e} by $(4\pi)^{\frac{1}{2}}\boldsymbol{J}/c$, $(4\pi)^{\frac{1}{2}}\rho_{\mathrm{e}}$. To obtain corresponding SI forms we replace $\boldsymbol{E}$, $\boldsymbol{B}$ by $(\varepsilon_0)^{\frac{1}{2}}\boldsymbol{E}$, $(\mu_0)^{-\frac{1}{2}}\boldsymbol{B}$ and $\boldsymbol{J}$, ρ_{e} by $(\mu_0)^{\frac{1}{2}}\boldsymbol{J}$, $(\varepsilon_0)^{-\frac{1}{2}}\rho_{\mathrm{e}}$.

Ampère–Maxwell $$\frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{B} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{J}, \tag{2}$$

Continuity $$\frac{\partial}{\partial \boldsymbol{x}} . \boldsymbol{J} + \frac{1}{c}\frac{\partial \rho_{\mathrm{e}}}{\partial t} = 0, \tag{3}$$

which lead to the Gauss equations

$$\frac{\partial}{\partial \boldsymbol{x}} . \boldsymbol{B} = 0, \qquad \frac{\partial}{\partial \boldsymbol{x}} . \boldsymbol{E} = \rho_{\mathrm{e}} . \tag{4a, b}$$

Under QS conditions the dependence of ρ_{e} and the scalar components of these vectors on $\phi - \Omega t$ leads to the operator relation

$$\partial/\partial t = -\Omega(\partial^*/\partial\phi),$$

where the asterisk indicates that the operation is to be applied only to such components. If we are to maintain the field equations in vector form we must first transform this QS relation appropriately. In the Appendix we show that

$$\partial/\partial t = -\Omega(\boldsymbol{k} \times \boldsymbol{x}) . (\partial^*/\partial \boldsymbol{x}), \tag{5}$$

which, when applied to a scalar function ψ, gives equation (A11) and hence the 'divergence' form

$$\partial\psi/\partial t = -(\partial/\partial \boldsymbol{x}) . \{\Omega(\boldsymbol{k} \times \boldsymbol{x})\psi\} . \tag{6}$$

The corresponding form for a vector function $\boldsymbol{X}$ is given in equation (A12), which may be transformed alternatively into the 'curl' form

$$\frac{\partial \boldsymbol{X}}{\partial t} = \frac{\partial}{\partial \boldsymbol{x}} \times \{\Omega(\boldsymbol{k} \times \boldsymbol{x}) \times \boldsymbol{X}\} - \Omega(\boldsymbol{k} \times \boldsymbol{x})\frac{\partial}{\partial \boldsymbol{x}} . \boldsymbol{X}, \tag{7}$$

or the 'gradient' form

$$\frac{\partial \boldsymbol{X}}{\partial t} = -\frac{\partial}{\partial \boldsymbol{x}}\{\Omega(\boldsymbol{k} \times \boldsymbol{x}) . \boldsymbol{X}\} + \Omega(\boldsymbol{k} \times \boldsymbol{x}) \times \left(\frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{X}\right) . \tag{8}$$

Applying the curl form (7) to equations (1) and (2) and the divergence form (6) to equation (3) we obtain, via equations (4), the QS vector forms

$$\frac{\partial}{\partial \boldsymbol{x}} \times \tilde{\boldsymbol{E}} = 0, \qquad \tilde{\boldsymbol{E}} = \boldsymbol{E} + \xi \times \boldsymbol{B}, \tag{9}$$

$$\frac{\partial}{\partial \boldsymbol{x}} \times \tilde{\boldsymbol{B}} = \tilde{\boldsymbol{J}}, \qquad \tilde{\boldsymbol{B}} = \boldsymbol{B} - \xi \times \boldsymbol{E}, \tag{10}$$

$$\frac{\partial}{\partial \boldsymbol{x}} . \tilde{\boldsymbol{J}} = 0, \qquad \tilde{\boldsymbol{J}} = \boldsymbol{J} - \rho_{\mathrm{e}}\xi, \tag{11}$$

where we have introduced the dimensionless toroidal vector

$$\boldsymbol{\xi} = (\Omega/c)\boldsymbol{k} \times \boldsymbol{x}. \tag{12}$$

These are obvious generalizations of the magnetohydrostatic equations.

We note that $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{B}}$ and $\tilde{\boldsymbol{J}}$ are the values of the electric and magnetic vectors and current density that would be measured in a frame rotating with the star as long as $\xi \ll 1$ (i.e. well within the light cylinder). More generally, if we denote toroidal and poloidal vector components by subscripts t and p, then $\tilde{\boldsymbol{E}}_{\mathrm{t}}$, $\tilde{\boldsymbol{E}}_{\mathrm{p}}(1-\xi^2)^{-\frac{1}{2}}$ and $\tilde{\boldsymbol{B}}_{\mathrm{t}}$, $\tilde{\boldsymbol{B}}_{\mathrm{p}}(1-\xi^2)^{-\frac{1}{2}}$ are the components of the field vectors and $\tilde{\boldsymbol{J}}_{\mathrm{t}}(1-\xi^2)^{-\frac{1}{2}}$, $\tilde{\boldsymbol{J}}_{\mathrm{p}}$ those of the current density that would be measured in an inertial frame instantaneously coincident with the rest frame of the star and having the local velocity $\boldsymbol{\xi}c$.

It is evident from equations (10) and (11) that $\tilde{\boldsymbol{B}}$ may be regarded as a 'vector potential' representation of $\tilde{\boldsymbol{J}}$. As shown in the Appendix, it follows that in the cases where there is independence of the axial coordinate z (cylindrical case) or the azimuthal coordinate ϕ (axisymmetric case) the perpendicular or poloidal components of the current may also be represented in terms of stream functions $-\tilde{B}_z$ or $-\varpi B_\phi$, where $\varpi = |\boldsymbol{k} \times \boldsymbol{x}|$ is the distance of $\boldsymbol{x}$ from the axis.

For the cylindrical case, by equations (A8) and (A9) of the Appendix, the component of the Ampère–Maxwell equation (10) perpendicular to the axis gives

$$\tilde{\boldsymbol{J}}_\perp = (\partial \tilde{B}_z/\partial \boldsymbol{x}) \times \boldsymbol{k}, \tag{13}$$

so that $-\tilde{B}_z$ is the corresponding two-dimensional current stream function χ in equation (32) of MWW.

In the axisymmetric case our QS model becomes static with

$$\partial^*/\partial\phi \equiv \partial/\partial t \equiv 0,$$

and equations (1)–(3) take the same forms as equations (9)–(11), and equations (6)–(8) provide certain identities with which these sets of equations can be reconciled. Again, by equations (A8) and (A9) the poloidal component of the current can be written

$$\boldsymbol{J}_{\mathrm{p}} = \frac{1}{\varpi}\frac{\partial(\varpi B_\phi)}{\partial \boldsymbol{x}} \times \boldsymbol{t}, \tag{14}$$

where $\boldsymbol{t}$ is the unit toroidal vector $\boldsymbol{k} \times \boldsymbol{x}/\varpi$. Thus, $-\varpi B_\phi$ is the Stokes stream function for the flow in an axial plane, equal to the function $-S$ defined in equations (2.11) and (2.12) of MPW.

3. QS and Static Potentials

The potentials usually taken to satisfy the Gauss equation (4a) and the Faraday–Neumann equation (1) are $\boldsymbol{A}$ and φ, such that

$$\boldsymbol{B} = \frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{A}, \qquad \boldsymbol{E} = -\frac{\partial \varphi}{\partial \boldsymbol{x}} - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}, \tag{15a, b}$$

which are arbitrary to a scalar function ψ according to well known gauge-transformation formulae. The arbitrariness is usually resolved by imposing the Lorentz condition, which has the effect of decoupling A from φ in the inhomogeneous equations obtained by substitution from (15) into the Ampère–Maxwell equation (2) and the Gauss equation (4b).

Under QS conditions, application of the gradient formula (8) to equation (15b) gives

$$E = -\frac{\partial}{\partial x}\left(\varphi - \xi . A\right) - \xi \times \left(\frac{\partial}{\partial x} \times A\right),$$

which, by equations (9), is equivalent to writing

$$\tilde{E} = -\partial\Phi/\partial x, \qquad \Phi = \varphi - \xi . A. \tag{16a, b}$$

The scalar potential Φ, appropriate to the corotating frame, was recognized by Mestel (1973) and exploited further in MWW and MPW, its constancy being a consequence of the assumption of perfect conductivity.

Although imposition of the Lorentz condition has the effect of decoupling the equations in A and φ, the cylindrical components of A remain mixed in the component equations. However, there is some advantage in physical understanding in retaining the corotating potential Φ, rather than φ, in our equations. For QS conditions we therefore choose to resolve the arbitrariness in our definitions (15) by imposing the gauge condition

$$\xi . A = 0, \tag{17}$$

i.e. by specifying that A is entirely poloidal. In the cylindrical case this has the effect of making the scalar potentials Φ and φ the same in both frames, while at the same time reducing the complexity of the equations in the scalar components of A and Φ, to an extent similar to that following the Lorentz condition.

In the static axisymmetric case A and φ are already independent and it is appropriate instead to impose the usual Coulomb gauge condition

$$(\partial/\partial x) . A = 0. \tag{18}$$

The corotating potential Φ can then be obtained from A and φ by means of equation (16b).

Cylindrical Case

It is appropriate, here, first to express E and $\tilde{B}$ in terms of $\tilde{E}$ and B. The required expression for E is in equations (9) and that for $\tilde{B}$ is found from equations (9) and (10),

$$\tilde{B} = B - \xi^2 B_{\mathrm{p}} - \xi \times \tilde{E}. \tag{19}$$

Introducing the potentials from equations (15) and (16) we have for the components parallel and perpendicular to the k axis

$$\tilde{E}_{\parallel} = 0, \qquad \tilde{E}_{\perp} = -\partial\Phi/\partial x, \tag{20}$$

and, by equations (A8)–(A10),

$$B_{\parallel} = \frac{\partial}{\partial x} \times A_{\perp}, \qquad B_{\perp} = \frac{\partial A_z}{\partial x} \times k. \tag{21}$$

Likewise, for the components of the Ampère–Maxwell equations (10) and Gauss equation (4b), we find

$$\frac{\partial}{\partial \boldsymbol{x}} \times (\boldsymbol{B}_\perp - \xi^2 \boldsymbol{B}_{\mathrm{p}\perp}) = \boldsymbol{J}_\parallel, \tag{22a}$$

$$\frac{\partial}{\partial \boldsymbol{x}} \times \{(1-\xi^2)\boldsymbol{B}_\parallel - \boldsymbol{\xi} \times \tilde{\boldsymbol{E}}\} = \boldsymbol{J}_\perp - \rho_{\mathrm{e}} \boldsymbol{\xi}, \tag{22b}$$

$$\frac{\partial}{\partial \boldsymbol{x}} \cdot (\tilde{\boldsymbol{E}} - \boldsymbol{\xi} \times \boldsymbol{B}_\parallel) = \rho_{\mathrm{e}}. \tag{23}$$

Evidently equation (22a) can be expressed entirely in terms of A_z and its source function J_z. We find

$$\frac{1}{\varpi}\frac{\partial}{\partial \varpi}\left(\varpi \frac{\partial A_z}{\partial \varpi}\right) + \frac{1 - \Omega^2 \varpi^2/c^2}{\varpi^2}\frac{\partial^2 A_z}{\partial \phi^2} = -J_z, \tag{24}$$

the same as the rationalized form of equation (36) of MWW. We denote the corresponding field 'mode I', having cylindrical components

$$\boldsymbol{E}_\parallel = \left(0, 0, \frac{\Omega}{c}\frac{\partial A_z}{\partial \phi}\right), \qquad \boldsymbol{B}_\perp = \left(\frac{1}{\varpi}\frac{\partial A_z}{\partial \phi}, -\frac{\partial A_z}{\partial \varpi}, 0\right). \tag{25}$$

The gauge choice (17) gives $\boldsymbol{A}_\perp = (A_\varpi, 0, 0)$, whence we obtain from equation (22b) two further mixed component equations in A_ϖ and Φ,

$$\left(\frac{1}{\varpi}\frac{\partial}{\partial \phi}, -\frac{\partial}{\partial \varpi}\right)\tilde{B}_z = \left(J_\varpi, J_\phi - \rho_{\mathrm{e}}\frac{\Omega \varpi}{c}\right), \tag{26a}$$

$$\tilde{B}_z = -\frac{1 - \Omega^2 \varpi^2/c^2}{\varpi}\frac{\partial A_\varpi}{\partial \phi} - \frac{\Omega \varpi}{c}\frac{\partial \Phi}{\partial \varpi}, \tag{26b}$$

which evidently satisfy the continuity equations (11). Equation (23) gives a further condition on these potentials,

$$\frac{1}{\varpi}\frac{\partial}{\partial \varpi}\left(\varpi \frac{\partial \Phi}{\partial \varpi} - \frac{\Omega \varpi}{c}\frac{\partial A_\varpi}{\partial \phi}\right) + \frac{1}{\varpi^2}\frac{\partial^2 \Phi}{\partial \phi^2} = -\rho_{\mathrm{e}}. \tag{27}$$

The field derived from A_ϖ and Φ we denote 'mode II', with cylindrical components

$$\boldsymbol{E}_\perp = \left(-\frac{\partial \Phi}{\partial \varpi} + \frac{\Omega}{c}\frac{\partial A_\varpi}{\partial \phi}, -\frac{1}{\varpi}\frac{\partial \Phi}{\partial \phi}, 0\right), \qquad \boldsymbol{B}_\parallel = \left(0, 0, -\frac{1}{\varpi}\frac{\partial A_\varpi}{\partial \phi}\right). \tag{28}$$

Equation (27) can be identified with the rationalized form of (37) of MWW, in which the stream function χ is equal to $-\tilde{B}_z$.

In a vacuum the source distributions vanish and the two modes are identical with those so denoted in MWW.

Axisymmetric Case

Under the static conditions which now apply the potentials A and φ (ϕ in MPW) are quite independent, with φ satisfying the usual Poisson equation

$$\nabla^2\varphi = -\rho_e, \tag{29}$$

and A, with the Coulomb gauge condition (18), satisfying the Poisson equation

$$\nabla^2 A = -\frac{\partial}{\partial \boldsymbol{x}} \times \left(\frac{\partial}{\partial \boldsymbol{x}} \times A\right) = -\boldsymbol{J}. \tag{30}$$

Proceeding as for the cylindrical case we have

$$\boldsymbol{E}_t = 0, \qquad \boldsymbol{E}_p = -\partial\varphi/\partial\boldsymbol{x}, \tag{31}$$

and, by equations (A8)–(A10),

$$\boldsymbol{B}_t = \frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{A}_p, \qquad \boldsymbol{B}_p = \frac{1}{\varpi}\frac{\partial(\varpi A_\phi)}{\partial \boldsymbol{x}} \times \boldsymbol{t}. \tag{32}$$

We note that $-\varpi A_\phi$ is equal to the Stokes stream function P defined in equation (2.1) of MPW, whence equations (16) and (29) are together equivalent to their equation (2.5); MPW utilizes no equivalent of $\boldsymbol{A}_p$.

The toroidal component of equation (30) can now be expressed entirely in terms of A_ϕ and its source function J_ϕ:

$$\frac{\partial}{\partial\varpi}\left(\frac{1}{\varpi}\frac{\partial(\varpi A_\phi)}{\partial\varpi}\right) + \frac{\partial^2 A_\phi}{\partial z^2} = -J_\phi, \tag{33}$$

the same as the rationalized form of equation (2.4) of MPW. We denote the corresponding field 'mode I', having cylindrical components

$$\boldsymbol{E}_t = (0,0,0), \qquad \boldsymbol{B}_p = \frac{1}{\varpi}\left(-\frac{\partial(\varpi A_\phi)}{\partial z}, 0, \frac{\partial(\varpi A_\phi)}{\partial\varpi}\right). \tag{34}$$

The poloidal component of equation (30) gives the standard unmixed component equations in A_ϖ and A_z and their respective source functions J_ϖ and J_z, while equation (29) does the same for φ and its source function ρ_e. The corresponding field we denote 'mode II', with cylindrical components

$$\boldsymbol{E}_p = \left(-\frac{\partial\varphi}{\partial\varpi}, 0, -\frac{\partial\varphi}{\partial z}\right), \qquad \boldsymbol{B}_t = \left(0, \frac{\partial A_\varpi}{\partial z} - \frac{\partial A_z}{\partial\varpi}, 0\right). \tag{35}$$

Evidently the electric field $\tilde{\boldsymbol{E}}$ associated with the potential Φ in the corotating frame is a mixture of these modes I and II.

Here we observe that the formal decomposition of the electromagnetic field into modes I and II under both cylindrical and axisymmetric conditions represents two uncoupled modes only in the vacuum regime. This is because the source functions ρ_e and $\boldsymbol{J}$ are not externally prescribed, but are subject to magnetohydrodynamical equations of motion under forces which are themselves dependent on the field vectors $\boldsymbol{E}$ and $\boldsymbol{B}$.

4. Plasma Equations of Motion

In classical plasma theory it is conventional and appropriate to neglect the inertial terms in the magnetohydrodynamical equations of motion for the several plasma species. Such a force-free perfectly conducting model of the pulsar magnetosphere, however, encounters difficulties in the neighbourhood of the light cylinder, where the plasma can no longer be expected to remain in corotation with the magnetic field. In particular, as the analysis in MWW shows, with corotation there can be no flow of electromagnetic energy across the light cylinder.

The effects of inertia can be readily incorporated into equations having the same form as the corresponding force-free equations. Burman and Mestel (1978) have shown that for a 'cold' plasma consisting of several species k of particles of masses m_k, bearing charges e_k, and having local mean velocities $\boldsymbol{v}_k$, the equation of motion for the species k leads to the form

$$\boldsymbol{u}_k \times \left(\frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{P}_k\right) = e_k \frac{\partial \Psi_k}{\partial \boldsymbol{x}}, \tag{36}$$

where

$$\boldsymbol{u}_k = \boldsymbol{v}_k - c\boldsymbol{\xi} \tag{37}$$

is the velocity in the corotating frame,

$$\boldsymbol{P}_k = \gamma_k m_k \boldsymbol{v}_k + e_k \boldsymbol{A}/c \tag{38}$$

is the generalized particle momentum,

$$\Psi_k = \Phi + \frac{\gamma_k m_k c^2}{e_k}\left(1 - \frac{\boldsymbol{\xi}\cdot\boldsymbol{v}_k}{c}\right), \tag{39}$$

and γ_k is the usual Lorentz factor $(1 - v_k^2/c^2)^{-\frac{1}{2}}$. From equation (36) we have the Bernoulli integral, namely Ψ_k is constant on the lines of $\boldsymbol{u}_k$; it is also constant on the lines of $(\partial/\partial\boldsymbol{x}) \times \boldsymbol{P}_k$ (Endean 1972*a*, 1972*b*; Burman and Mestel 1978, 1979). The quantity $e_k \Psi_k$ can be recognized as the combination of the energy and angular momentum integrals Γmc^2 defined in equation (2.17) of MPW.

In the corotating frame we have the partial current and charge densities

$$\tilde{\boldsymbol{J}}_k = \rho_{ek} \boldsymbol{u}_k/c, \qquad \rho_{ek} = n_k e_k, \tag{40}$$

where n_k is the number density of the species, subject to the QS equation of continuity

$$(\partial/\partial\boldsymbol{x}) \cdot \tilde{\boldsymbol{J}}_k = 0. \tag{41}$$

Thus, by analogy with the relations (10) we may introduce the 'vector potential' $\tilde{\boldsymbol{B}}_k$ via the relation

$$\tilde{\boldsymbol{J}}_k = (\partial/\partial\boldsymbol{x}) \times \tilde{\boldsymbol{B}}_k, \tag{42}$$

so that, since $\tilde{\boldsymbol{J}} = \Sigma_k \tilde{\boldsymbol{J}}_k$, $\tilde{\boldsymbol{B}}$ will differ from $\Sigma_k \tilde{\boldsymbol{B}}_k$ by, at most, the gradient of some arbitrary scalar function. Again, by writing equation (38) in the form

$$A_k^* = \boldsymbol{P}_k c/e_k = \boldsymbol{A} + (\gamma_k m_k c/e_k)\boldsymbol{v}_k\,, \tag{43}$$

equations (36) and (40) give

$$\tilde{\boldsymbol{J}}_k \times \{(\partial/\partial \boldsymbol{x}) \times \boldsymbol{A}_k^*\} = n_k e_k \partial \Psi_k/\partial \boldsymbol{x}\,. \tag{44}$$

For negligible inertia $\boldsymbol{A}_k^* = \boldsymbol{A}$ and $\Psi_k = \Phi + m_k c^2/e_k$, so that for a charge-separated plasma

$$\tilde{\boldsymbol{J}} \times \{(\partial/\partial \boldsymbol{x}) \times \boldsymbol{A}\} = \rho_e \partial \Phi/\partial \boldsymbol{x}\,, \tag{45}$$

which can readily be identified with the force-free equation (10) of MWW. Moreover, under conditions of perfect conductivity, $\Phi = \text{const.}$, so that for an individual particle and for the charge-separated plasma we have relations equivalent to equations (8) and (12) of MWW,

$$\boldsymbol{u} = \kappa \boldsymbol{B}\,, \qquad \tilde{\boldsymbol{J}} = \psi \boldsymbol{B}\,, \tag{46}$$

where κ and ψ are scalar functions such that ψ is subject to the equations of Section 3; in particular, ψ is constant on the lines of $\boldsymbol{B}$.

Burman and Mestel (1978, 1979) introduced the assumption that near the pulsar surface the particle motions are nonrelativistic. Equation (39) then gives for particles of species k

$$\Psi_k = \Phi_0 + m_k c^2/e_k\,, \tag{47}$$

where $\Phi = \Phi_0$ within and on the pulsar surface. They observed that this constant value Ψ_k will be propagated out along the lines of $\boldsymbol{u}_k$ or $\tilde{\boldsymbol{J}}_k$ (which are now identical with those of $(\partial/\partial \boldsymbol{x}) \times \boldsymbol{P}_k$ or $(\partial/\partial \boldsymbol{x}) \times \boldsymbol{A}_k^*$) throughout the region occupied by that species in contact with the pulsar surface. Corresponding to the perfectly conducting force-free relations (46), we now have

$$\boldsymbol{u}_k = \kappa_k \boldsymbol{B}_k^*\,, \qquad \tilde{\boldsymbol{J}}_k = \psi_k \boldsymbol{B}_k^*\,, \tag{48}$$

where

$$\boldsymbol{B}_k^* = \frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{A}_k^* = \boldsymbol{B} + \frac{m_k c}{e_k}\frac{\partial}{\partial \boldsymbol{x}} \times (\gamma_k \boldsymbol{v}_k)\,. \tag{49}$$

By virtue of equations (41), (48) and (49) we have

$$\boldsymbol{B}_k^* \cdot \partial \psi_k/\partial \boldsymbol{x} = 0\,, \tag{50}$$

so that ψ_k is constant on the lines of $\boldsymbol{B}_k^*$ and $\tilde{\boldsymbol{J}}_k$. As pointed out by Burman and Mestel, the motion along these lines adds an 'inertial drift' to the streaming of particles along the lines of $\boldsymbol{B}$. This will be significant where γ_k and its gradient become very large, presumably in the neighbourhood of the light cylinder.

In this paper we go on to show that, in the cylindrical and axisymmetric cases, with Ψ_k given by the constant value (47), we can obtain integrals of the equations of motion (44) which more closely specify the scalar function ψ_k in equation (48). Thus, we have

$$\tilde{\boldsymbol{J}}_k \times \boldsymbol{B}_k^* = 0\,, \tag{51}$$

in which both $\tilde{\boldsymbol{J}}_k$ and $\boldsymbol{B}_k^*$ are solenoidal, according to equations (42) and (49).

Cylindrical Case

First we consider the component of equation (51) parallel to $\boldsymbol{k}$. From equations (A8) and (A9) we find

$$\tilde{\boldsymbol{J}}_{k\perp} \times \boldsymbol{B}^*_{k\perp} = \frac{\partial \tilde{B}_{kz}}{\partial \boldsymbol{x}} \times \frac{\partial A^*_{kz}}{\partial \boldsymbol{x}},$$

whence we obtain the integral

$$\tilde{B}_{kz} = F_k(A^*_{kz}), \tag{52}$$

where F_k is a differentiable function subject to the equations of Section 3. We note that, since $-\tilde{B}_{kz}$ is the stream function for the species k and $A^*_{kz} = P_{kz}c/e_k$, this is the inverse form of equations (48) and (49) of MWW and equation (10) of Burman and Mestel (1979).

For the perpendicular component we find

$$\begin{aligned}\tilde{\boldsymbol{J}}_{k\parallel} \times \boldsymbol{B}^*_{k\perp} + \tilde{\boldsymbol{J}}_{k\perp} \times \boldsymbol{B}^*_{k\parallel} &= \boldsymbol{k}.\left\{\left(\frac{\partial}{\partial \boldsymbol{x}} \times \tilde{\boldsymbol{B}}_k\right)\frac{\partial A^*_{kz}}{\partial \boldsymbol{x}} - \left(\frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{A}^*_k\right)\frac{\partial \tilde{B}_{kz}}{\partial \boldsymbol{x}}\right\} \\ &= \boldsymbol{k}.\left(\frac{\partial}{\partial \boldsymbol{x}} \times \tilde{\boldsymbol{B}}_k - F'_k(A^*_{kz})\frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{A}^*_k\right)\frac{\partial A^*_{kz}}{\partial \boldsymbol{x}},\end{aligned}$$

by equation (52). Since, in general, $\partial A^*_{kz}/\partial \boldsymbol{x} \neq 0$ we must have

$$\left(\frac{\partial}{\partial \boldsymbol{x}} \times \tilde{\boldsymbol{B}}_k\right)_{\parallel} = F'_k(A^*_{kz})\left(\frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{A}^*_k\right)_{\parallel};$$

a similar result for the perpendicular components follows from equation (52). We therefore have

$$\tilde{\boldsymbol{J}}_k = F'_k(A^*_{kz})\,\boldsymbol{B}^*_k, \tag{53}$$

so that here ψ_k in equation (48) has the form $F'_k(A^*_{kz})$ and the lines of $\tilde{\boldsymbol{J}}_k$ and $\boldsymbol{B}^*_k$ lie on the surfaces A^*_{kz} = const., which are also surfaces of constant axial momentum P_{kz}.

The results (52) and (53) are hitherto unobtained generalizations of the force-free results given in equations (18) and (19) of MWW. As was to be hoped, with the inclusion of inertial terms the flow of electromagnetic energy across the light cylinder $\xi = 1$ is no longer proscribed. With the force-free assumption it will be recalled that $A^*_{kz} = A_z$ which is constant on the light cylinder, and that Φ is constant throughout the region occupied by the plasma species; consequently, $\boldsymbol{B}$ is parallel to $\boldsymbol{k}$ (i.e. tangential to the light cylinder) while $\boldsymbol{E}$ is normal, leading to a tangential Poynting vector.

Axisymmetric Case

Proceeding as for the cylindrical case we readily obtain results analogous to equations (52) and (53), with the toroidal and poloidal components taking the place of the parallel and perpendicular components of equation (51) and the corresponding vector quantities. We find, corresponding to equation (52),

$$\varpi\tilde{B}_{k\phi} = G_k(\varpi A^*_{k\phi}), \tag{54}$$

where G_k is a differentiable function again subject to the equations of Section 3. Since $-\varpi\tilde{B}_{k\phi}$ is the Stokes stream function for the species k, equal to $-\varpi B_\phi$ in a charge-separated plasma, and $\varpi A^*_{k\phi} = \varpi P_{k\phi}\, c/e_k$, this is the inverse form of the generalized angular momentum integral (2.16) of MPW.

Again, corresponding to equation (53), we find

$$\tilde{\boldsymbol{J}}_k = G_k'(\varpi A^*_{k\phi})\,\boldsymbol{B}^*_k\,, \tag{55}$$

so that here ψ_k has the form $G_k'(\varpi A^*_{k\phi})$ and the lines of $\tilde{\boldsymbol{J}}_k$ and $\boldsymbol{B}^*_k$ lie on the surfaces $\varpi A^*_{k\phi} =$ const., which are also surfaces of constant angular momentum $\varpi P_{k\phi}$. This result can be identified with equation (2.27) of MPW in the particular case $\tilde{\alpha}(\tilde{P}) = \alpha$.

5. Conclusions

Formally, the solution of the problem of the pulsar magnetosphere in the cylindrical and axisymmetric cases can now proceed by combining the equations for the electromagnetic potentials $\boldsymbol{A}$, Φ and φ obtained in Section 3 with the integrals obtained in Section 4 for the partial currents $\tilde{\boldsymbol{J}}_k = \rho_{ek}(\boldsymbol{v}_k/c - \boldsymbol{\xi})$ in terms of $\boldsymbol{A}$ and the local mean velocities $\boldsymbol{v}_k$ of each species.

We leave the development of this approach to a future paper. Apart from obtaining a new integral for the cylindrical case with particle inertia, the purpose of the present paper has been achieved with the systematic presentation of a theoretical framework on which further attacks can be made on the problem of the structure of the pulsar magnetosphere, from the surface of the star out into interstellar space.

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Appendix. Vector Analysis in Cylindrical Coordinate Systems

The base vectors appropriate to a cylindrical system of coordinates ϖ, ϕ, z where

$$\boldsymbol{x} = \varpi\boldsymbol{e}_\varpi + z\boldsymbol{e}_z\,, \tag{A1}$$

are $\boldsymbol{e}_z = \boldsymbol{k}$ along the axis of the system, $\boldsymbol{e}_\phi = \boldsymbol{t}$ along the toroidal direction of $\boldsymbol{k}\times\boldsymbol{x} = \varpi\boldsymbol{t}$, and $\boldsymbol{e}_\varpi = \boldsymbol{t}\times\boldsymbol{k}$. The gradient operator, applicable to any scalar function ψ, is then

$$\frac{\partial}{\partial\boldsymbol{x}} = \boldsymbol{e}_\varpi\frac{\partial}{\partial\varpi} + \frac{\boldsymbol{e}_\phi}{\varpi}\frac{\partial}{\partial\phi} + \boldsymbol{e}_z\frac{\partial}{\partial z}\,. \tag{A2}$$

When applied to the base vectors, it gives the dyads

$$\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{e}_\varpi, \boldsymbol{e}_\phi, \boldsymbol{e}_z) = \frac{\boldsymbol{t}}{\varpi}\boldsymbol{k} \times (\boldsymbol{e}_\varpi, \boldsymbol{e}_\phi, \boldsymbol{e}_z), \tag{A3}$$

and hence, for a vector function

$$\boldsymbol{X} = X_\varpi \boldsymbol{e}_\varpi + X_\phi \boldsymbol{e}_\phi + X_z \boldsymbol{e}_z, \tag{A4}$$

$$\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{X} = \frac{\partial^*}{\partial \boldsymbol{x}}\boldsymbol{X} + \frac{\boldsymbol{t}}{\varpi}\boldsymbol{k} \times \boldsymbol{X}, \tag{A5}$$

where, in terms of the gradients of the scalar components X_ϖ, X_ϕ, X_z,

$$\frac{\partial^*}{\partial \boldsymbol{x}}\boldsymbol{X} = \frac{\partial X_\varpi}{\partial \boldsymbol{x}}\boldsymbol{e}_\varpi + \frac{\partial X_\phi}{\partial \boldsymbol{x}}\boldsymbol{e}_\phi + \frac{\partial X_z}{\partial \boldsymbol{x}}\boldsymbol{e}_z. \tag{A6}$$

The usual formulae for the divergence and curl of the vector $\boldsymbol{X}$ are formally obtained by inserting a $\cdot$ and a $\times$ between the elements of the dyads in equations (A5) and (A6). Thus, for the curl we have

$$\frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{X} = \frac{\partial X_\varpi}{\partial \boldsymbol{x}} \times \boldsymbol{e}_\varpi + \frac{\partial X_\phi}{\partial \boldsymbol{x}} \times \boldsymbol{e}_\phi + \frac{\partial X_z}{\partial \boldsymbol{x}} \times \boldsymbol{e}_z + \frac{X_\phi}{\varpi}\boldsymbol{e}_z, \tag{A7}$$

leading to the standard form when the operator (A2) is applied to the components.

The form (A7) leads to two particular formulae which are useful in the cylindrical and axisymmetric cases. For the axial and toroidal components $\boldsymbol{X}_\parallel = X_z \boldsymbol{k}$ and $\boldsymbol{X}_\mathrm{t} = X_\phi \boldsymbol{t}$ we find

$$\frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{X}_\parallel = \frac{\partial X_z}{\partial \boldsymbol{x}} \times \boldsymbol{k}, \qquad \frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{X}_\mathrm{t} = \frac{1}{\varpi}\frac{\partial(\varpi X_\phi)}{\partial \boldsymbol{x}} \times \boldsymbol{t}. \tag{A8}$$

These results, which are generally valid, establish $-X_z$ as a stream function in the cylindrical case of z independence and $-\varpi X_\phi$ as a Stokes stream function in the case of ϕ independence. For then it can readily be seen that

$$\left(\frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{X}\right)_\perp = \frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{X}_\parallel, \qquad \left(\frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{X}\right)_\mathrm{p} = \frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{X}_\mathrm{t}, \tag{A9}$$

whence, also,

$$\left(\frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{X}\right)_\parallel = \frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{X}_\perp, \qquad \left(\frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{X}\right)_\mathrm{t} = \frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{X}_\mathrm{p}. \tag{A10}$$

QS Condition

Here the scalar components of the field vectors and potentials depend on the azimuth ϕ and time t through the quantity $\phi - \Omega t$. Then, for such scalars, we have

$$\partial\psi/\partial t = -\Omega\,\partial\psi/\partial\phi = -\Omega\varpi\,\boldsymbol{e}_\phi \cdot \partial\psi/\partial \boldsymbol{x}.$$

We can write this in the coordinate-independent form

$$\partial\psi/\partial t = -\Omega(\boldsymbol{k} \times \boldsymbol{x}) . \partial\psi/\partial\boldsymbol{x}, \tag{A11}$$

whence, for a vector $\boldsymbol{X}$ with such components,

$$\partial\boldsymbol{X}/\partial t = -\Omega(\boldsymbol{k} \times \boldsymbol{x}) . (\partial^*/\partial\boldsymbol{x})\boldsymbol{X},$$

or, by equation (A5),

$$\partial\boldsymbol{X}/\partial t = -\Omega(\boldsymbol{k} \times \boldsymbol{x}) . (\partial/\partial\boldsymbol{x})\boldsymbol{X} + \Omega\,\boldsymbol{k} \times \boldsymbol{X}. \tag{A12}$$

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