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Energy Loss by Energetic Test Ions in a Plasma

T. Hamada

Department of Theoretical Physics, University of New South Wales, P.O. Box 1, Kensington, N.S.W. 2033; on leave from the Department of Physics, Ibaraki University, Mito 310, Japan.

Abstract

A study is made of the energy loss by an energetic (but nonrelativistic) test ion of charge Ze to plasma electrons in thermal equilibrium at temperature T. The classical region $\kappa T \ll Z^2$ Ry and the quantum region $\kappa T \gg Z^2$ Ry are both considered. The loss rates are given in closed form in terms of known functions of the ion speed. May's earlier work concerning the classical region is revised. Certain limiting cases are discussed in connection with the Bohr–Bethe–Bloch theory of the stopping power of ordinary matter.

1. Introduction

There arise three characteristic lengths in the discussion of collisions of a test ion of charge Ze with plasma electrons in thermal equilibrium at temperature T:

$$b_{\mathrm{D}} = k_{\mathrm{D}}^{-1} = \left(\frac{\kappa T}{m\omega_{\mathrm{p}}^2}\right)^{\frac{1}{2}}, \qquad b_0 = k_0^{-1} = \frac{Ze^2}{\kappa T}, \qquad b_{\mathrm{Q}} = k_{\mathrm{Q}}^{-1} = \left(\frac{\hbar^2}{m\kappa T}\right)^{\frac{1}{2}}. \qquad (1)$$

Here m is the electron mass and $\omega_{\mathrm{p}} = (4\pi ne^2/m)^{\frac{1}{2}}$, with n the electron number density, is the electron plasma frequency. In most cases of interest the Debye length b_{D} greatly exceeds b_0. The length b_{Q} is essentially the thermal de Broglie wavelength of a plasma electron characterizing quantum diffraction effects. If we have $b_0 \gg b_{\mathrm{Q}}$, that is, $\kappa T \ll Z^2$ Ry (1 Ry $= e^4m/2\hbar^2 = 13\cdot605$ eV), such diffraction effects are negligible, and we call this region of plasma parameters the classical region. The lengths b_{D} and b_0 then characterize classical maximum and minimum collision impact parameters respectively, leading to the ubiquitous Coulomb logarithm $\ln(b_{\mathrm{D}}/b_0) = \ln(k_0/k_{\mathrm{D}})$. If on the contrary we have $b_{\mathrm{Q}} \gg b_0$, that is, $\kappa T \gg Z^2$ Ry, quantum diffraction effects dominate close collisions and we are in the quantum region. The Coulomb logarithm in this region should be $\ln(b_{\mathrm{D}}/b_{\mathrm{Q}}) = \ln(k_{\mathrm{Q}}/k_{\mathrm{D}})$ (Kihara 1964; Frankel 1965; DeWitt 1966). The quantum region covers a considerable range of plasma parameters of practical importance; e.g. for an α particle with $\kappa T > 100$ eV.

It is the purpose of the present work to study the energy loss by energetic test ions in both the classical and quantum regions. By energetic we mean that the kinetic energy of the ion is much higher than the plasma thermal energy, that is, $MV^2 \gg \kappa T$,

M and V being respectively the mass and speed of the ion. This does not necessarily mean, however, that the parameter

$$x = (mV^2/2\kappa T)^{\frac{1}{2}} \tag{2}$$

is much larger than unity, since $M \gg m$. The energy loss rate will be given here in a closed form in terms of known functions of x. Such an analysis in the classical region was presented by May (1969). As we shall see, however, it appears that his analysis does not give correct nondominant terms in the loss rate. In the quantum region, the calculations available so far either are for a limited range of the parameter x (Larkin 1960; Honda 1964) or are overly simplified (Frankel 1965).

The classical and quantum calculations are carried out in Sections 2 and 3 respectively. Section 4 is devoted to a discussion of the results and, in particular, the connections with the Bohr–Bethe–Bloch theory (Bloch 1933*a*, 1933*b*) of the stopping power of ordinary matter are pointed out.

2. Classical Region ($\kappa T \ll Z^2$ Ry)

There exist various equivalent plasma kinetic equations that are free from divergences in the classical region (Aono 1968*a*). For our purpose the formulation by Kihara and Aono (1963) finds most direct application. It combines the Boltzmann equation for the Coulomb potential (impact theory; denoted by subscript I throughout) with the dielectric formulation (wave theory; subscript W) and gives the average energy loss rate in the form

$$(\mathrm{d}E/\mathrm{d}t)_{\mathrm{C}} = (\mathrm{d}E/\mathrm{d}t)_{\mathrm{W}} + (\mathrm{d}E/\mathrm{d}t)_{\mathrm{I}}. \tag{3}$$

Let us assume that the parameter x defined by equation (2) satisfies $x > (m/M)^{\frac{1}{3}}$ so that the test ion loses its energy predominantly to plasma electrons (Butler and Buckingham 1962). Let us further omit the terms of order m/M, taking them to be negligible compared with unity. The two terms in equation (3) are then given by

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{W}} = \frac{2(Ze^2)^2}{m} \iiint \mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega\, \frac{\delta(\omega - \boldsymbol{k}\,.\,\boldsymbol{V})\,\delta(\boldsymbol{k}\,.\,\boldsymbol{g})}{|\,k^2\varepsilon_{\mathrm{C}}(k,\omega)\,|^2} \times \left(\boldsymbol{k}\,.\,\boldsymbol{V}\right)\left(\boldsymbol{k}\,.\,\frac{\partial f(\boldsymbol{v})}{\partial \boldsymbol{v}}\right) \exp\{-\tfrac{1}{2}(kb_1)^2\}, \tag{4}$$

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{I}} = -\frac{4\pi(Ze^2)^2}{m} \int \mathrm{d}\boldsymbol{v}\, \frac{(\boldsymbol{g}\,.\,\boldsymbol{V})f(\boldsymbol{v})}{g^3} \int_0^\infty \mathrm{d}b\, \frac{b\exp\{-\frac{1}{2}(b/b_1)^2\}}{b^2 + (Ze^2/mg^2)^2}. \tag{5}$$

Here b_1 is an intermediate length such that

$$b_{\mathrm{D}} \gg b_1 \gg b_0 \gg b_{\mathrm{Q}}, \tag{6}$$

$f(\boldsymbol{v})$ is the electron velocity distribution function given by

$$f(\boldsymbol{v}) = n(m/2\pi\kappa T)^{3/2} \exp(-mv^2/2\kappa T), \tag{7}$$

with n the electron number density as before, and $\boldsymbol{g} = \boldsymbol{V} - \boldsymbol{v}$ is the relative velocity.

The classical dielectric constant $\varepsilon_C(k,\omega)$ which characterizes the wave theory is given by

$$\varepsilon_C(k,\omega) = 1+\frac{4\pi e^2}{mk^2}\int \mathrm{d}\boldsymbol{u}\,\frac{\boldsymbol{k}.\{\partial f(\boldsymbol{u})/\partial \boldsymbol{u}\}}{\omega-(\boldsymbol{k}.\boldsymbol{u})+\mathrm{i}\delta}. \tag{8}$$

Let us first evaluate $(\mathrm{d}E/\mathrm{d}t)_\mathrm{W}$. The integration over $\boldsymbol{v}$ in equation (4) can be easily carried out and we arrive at

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_\mathrm{W} = -\frac{4\pi n(Ze^2)^2}{mV}\frac{4}{\sqrt{\pi}}\int_0^x \mathrm{d}s\, s^2\exp(-s^2) \times \int_0^\infty \mathrm{d}k\,\frac{k^3\exp\{-\frac{1}{2}(kb_1)^2\}}{(k^2+k_\mathrm{D}^2 X)^2+(k_\mathrm{D}^2 Y)^2}, \tag{9}$$

where we have changed the variable to $s = (\omega/kV)x$ and (Kihara and Aono 1963)

$$X(s) = 1-2s\exp(-s^2)\int_0^s \mathrm{d}t\,\exp(t^2), \qquad Y(s) = \sqrt{\pi}\,s\exp(-s^2). \tag{10}$$

The integral over k in equation (9) yields (Abramowitz and Stegun 1972, formulae 5.1.11, 5.1.43 and 5.1.44)

$$\ln(b_\mathrm{D}/b_1)+\tfrac{1}{2}\ln 2-\tfrac{1}{2}\gamma-\tfrac{1}{4}\ln(X^2+Y^2) -\tfrac{1}{2}(X/Y)\arctan(Y/X)+O\{(b_1/b_\mathrm{D})^2\ln(b_1/b_\mathrm{D})\},$$

where $\gamma = 0\cdot57721...$ is Euler's constant.

Following May (1969) we now define the two functions

$$\Psi(x) = 4\pi^{-\frac{1}{2}}\int_0^x \mathrm{d}s\,s^2\exp(-s^2) = \operatorname{erf}(x)-2\pi^{-\frac{1}{2}}x\exp(-x^2), \tag{11}$$

$$\Delta_1(x) = -\{\pi\Psi(x)\}^{-1}\int_0^x \mathrm{d}s\,s\{Y\ln(X^2+Y^2)+2X\arctan(Y/X)\}. \tag{12}$$

The inverse tangent is to be evaluated in the range 0–π. The function $\Delta_1(x)$ has been tabulated by May (1969). In terms of these functions, equation (9) takes the form

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_\mathrm{W} = -\frac{4\pi n(Ze^2)^2}{mV}\Psi(x)\left\{\ln\left(\frac{b_\mathrm{D}}{b_1}\right)+\Delta_1(x)+\tfrac{1}{2}\ln 2-\tfrac{1}{2}\gamma\right\}. \tag{13}$$

The error involved is at most of the order $(b_1/b_\mathrm{D})^2\ln(b_1/b_\mathrm{D})$, which is much smaller than the terms retained as long as the conditions (6) hold.

We next turn to the evaluation of $(\mathrm{d}E/\mathrm{d}t)_\mathrm{I}$. Carrying out the integration over b in equation (5), we find (Abramowitz and Stegun 1972, formula 5.1.28)

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_\mathrm{I} = -\frac{4\pi(Ze^2)^2}{m}\int \mathrm{d}\boldsymbol{v}\,\frac{(\boldsymbol{g}.\boldsymbol{V})f(\boldsymbol{v})}{g^3}\left\{\ln\left(\frac{mg^2b_1}{Ze^2}\right)+\tfrac{1}{2}\ln 2-\tfrac{1}{2}\gamma\right\}. \tag{14}$$

Here the error involved is at most of the order $(b_0/b_1)^2\ln(b_0/b_1)$. Using the function

$\Delta_2(x)$ defined and tabulated by May (1969), namely (with some small rearrangement of terms)

$$\Delta_2(x) = \frac{4}{\sqrt{\pi}\,\Psi(x)}\left[\int_0^x \mathrm{d}s\, s^2 \exp(-s^2)\ln(x^2-s^2)\right.$$
$$\left. + \int_x^\infty \mathrm{d}s\, s^2 \exp(-s^2)\left\{\ln\left(\frac{s+x}{s-x}\right) - \frac{2x}{s}\right\}\right] - \ln 2 - 1\,, \qquad (15)$$

we obtain

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{I}} = -\frac{4\pi n(Ze^2)^2}{mV}\Psi(x)\left\{\ln\left(\frac{4b_1}{b_0}\right) + \Delta_2(x) + 1 + \tfrac{1}{2}\ln 2 - \tfrac{1}{2}\gamma\right\}. \qquad (16)$$

According to equation (3), the average energy loss rate in the classical region from equations (13) and (16) is now given by

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{C}} = -\frac{4\pi n(Ze^2)^2}{mV}\Psi(x)\left\{\ln\left(\frac{4k_0}{k_{\mathrm{D}}}\right) + \Delta_1(x) + \Delta_2(x) + 1 + \ln 2 - \gamma\right\}. \qquad (17)$$

If we make the choice $b_1 \sim (b_0 b_{\mathrm{D}})^{\frac{1}{2}}$ then the error involved in this result is at most of relative order $(b_0/b_{\mathrm{D}})\ln(b_0/b_{\mathrm{D}})$. The rate (17) differs from the one obtained by May (1969) by $1 + \ln 2 - \gamma = 1\cdot 116$ in the factor within the braces. There are reasons to believe that the kinetic equations used by him do not give correct nondominant terms; it is known that a certain divergence-free kinetic equation does give a wrong result (Gould and DeWitt 1967). Further discussion on this point is given in Section 4.

3. Quantum Region ($\kappa T \gg Z^2$ Ry)

Let $\overline{W}(\boldsymbol{p}\to\boldsymbol{p}')$ be the probability per unit time of a test ion of momentum $\boldsymbol{p}$ being scattered to $\boldsymbol{p}'$ by plasma electrons. The average rate of energy change of the ion will then be given by

$$\mathrm{d}E/\mathrm{d}t = \sum_{\boldsymbol{p}'} (E_{p'} - E_p)\, \overline{W}(\boldsymbol{p}\to\boldsymbol{p}')\,, \qquad (18)$$

where $E_p = p^2/2M$. It is a simple matter to show that equation (18) follows from the Boltzmann equation for the ion momentum distribution function if the latter is proportional to $\delta_{p'p(t)}$, which is the case here since we are assuming $E_p \gg \kappa T$ (May 1964).

Turning now to the transition rate $W(\boldsymbol{p}\to\boldsymbol{p}')$, we first note that the condition $b_{\mathrm{Q}} \gg b_0$ (that is, $\kappa T \gg Z^2$ Ry) assures the validity of the Born approximation. We then have (e.g. Wyld and Pines 1962)

$$\overline{W}(\boldsymbol{p}\to\boldsymbol{p}+\hbar\boldsymbol{k}) = \sum_{\boldsymbol{q}} w(\boldsymbol{p},\boldsymbol{q}\to\boldsymbol{p}+\hbar\boldsymbol{k},\boldsymbol{q}-\hbar\boldsymbol{k})\,N(\boldsymbol{q})\,, \qquad (19)$$

where $N(\boldsymbol{q})$ is the number of electrons of momentum $\boldsymbol{q}$ and

$$w(\boldsymbol{p},\boldsymbol{q}\to\boldsymbol{p}+\hbar\boldsymbol{k},\boldsymbol{q}-\hbar\boldsymbol{k}) = \frac{2\pi}{\hbar}\left|\frac{1}{\Omega}\left(-\frac{4\pi Ze^2}{k^2}\right)\frac{1}{\varepsilon_{\mathrm{Q}}\big(k,\hbar^{-1}(E_{p+\hbar k}-E_p)\big)}\right|^2$$
$$\times\,\delta(E_{p+\hbar k}+\varepsilon_{q-\hbar k}-E_p-\varepsilon_q)\,, \qquad (20)$$

with

$$\varepsilon_{\mathrm{Q}}(k,\omega) = 1 + \frac{4\pi e^2}{k^2}\frac{1}{\Omega}\sum_{k'}\frac{N(\hbar\boldsymbol{k}+\hbar\boldsymbol{k}') - N(\hbar\boldsymbol{k}')}{\hbar\omega - (\varepsilon_{\hbar k+\hbar k'} - \varepsilon_{\hbar k'}) + \mathrm{i}\delta}\,, \qquad (21)$$

$\varepsilon_q = q^2/2m$ and Ω the normalization volume. As in Section 2 we have assumed that the test ion is losing its energy predominantly to plasma electrons. In the continuum limit we have

$$\Omega^{-1}\sum_{k} \to (2\pi)^{-3}\int \mathrm{d}\boldsymbol{k}, \qquad \Omega^{-1}\sum_{q} \ldots N(q) \to \int \mathrm{d}\boldsymbol{v} \ldots f(\boldsymbol{v}). \tag{22}$$

Introducing equations (19) and (20) into (18) and using the results (22), we obtain the average energy loss rate in the quantum region as

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{Q}} = \frac{4(Ze^2)^2}{\hbar}\int \mathrm{d}\boldsymbol{v}\, f(\boldsymbol{v}) \int \mathrm{d}\boldsymbol{k}\, \frac{\boldsymbol{k}\,.\,\boldsymbol{V}}{|\,k^2\,\varepsilon_{\mathrm{Q}}(k,\omega)\,|^2}\,\delta(\boldsymbol{k}\,.\,\boldsymbol{g} + \hbar k^2/2m), \tag{23}$$

where $\hbar\omega = E_{p+\hbar k} - E_p$, and m/M has been taken to be negligible compared with unity, as in Section 2. The rate (23) is free from divergences as it stands, because of the presence of $\varepsilon_{\mathrm{Q}}(k,\omega)$ for small k and of the δ function for large k.

In order to evaluate the integral over $\boldsymbol{k}$ in equation (23), let us introduce an intermediate wave number k_1 such that

$$k_0 \gg k_{\mathrm{Q}} \gg k_1 \gg k_{\mathrm{D}}, \tag{24}$$

and divide the integration into the two regions (i) $k_1 > k > 0$ and (ii) $\infty > k > k_1$. It will be seen that the final result is independent of k_1 to a very good approximation as long as the conditions (24) are satisfied.

In region (i) the δ function in equation (23) may be approximated by

$$\delta(\boldsymbol{k}\,.\,\boldsymbol{g} + \hbar k^2/2m) = \delta(\boldsymbol{k}\,.\,\boldsymbol{g}) + (\hbar \boldsymbol{k}/2m)\,.\,(\partial/\partial \boldsymbol{g})\delta(\boldsymbol{k}\,.\,\boldsymbol{g}), \tag{25}$$

and the dielectric constant (21) may be replaced by its classical counterpart (8). The integrations in equation (23) can then be carried out in much the same way as in Section 2 and we arrive at

$$\left(\frac{\mathrm{d}E(\mathrm{i})}{\mathrm{d}t}\right)_{\mathrm{Q}} = -\frac{4\pi n(Ze^2)^2}{mV}\,\Psi(x)\left\{\ln\left(\frac{k_1}{k_{\mathrm{D}}}\right) + \Delta_1(x)\right\}, \tag{26}$$

where $\Psi(x)$ and $\Delta_1(x)$ are given by equations (11) and (12) respectively. We note that there is no place for quantum effects to enter in region (i).

In region (ii) the expansion (25) cannot be used but $\varepsilon_{\mathrm{Q}}(k,\omega)$ may be replaced by unity. Evaluation of the $\boldsymbol{k}$-integration in equation (23) yields

$$\left(\frac{\mathrm{d}E(\mathrm{ii})}{\mathrm{d}t}\right)_{\mathrm{Q}} = -\frac{4\pi(Ze^2)^2}{m}\int \mathrm{d}\boldsymbol{v}\, f(\boldsymbol{v})\,\frac{\boldsymbol{V}\,.\,\boldsymbol{g}}{g^3}\,\ln\left(\frac{2mg}{\hbar k_1}\right). \tag{27}$$

To be rigorous, the range of integration over $\boldsymbol{v}$ is limited by $g = |\,\boldsymbol{V} - \boldsymbol{v}\,| > \hbar k_1/2m$. It is easy to see, however, that the error caused by ignoring this restriction is at most of order $(k_1/k_{\mathrm{Q}})^2 \ln(k_1/k_{\mathrm{Q}})$. Carrying out the integration over $\boldsymbol{v}$ we find

$$\left(\frac{\mathrm{d}E(\mathrm{ii})}{\mathrm{d}t}\right)_{\mathrm{Q}} = -\frac{4\pi n(Ze^2)^2}{mV}\,\Psi(x)\left\{\ln\left(\frac{4k_{\mathrm{Q}}}{k_1}\right) + \tfrac{1}{2}\Delta_2(x) + \tfrac{1}{2}\right\}, \tag{28}$$

where $\Delta_2(x)$ is defined by equation (15).

Adding equations (26) and (28) we arrive at the desired loss rate in the quantum region:

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{Q}} = -\frac{4\pi n(Ze^2)^2}{mV}\Psi(x)\left\{\ln\left(\frac{4k_{\mathrm{Q}}}{k_{\mathrm{D}}}\right) + \Delta_1(x) + \tfrac{1}{2}\Delta_2(x) + \tfrac{1}{2}\right\}. \tag{29}$$

If we choose $k_1 \sim (k_{\mathrm{D}} k_{\mathrm{Q}})^{\frac{1}{2}}$ then the error involved in this result is at most of the relative order $(k_{\mathrm{D}}/k_{\mathrm{Q}})\ln(k_{\mathrm{D}}/k_{\mathrm{Q}})$. (We note that $k_{\mathrm{Q}}/k_{\mathrm{D}} = \kappa T/\hbar\omega_{\mathrm{p}}$.) The dominant Coulomb logarithm is not $\ln(k_0/k_{\mathrm{D}})$ but $\ln(k_{\mathrm{Q}}/k_{\mathrm{D}})$ when we have $\kappa T \gg Z^2$ Ry. Unless $Z \gg 1$ the use of the classical rate (17) substantially overestimates the loss rate if $\kappa T \gtrsim 1$ keV.

4. Discussion

Equation (17) for $\kappa T \ll Z^2$ Ry and equation (29) for $\kappa T \gg Z^2$ Ry summarize the results of the present work. They are valid when (*a*) the kinetic energy of the test ion is much higher than the plasma thermal energy, that is, $E \gg \kappa T$, but $(V/c)^2 \ll 1$, and (*b*) $x \gtrsim (m/M)^{\frac{1}{3}}$ so that the energy is lost predominantly to plasma electrons.

Let us first consider the limiting case $x \gg 1$. Using the asymptotic forms of $\Delta_1(x)$ and $\Delta_2(x)$ given by May (1969), we find

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{C},x>>1} = -\frac{4\pi n(Ze^2)^2}{mV}\left\{\ln\left(\frac{2mV^3}{Ze^2\omega_{\mathrm{p}}}\right) - \gamma\right\}, \tag{30}$$

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{Q},x>>1} = -\frac{4\pi n(Ze^2)^2}{mV}\ln\left(\frac{2mV^2}{\hbar\omega_{\mathrm{p}}}\right). \tag{31}$$

Equation (30) is in complete agreement with the results of Jackson (1962) and Kihara and Aono (1963). This limiting form has a very clear physical interpretation, as shown by Aono (1968*b*). It is also interesting to see that the rate (30) can be obtained by simply replacing the harmonic atomic frequency ω by ω_{p} in the classical formula for the stopping power of ordinary matter due originally to Bohr (1913) (see Bloch 1933*a*, 1933*b*). May's (1969) formula does not have this limiting form (Gould 1972); in fact this is the reason why it was considered that his theory needed revision in the first place.

The quantum loss rate as given by equation (31) is in complete agreement with the result first obtained by Larkin (1960). We observe that the energy lost to completely degenerate electrons by ions with $V \gg v_{\mathrm{F}}$ (the Fermi speed) is also given by equation (31) (Kramers 1947; Ritchie 1959). Apparently, sufficiently fast ions do not distinguish between Maxwell–Boltzmann and Fermi–Dirac statistics. It is also satisfying to see that the rate (31) can be obtained from the Bethe–Bloch formula (Bloch 1933*a*, 1933*b*) for the quantum stopping power of ordinary matter by replacing the average atomic binding energy by $\hbar\omega_{\mathrm{p}}$. These observations do not justify the modification of the Bethe–Bloch formula considered by Bagge and Hora (1974).

Consider next the case $x < 1$. Neglecting terms of order x^2 we have

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{C},x<1} = -\frac{4n(Ze^2)^2(2\pi m)^{\frac{1}{2}}V^2}{3(\kappa T)^{3/2}}\left\{\ln\left(\frac{4(\kappa T)^{3/2}}{Ze^2 m^{\frac{1}{2}}\omega_{\mathrm{p}}}\right) - 2\gamma - \tfrac{1}{2}\right\}, \tag{32}$$

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{Q},x<1} = -\frac{4n(Ze^2)^2(2\pi m)^{\frac{1}{2}}V^2}{3(\kappa T)^{3/2}}\left\{\ln\left(\frac{2^{3/2}\kappa T}{\hbar\omega_{\mathrm{p}}}\right)-\tfrac{1}{2}(1+\gamma)\right\}. \tag{33}$$

The limiting forms (32) and (33) are in complete agreement with the results of Kihara and Aono (1963) and Honda (1964) respectively, for $E \gg \kappa T$.

We conclude with a remark on the comparison of the classical energy loss rate (17) with experiment. Halverson's (1968) preliminary report on the measurement of the energy loss rate of 5 keV protons in a lithium plasma with $n \sim 4\times10^{12}\ \mathrm{cm}^{-3}$ and $\kappa T \sim 1\cdot5$ eV indicated that the necessary correction to the dominant Coulomb logarithm $\ln(4k_0/k_{\mathrm{D}})$ in the theory amounted to -40%. May's (1969) formula gave a correction of -14%, and he thus pointed out that although his result diminished the difference there was still a significant discrepancy between the theory and experiment. This problem has led to a recent study by Swami and Sharma (1977) on the effect of possible turbulence within the plasma which could appreciably reduce the loss rate. The present revision of May's formula represented by equation (17) yields a correction of only -3% to the simple theory, so that it might appear that the discrepancy is even more serious. However, a full account of the same experiment by Caby-Eyraud (1970) shows that $\ln(4k_0/k_{\mathrm{D}})$ alone gives a loss rate well within the experimental uncertainty of 40%. Our equation (17) is therefore not at variance with experiment. There exist no experimental data in the quantum region where equation (29) is applicable.

Notes added in proof

If $x < 1$ then equation (17) applies when $\kappa T \ll Z^2$ Ry and equation (29) when $\kappa T \gg Z^2$ Ry, as described in the text. If $x > 1$, however, equation (17) is valid when $\kappa T \ll (Z^2/x^2)$ Ry and equation (29) when $\kappa T \gg (Z^2/x^2)$ Ry. These points are discussed in a forthcoming paper by George and Hamada (1978).

For more recent comparisons of the theory and experiment, the reader is referred to the paper by Burke and Post (1974).

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