Aust. J. Phys., 1976, **29**, 363–74

A Collective Rotational State of Spherical Nuclei

J. R. Henderson and J. Lekner

Department of Physics, Victoria University of Wellington, Wellington, New Zealand.

Abstract

We consider a particular many-body rotational excitation Ψ of a spherical self-bound system of particles, of the form studied by Lekner (1974). This angular momentum eigenstate is translationally invariant and thus is not a spurious state. The energy of Ψ is found from first principles to be substantially larger than that of the first 2^+ excited states of even–even nuclei, with the exception of ^{208}Pb. The quadrupole moment is negative, the g-factor is approximately Z/A and the lifetime is shorter than the single-particle (Weisskopf) value by a factor of the order of A/Z^2. It is suggested that these states are the finite system rotational analogues of Feynman's phonons and rotons.

1. Introduction

Let $\Phi(r_1, ..., r_A)$ be a translationally and rotationally invariant ground or vibrational state of a self-bound system of A particles, with $H\Phi = E_0 \Phi$. It has been shown (Lekner 1974; hereinafter referred to as Paper I) that $\Psi = F\Phi$, where

$$F = \tfrac{1}{2} \sum_{i=1}^{A} \sum_{j=1}^{A} (x_{ij} + \mathrm{i}y_{ij})^L f(r_{ij}) \qquad (L \text{ even}) \tag{1}$$

is an eigenstate of $\boldsymbol{L}^2$ and L_z with eigenvalues $L(L+1)\hbar^2$ and $L\hbar$. The wavefunctions Ψ have the same permutation and inversion symmetry as Φ, and they are translationally invariant. These properties hold irrespective of the masses or permutation symmetries of the constituent particles, but it is clear that, since equation (1) treats each pair of particles in the same way, Ψ is best suited to describe rotational states of systems composed of particles with nearly identical masses and pair interactions, e.g. nuclei and helium microdroplets. We note in passing that Karl and Obryk (1968) and Castilho Alcarás and Leal Ferreira (1971) have found only one symmetric $L = 2$ eigenstate for the three-body system, namely the state Ψ with $f = 1$. The translational invariance of the Ψ considered here guarantees that it is not a spurious state (Elliott and Skyrme 1955; Lipkin 1958; Avilés 1968), i.e. we can be sure that Ψ describes a genuine internal motion of the self-bound system.

In Paper I it was further shown that for harmonic pairwise interactions between A identical Bose particles, that is,

$$H = -(\hbar^2/2m) \sum_{i=1}^{A} \nabla_i^2 + V(\boldsymbol{r}_1, ..., \boldsymbol{r}_A) \tag{2a}$$

with

$$V(\boldsymbol{r}_1, ..., \boldsymbol{r}_A) = \sum_{i<j}\sum v(1 + r_{ij}^2/a^2), \tag{2b}$$

the state Ψ with $f = 1$ is an exact energy eigenstate, with

$$E_L = E_0 + L(2\hbar^2 vA/ma^2)^{\frac{1}{2}} \tag{3}$$

(this equation corrects an error of a factor of $\sqrt{2}$ in Paper I (eqn 30), arising from the same erroneous factor in I(28)). The purpose of the present paper is to carry the analysis of Paper I further, by evaluating the expectation values of the energy, quadrupole moment, magnetic dipole moment and lifetime of the state with $L = 2$ and $f = 1$. These are then compared with experimental values for nuclei with spherical ground states.

2. Expectation Value of Energy

We will assume here and in the remainder of this paper that Φ^2 has complete permutation symmetry, e.g. we neglect the differences between proton and neutron masses and interactions. We also assume that the spins are paired up to give zero $\boldsymbol{S}$, because $\boldsymbol{L} = 0$ as well as $\boldsymbol{J} = 0$ in the ground state Φ. Thus we are discussing the nondeformed even–even nuclei. These assumptions also imply that, for example, $\langle x_{12}^2\rangle = \langle y_{12}^2\rangle = \langle z_{12}^2\rangle$ where the expectation value is defined in equation (5) below.

Since Ψ is an eigenstate of angular momentum with $L = 2$, it is orthogonal to Φ and thus the expectation value of the Hamiltonian in the state $\Psi = F\Phi$ gives a variational bound

$$\begin{aligned} E - E_0 &\leqslant \int \mathrm{d}1 \ldots \mathrm{d}A\, \Psi^*(H-E_0)\Psi \Big/ \int \mathrm{d}1 \ldots \mathrm{d}A\, |\Psi|^2 \\ &= \frac{A\hbar^2}{2m} \int \mathrm{d}1 \ldots \mathrm{d}A\, |\nabla_1 F|^2 \Phi^2 \Big/ \int \mathrm{d}1 \ldots \mathrm{d}A\, |F|^2 \Phi^2 \\ &= \frac{A\hbar^2}{2m} \frac{\langle |\nabla_1 F|^2\rangle}{\langle |F|^2\rangle}, \end{aligned} \tag{4}$$

where

$$\langle B\rangle = \int \mathrm{d}1 \ldots \mathrm{d}A\, B\Phi^2 \Big/ \int \mathrm{d}1 \ldots \mathrm{d}A\, \Phi^2 \tag{5}$$

denotes a ground-state expectation value. The second step in obtaining the expression (4) comes from Paper I(25), and is valid for any Hamiltonian of the form (2a) with an interaction $V(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_A)$ which is completely symmetric and independent of spins and momenta.

We take $f = 1$ in the wavefunction (1), since this gives exact energy eigenstates for harmonic interactions, and also because this is mathematically the simplest and most tractable. A further reason for taking $f = 1$ is given in Section 6. We find

$$\langle |\nabla_1 F|^2\rangle = 4A(A-1)\langle x_{12}^2 + y_{12}^2\rangle = \tfrac{8}{3} A(A-1)\langle r_{12}^2\rangle \tag{6}$$

and

$$\begin{aligned} \langle |F|^2\rangle = {}& \tfrac{1}{2}A(A-1)\langle (x_{12}^2+y_{12}^2)^2\rangle \\ & + A(A-1)(A-2)\langle (x_{12}^2-y_{12}^2)(x_{13}^2-y_{13}^2) + 4x_{12}y_{12}x_{13}y_{13}\rangle \\ & + \tfrac{1}{4}A(A-1)(A-2)(A-3)\langle (x_{12}^2-y_{12}^2)(x_{34}^2-y_{34}^2) + 4x_{12}y_{12}x_{34}y_{34}\rangle \end{aligned} \tag{7}$$

(this equation corrects two counting errors in Paper I(39), namely a factor 2 in the three-body term and a factor $\frac{1}{2}$ in the four-body term).

In the remainder of this section, the expression for $\langle |F|^2\rangle$ and hence also that for E_2-E_0 is reduced to expectation values involving only r_{ij} terms. We first note that the energy cannot depend on the azimuthal quantum number. Thus

$$\Psi_{22} = \tfrac{1}{2}\sum_i\sum_j (x_{ij}+\mathrm{i}y_{ij})^2\Phi, \qquad \Psi_{20} = \tfrac{1}{2}\sum_i\sum_j (3z_{ij}^2-r_{ij}^2)\Phi \tag{8,9}$$

must have the same value of $\langle|\nabla_1 F|^2\rangle/\langle|F|^2\rangle$. For Ψ_{20} we find

$$\langle|\nabla_1 F|^2\rangle = 4A(A-1)\langle r_{12}^2\rangle \tag{10}$$

and

$$\begin{aligned}\langle|F|^2\rangle = {}& \tfrac{1}{2}A(A-1)\langle(3z_{12}^2-r_{12}^2)^2\rangle + A(A-1)(A-2)\langle(3z_{12}^2-r_{12}^2)(3z_{13}^2-r_{13}^2)\rangle \\ & +\tfrac{1}{4}A(A-1)(A-2)(A-3)\langle(3z_{12}^2-r_{12}^2)(3z_{34}^2-r_{34}^2)\rangle .\end{aligned} \tag{11}$$

Thus equations (6), (7), (10) and (11) give the equality

$$\begin{aligned}&\langle(3z_{12}^2-r_{12}^2)^2+2(A-2)(3z_{12}^2-r_{12}^2)(3z_{13}^2-r_{13}^2)+\tfrac{1}{2}(A-2)(A-3)(3z_{12}^2-r_{12}^2)(3z_{34}^2-r_{34}^2)\rangle \\ &\quad = \tfrac{3}{2}\langle(x_{12}^2+y_{12}^2)^2+2(A-2)\{(x_{12}^2-y_{12}^2)(x_{13}^2-y_{13}^2)+4x_{12}y_{12}x_{13}y_{13}\} \\ &\qquad +\tfrac{1}{2}(A-2)(A-3)\{(x_{12}^2-y_{12}^2)(x_{34}^2-y_{34}^2)+4x_{12}y_{12}x_{34}y_{34}\}\rangle .\end{aligned} \tag{12}$$

We will now prove the equality of the two- and three-body terms in equation (12), and thus show the equality of the four-body terms.

In the two-body terms, use of the facts that

$$\langle z_{12}^2 r_{12}^2\rangle = \tfrac{1}{3}\langle(x_{12}^2+y_{12}^2+z_{12}^2)r_{12}^2\rangle = \tfrac{1}{3}\langle r_{12}^4\rangle \tag{13}$$

and

$$\langle z_{12}^4\rangle = \tfrac{1}{5}\langle r_{12}^4\rangle \tag{14}$$

(obtained by angular integration) demonstrates equality. The value of the two-body term is

$$\langle(3z_{12}^2-r_{12}^2)^2\rangle = \tfrac{4}{5}\langle r_{12}^4\rangle . \tag{15}$$

In the three-body term, we use in addition the identity

$$2z_{12}z_{13} = z_{12}^2+z_{13}^2-z_{23}^2 \tag{16}$$

to show that

$$\langle z_{12}^2 z_{13}^2\rangle = \tfrac{1}{10}\langle r_{12}^4\rangle . \tag{17}$$

Thus

$$\langle(3z_{12}^2-r_{12}^2)(3z_{13}^2-r_{13}^2)\rangle = \tfrac{9}{10}\langle r_{12}^4\rangle-\langle r_{12}^2 r_{13}^2\rangle . \tag{18}$$

In the same way the three-body expectation value on the right-hand side of equation (12) is given by (omitting the factor $2(A-2)$)

$$\tfrac{3}{2}\langle 2x_{12}^2x_{13}^2+3x_{12}^2y_{12}^2-4x_{12}^2y_{13}^2\rangle . \tag{19}$$

The first term we know from equation (17). The second term we find from $\langle r_{12}^4\rangle = \langle(x_{12}^2+y_{12}^2+z_{12}^2)^2\rangle$ and equation (14) to be

$$\langle x_{12}^2 y_{12}^2\rangle = \tfrac{1}{15}\langle r_{12}^4\rangle . \tag{20}$$

The last term we obtain by expanding $\langle r_{12}^2 r_{13}^2\rangle$:

$$\langle x_{12}^2 y_{13}^2\rangle = \tfrac{1}{6}\langle r_{12}^2 r_{13}^2\rangle - \tfrac{1}{20}\langle r_{12}^4\rangle . \tag{21}$$

These identities reduce the expression (19) to the right-hand side of equation (18), so that we have demonstrated the equality of the three-body terms. We may thus put

$$\langle(3z_{12}^2-r_{12}^2)(3z_{34}^2-r_{34}^2)\rangle = \tfrac{3}{2}\langle(x_{12}^2-y_{12}^2)(x_{34}^2-y_{34}^2) + 4x_{12}\,y_{12}\,x_{34}\,y_{34}\rangle . \tag{22}$$

By use of the identity

$$2x_{12}\,x_{34} = x_{14}^2+x_{23}^2-x_{13}^2-x_{24}^2 , \tag{23}$$

the right-hand side of equation (22) reduces to

$$3\langle x_{12}^2 x_{34}^2 + x_{12}^2 y_{34}^2 + 2x_{14}^2 y_{14}^2 - 4x_{14}^2 y_{13}^2\rangle . \tag{24}$$

The last two terms of this expression we know from equations (20) and (21). We can find a relationship between the first two by expanding $r_{12}^2 r_{34}^2$:

$$\langle r_{12}^2 r_{34}^2\rangle = \langle(x_{12}^2+y_{12}^2+z_{12}^2)(x_{34}^2+y_{34}^2+z_{34}^2)\rangle = \langle 3x_{12}^2 x_{34}^2 + 6x_{12}^2 y_{34}^2\rangle . \tag{25}$$

Now the left-hand side of equation (22) is equal to

$$\langle 9z_{12}^2 z_{34}^2 - 6z_{12}^2 r_{34}^2 + r_{12}^2 r_{34}^2\rangle = \langle 9z_{12}^2 z_{34}^2 - r_{12}^2 r_{34}^2\rangle . \tag{26}$$

We thus have, equating (26) to the expression (24),

$$\langle 9x_{12}^2 x_{34}^2 - r_{12}^2 r_{34}^2\rangle = 3\langle x_{12}^2 x_{34}^2 + x_{12}^2 y_{34}^2 + \tfrac{1}{3}r_{12}^4 - \tfrac{2}{3}r_{12}^2 r_{13}^2\rangle . \tag{27}$$

We can now evaluate the x and y terms using equations (25) and (27):

$$\langle x_{12}^2 x_{34}^2\rangle = \tfrac{1}{15}\langle 2r_{12}^4 - 4r_{12}^2 r_{13}^2 + 3r_{12}^2 r_{34}^2\rangle , \tag{28}$$

$$\langle x_{12}^2 y_{34}^2\rangle = \tfrac{1}{15}\langle -r_{12}^4 + 2r_{12}^2 r_{13}^2 + r_{12}^2 r_{34}^2\rangle . \tag{29}$$

The four-body term in equation (12) is thus

$$\langle(3z_{12}^2-r_{12}^2)(3z_{34}^2-r_{34}^2)\rangle = \tfrac{2}{5}\langle 3r_{12}^4 - 6r_{12}^2 r_{13}^2 + 2r_{12}^2 r_{34}^2\rangle , \tag{30}$$

and the variational bound for the excitation energy of the state Ψ is

$$\frac{\hbar^2}{m}\,\frac{2A\langle r_{12}^2\rangle}{\langle \frac{2}{5}r_{12}^4 + (A-2)(\frac{9}{10}r_{12}^4 - r_{12}^2 r_{13}^2) + \frac{1}{4}(A-2)(A-3)\frac{2}{5}(3r_{12}^4 - 6r_{12}^2 r_{13}^2 + 2r_{12}^2 r_{34}^2)\rangle} . \tag{31}$$

No approximations have been made to this stage; the result (31) is a rigorous expectation value of the energy in the state Ψ. To evaluate this exactly, however, we would need to know the two-, three- and four-particle correlation functions of the system. In the next section we evaluate (31) in the simplest physically meaningful approximation, namely that in which the particles are correlated simply by coexisting in a finite system.

3. Weak Correlation Approximation

As a first approximation we assume that the A particles are correlated by virtue of the finiteness of the self-bound system, i.e. we assume the system to be characterized solely by a number density $n(r)$ (measured relative to the centre of mass of the system). The use of this 'weak correlation' approximation is supported to some extent by the fact that in nuclei the hard core of the nucleons occupies only about 1% of the total volume (Irvine 1972; de Shalit and Feshbach 1974) so that pair correlations due to nucleon–nucleon interactions can be expected not to be dominant in the evaluation of the expectation values in the result (31). It turns out that the weak correlation approximation is sufficient to make the three-body term

$$\langle (3z_{12}^2 - r_{12}^2)(3z_{13}^2 - r_{13}^2)\rangle$$

nonzero (and in fact positive–definite in this approximation), whereas in a completely uncorrelated (infinite) system it would be zero.

When the system is characterized solely by a single-particle density $n(r)$, depending only on the radial distance from the centre of mass, the expectation values needed for the evaluation of the energy bound (31) can be found by working in spherical bipolar coordinates (Hill 1956). We have

$$\langle r_{12}^2\rangle = \frac{\int_0^\infty \mathrm{d}r_1\, r_1\, n(r_1) \int_0^\infty \mathrm{d}r_2\, r_2\, n(r_2) \int_{|r_1-r_2|}^{r_1+r_2} \mathrm{d}r_{12}\, r_{12}^3}{\int_0^\infty \mathrm{d}r_1\, r_1\, n(r_1) \int_0^\infty \mathrm{d}r_2\, r_2\, n(r_2) \int_{|r_1-r_2|}^{r_1+r_2} \mathrm{d}r_{12}\, r_{12}}$$

$$= \frac{\int_0^\infty \mathrm{d}r_1\, r_1^2\, n(r_1) \int_0^\infty \mathrm{d}r_2\, r_2^2\, n(r_2)\, (r_1^2+r_2^2)}{\left(\int_0^\infty \mathrm{d}r\, r^2\, n(r)\right)^2} = 2[\![r^2]\!]\,, \tag{32}$$

where

$$[\![f(r)]\!] \equiv \int_0^\infty \mathrm{d}r\, r^2\, n(r)\, f(r) \Big/ \int_0^\infty \mathrm{d}r\, r^2\, n(r)\,. \tag{33}$$

Similarly

$$\langle r_{12}^4\rangle = 2[\![r^4]\!] + \tfrac{10}{3}[\![r^2]\!]^2\,. \tag{34}$$

The three-particle correlations are a little more complicated: $\langle r_{12}^2\, r_{13}^2\rangle$ is given by

$$\frac{\int_0^\infty \mathrm{d}r_1\, n(r_1) \int_0^\infty \mathrm{d}r_2\, r_2\, n(r_2) \int_{|r_1-r_2|}^{r_1+r_2} \mathrm{d}r_{12}\, r_{12}^3 \int_0^\infty \mathrm{d}r_3\, r_3\, n(r_3) \int_{|r_1-r_3|}^{r_1+r_3} \mathrm{d}r_{13}\, r_{13}^3}{\int_0^\infty \mathrm{d}r_1\, n(r_1) \int_0^\infty \mathrm{d}r_2\, r_2\, n(r_2) \int_{|r_1-r_2|}^{r_1+r_2} \mathrm{d}r_{12}\, r_{12} \int_0^\infty \mathrm{d}r_3\, r_3\, n(r_3) \int_{|r_1-r_3|}^{r_1+r_3} \mathrm{d}r_{13}\, r_{13}}$$

$$= \frac{\int_0^\infty \mathrm{d}r_1\, r_1^2\, n(r_1) \int_0^\infty \mathrm{d}r_2\, r_2^2\, n(r_2) \int_0^\infty \mathrm{d}r_3\, r_3^2\, n(r_3)\, (r_1^2+r_2^2)(r_1^2+r_3^2)}{\left(\int_0^\infty \mathrm{d}r\, r^2\, n(r)\right)^3}$$

$$= [\![r^4]\!] + 3[\![r^2]\!]^2\,. \tag{35}$$

Thus the three-body term (18) is positive–definite:

$$\langle \tfrac{9}{10} r_{12}^4 - r_{12}^2 r_{13}^2 \rangle = \tfrac{4}{5} [\![r^4]\!] . \tag{36}$$

In the weak correlation approximation, the four-particle term $\langle r_{12}^2 r_{34}^2 \rangle$ factorizes as

$$\langle r_{12}^2 r_{34}^2 \rangle = \langle r_{12}^2 \rangle^2 = 4 [\![r^2]\!]^2 . \tag{37}$$

It follows that the total four-body term in the expectation value of the energy is zero:

$$\langle 3r_{12}^4 - 6r_{12}^2 r_{13}^2 + 2r_{12}^2 r_{34}^2 \rangle = 6[\![r^4]\!] + 10[\![r^2]\!]^2 - 6[\![r^4]\!] - 18[\![r^2]\!]^2 + 8[\![r^2]\!]^2 = 0 . \tag{38}$$

These results are true for arbitrary radial variation of the density.

In the weak correlation approximation, the energy of the $L = 2$ state thus reduces to

$$\Delta E_2 \equiv E_2 - E_0 \leqslant \frac{\hbar^2}{m} \frac{A[\![r^2]\!]}{\frac{1}{3}[\![r^2]\!]^2 + (A-1)\frac{1}{5}[\![r^4]\!]} . \tag{39}$$

When $A \gtrsim 20$, we can write

$$\Delta E_2 \lesssim (5\hbar^2/m)[\![r^2]\!]/[\![r^4]\!] . \tag{40}$$

If we further assume that the system has a uniform density up to a sharp cutoff at radius R, we have

$$[\![r^2]\!] = \tfrac{3}{5} R^2 , \qquad [\![r^4]\!] = \tfrac{3}{7} R^4 , \tag{41}$$

so that

$$\Delta E_2 \lesssim 7\hbar^2/mR^2 . \tag{42}$$

In the next section we see how these results may be obtained much more simply by breaking the translational invariance of the wavefunction.

4. Wavefunctions with Broken Translational Invariance

In Paper I it was pointed out that (i) the orbital angular momentum of a system of particles is independent of the choice of origin if and only if the system has zero total momentum (i.e. its wavefunction is translationally invariant) and (ii) because of the uncertainty principle it is impossible in quantum mechanics to fix the centre of mass of a system (at the origin, for example) when the system has zero total momentum. Thus the only rigorous way to treat the problem of rotational excitations of a self-bound system which is not fixed in space by external forces is to deal exclusively with translationally invariant wavefunctions. We have done so here (up to this point) thus ensuring that the wavefunction considered does correspond to an actual internal excitation, and not to a spurious state.

Having set up the excitation in a translationally invariant way, however, we are free to break the translational invariance of that wavefunction without risk of spurious states; that is, we are sure from its origins that the translationally *variant* wavefunction represents an internal excitation. We find, in a simple calculation, that breaking the translational invariance leads to an error in the energy of order A^{-1}. We are also able to readily obtain the quadrupole moment and the lifetime.

Consider Ψ_{22} given by equation (8). Since

$$\tfrac{1}{2}\sum_{i=1}^{A}\sum_{j=1}^{A} x_{ij}^2 = A\sum_{i=1}^{A} x_i^2 - A^2X^2 \tag{43}$$

and

$$\sum_{i=1}^{A}\sum_{j=1}^{A} x_{ij}y_{ij} = 2A\sum_{i=1}^{A} x_i y_i - 2A^2XY, \tag{44}$$

where

$$X = A^{-1}\sum_{i=1}^{A} x_i \quad \text{etc.,} \qquad \boldsymbol{R} = (X, Y, Z),$$

we have

$$\tfrac{1}{2}\sum_i\sum_j (x_{ij}+\mathrm{i}y_{ij})^2 = A\sum_j (x_j+\mathrm{i}y_j)^2 - A^2(X+\mathrm{i}Y)^2. \tag{45}$$

Similarly

$$\tfrac{1}{2}\sum_i\sum_j (3z_{ij}^2 - r_{ij}^2) = A\sum_j (3z_j^2 - r_j^2) - A^2(3Z^2 - R^2). \tag{46}$$

Thus when we break the translational invariance by fixing the centre of mass at the origin of the coordinate system $X=0$, $Y=0$, $Z=0$, the wavefunctions Ψ_{22} and Ψ_{20} (equations 8, 9) become

$$\Psi'_{22} = \sum_j (x_j+\mathrm{i}y_j)^2\Phi \equiv F_{22}\Phi, \qquad \Psi'_{20} = \sum_j (3z_j^2 - r_j^2)\Phi \equiv F_{20}\Phi \tag{47, 48}$$

(we have dropped the factor A for simplicity). These states are angular momentum eigenstates as before, with energy ΔE_2 given by (4) above. For the Ψ'_{20} state we have

$$\langle(\nabla_1 F)^2\rangle = 8\langle r_1^2\rangle \tag{49}$$

and

$$\langle F^2\rangle = A\langle(3z_1^2 - r_1^2)^2\rangle + A(A-1)\langle(3z_1^2 - r_1^2)(3z_2^2 - r_2^2)\rangle. \tag{50}$$

In the weak correlation approximation (and for spherical Φ) the second term is zero, so we have

$$\langle F^2\rangle = A\tfrac{4}{5}\langle r_1^4\rangle. \tag{51}$$

Thus the weak correlation approximation gives

$$\Delta E_2 \leqslant (5\hbar^2/m)\langle r_1^2\rangle/\langle r_1^4\rangle. \tag{52}$$

Since these expectation values are taken with the centre of mass fixed at the origin, we have $\langle r^n\rangle = [\![r^n]\!]$ as defined by equation (33), so that the result (52) is the same as (40), and differs from (39) by terms of order A^{-1}. We have thus shown that, in the weak correlation approximation, breaking translational invariance leads to an error of order A^{-1} only, as could be expected.

It is also easy to calculate the quadrupole moment Q in the same approximation. We have

$$\begin{aligned} Q &= e\int \mathrm{d}1 \ldots \mathrm{d}A \sum_{j=1}^{Z} (3z_j^2 - r_j^2)\,|\Psi'_{22}|^2 \Big/ \int \mathrm{d}1 \ldots \mathrm{d}A\,|\Psi'_{22}|^2 \\ &= Ze\langle(3z_1^2 - r_1^2)(x_1^2+y_1^2)^2\rangle / A\langle(x_1^2+y_1^2)^2\rangle \\ &= -\tfrac{4}{7}ZeA^{-1}\langle r_1^6\rangle/\langle r_1^4\rangle, \end{aligned} \tag{53}$$

so that the excited state is oblate (pancake-shaped), with a small negative deformation parameter, proportional to A^{-1}. For uniform density up to a cutoff radius R, equation (53) becomes

$$Q = -\tfrac{4}{9} ZeA^{-1}R^2. \tag{54}$$

We can also estimate the magnetic dipole moment on the assumption that the spin contribution (for even–even nuclei) is negligible. The orbital contribution is, irrespective of translational invariance,

$$\boldsymbol{\mu} = (e/2mc) \sum_{j=1}^{Z} \langle \boldsymbol{l}_j \rangle \approx (e/2mc)ZA^{-1}\langle \boldsymbol{L} \rangle,$$

so that

$$\mu = (e\hbar/mc)ZA^{-1}, \qquad g = ZA^{-1}$$

in the Ψ_{22} state.

The lifetime of the state is readily obtained from the transition rate (Blatt and Weisskopf 1952, p. 595)

$$T_E(L, M) = \frac{8\pi(L+1)}{L\{(2L+1)!!\}^2} \frac{k^{2L+1}}{\hbar} |\langle f \,|\, Q_{LM} \,|\, i \rangle|^2.$$

We have

$$Q_{20} = \tfrac{1}{4}(5/\pi)^{\frac{1}{2}} e \sum_{i=1}^{Z} (3z_i^2 - r_i^2),$$

with

$$|\,i\rangle = N_i^{-1} \sum_j (3z_j^2 - r_j^2)\Phi, \qquad |\,f\rangle = N_f^{-1}\Phi,$$

where N_i and N_f are normalization factors. In the weak correlation approximation we find

$$T_E(2, 0) = (\tfrac{1}{75} Z^2 e^2 k^5/A\hbar)\langle r_1^4 \rangle. \tag{55}$$

From equations (55) and (41), the linewidth Γ of the 2^+ state is

$$\Gamma \approx 1{\cdot}2(Z^2/A)\Gamma_{\mathrm{W}}, \tag{56}$$

where Γ_{W} is the Weisskopf width (de Shalit and Feshbach 1974, p. 702). Thus the width is large relative to the single-particle value, as befits a collective state.

5. Effect of Interactions on Correlations

We saw in Section 3 that the four-particle term given by equation (30) is zero in the weak correlation approximation, i.e. when the particles are correlated solely because of the finiteness of the self-bound system. The validity of this 'zeroth' approximation needs to be examined in more detail, since the four-particle term in the exact expression (31) for the energy is multiplied by the factor $\frac{1}{4}(A-3)$ relative to the three-particle term.

Let the pair interactions be characterized by a range a. If one particle is placed randomly in the system (of radius R) the probability of placing a second particle in the range of interaction with the first is of order $(a/R)^3$. This leads us to expect that, for example,

$$\langle r_{12}^n \rangle - \langle r_{12}^n \rangle_0 \sim \langle r_{12}^n \rangle_0 (a/R)^3, \tag{57}$$

where $\langle r_{12}^n \rangle_0$ is the expectation value calculated in the weak correlation approximation. A precise formulation in terms of the pair correlation function $g(r)$ gives (c.f. equation 32)

$$\langle r_{12}^n \rangle = \frac{\int_0^\infty dr_1\, r_1\, n(r_1) \int_0^\infty dr_2\, r_2\, n(r_2) \int_{|r_1-r_2|}^{r_1+r_2} dr_{12}\, r_{12}\, g(r_{12})\, r_{12}^n}{\int_0^\infty dr_1\, r_1\, n(r_1) \int_0^\infty dr_2\, r_2\, n(r_2) \int_{|r_1-r_2|}^{r_1+r_2} dr_{12}\, r_{12}\, g(r_{12})}; \tag{58}$$

$\langle r_{12}^n \rangle_0$ is the same expression with $g = 1$. Thus we see that, writing $g = 1+(g-1)$ and using equation (32),

$$\langle r_{12}^2 \rangle = \frac{2\int_0^\infty dr_1\, r_1^2\, n(r_1) \int_0^\infty dr_1\, r_1^2\, n(r_1)(r_1^2+r_2^2) \;+K_2}{2\left(\int_0^\infty dr\, r^2\, n(r)\right)^2 \;+K_0}, \tag{59}$$

where

$$K_n \equiv \int_0^\infty dr\, r\, n(r) \int_0^\infty ds\, s\, n(s)\, G_n(r,s), \tag{60}$$

with

$$G_n(r,s) \equiv \int_{|r-s|}^{r+s} dt\, t\{g(t)-1\}\, t^n. \tag{61}$$

We can evaluate G_n explicitly for a simple model where $g = 0$ for $r < a$ and $g = 1$ for $r > a$ (a correlation hole arising out of hard core repulsions):

$$G_n(r,s) = 0, \qquad |r-s| > a; \tag{62a}$$

$$= -(n+2)^{-1}\{(r+s)^{n+2}-|r-s|^{n+2}\}, \qquad |r-s| < a,\ r+s < a; \tag{62b}$$

$$= -(n+2)^{-1}\{a^{n+2}-|r-s|^{n+2}\}, \qquad |r-s| < a,\ r+s > a. \tag{62c}$$

If we further assume that the density $n(r)$ is a constant n_0 up to $r = R$ and zero for $r > R$, we find

$$K_n = n_0^2\{-\tfrac{4}{3}(n+3)^{-1}R^3a^{n+3}+\tfrac{1}{2}(n+4)^{-1}R^2a^{n+4}+O(a^{n+6})\}. \tag{63}$$

Substitution into equation (59) gives

$$\langle r_{12}^2 \rangle = \tfrac{6}{5}R^2\{1+(a/R)^3+HO(a/R)\}, \qquad \text{while} \qquad \langle r_{12}^2 \rangle_0 = \tfrac{6}{5}R^2, \tag{64}$$

where $HO(a/R)$ indicates terms of higher order in a/R. Thus we have justified the approximation (57) in detail for a specific case.

We now see that, for this simple model, the total four-body term appearing in the result (31) is of the order of the three-body term multiplied by $A(a/R)^3$. For nuclei and helium droplets we have $R = r_0 A^{\frac{1}{3}}$, where r_0 is approximately independent of A. It is thus plausible that the neglected four-body term is smaller than the three-body term by the factor $(a/r_0)^3$. For nuclei the core size is of order 0·4 fm, with $r_0 \approx 1{\cdot}2$ fm $\approx 3a$, while for liquid helium $a = 2{\cdot}56$ Å and $r_0 \approx 0{\cdot}85a$. Thus we expect the above approximations to give reasonable results for nuclei but only a rough estimate for helium microdroplets.

6. Comparison with Experiment

The weak correlation approximation gives the formula (39) for the energy ΔE_2 of the 2^+ state. For large A, and on the assumption of a fairly sharp surface, this result simplifies to $\Delta E_2 \lesssim 7\hbar^2/mR^2$ (equation 42). If we put $R = r_0 A^{\frac{1}{3}}$ with $r_0 = 1\cdot 2$ fm (Irvine 1972), we have for nuclei

$$\Delta E_2 \lesssim 200 A^{-\frac{2}{3}} \text{ MeV}. \tag{65}$$

This is a large excitation energy for nuclei. For $A = 208$ we have $\Delta E_2 \lesssim 5\cdot 7$ MeV, while the first 2^+ state of ^{208}Pb is at $4\cdot 085$ MeV. For all other even–even nuclei, the first 2^+ state is considerably below the bound (65), although the $A^{-\frac{2}{3}}$ trend is roughly followed by spherical nuclei.

There are (at least) three possible explanations of the above discrepancy:

(i) correlations may make the four-body term significant,

(ii) the trial wavefunction we have used needs to be improved, or

(iii) the first 2^+ excited states of most spherical even–even nuclei (except perhaps ^{208}Pb) are not collective rotations of the type described by our wavefunction.

Since four-particle correlations are difficult to discuss rigorously, we have given a heuristic discussion of (i) in the previous section. We can test the explanation (ii) as follows. Consider, instead of Ψ'_{20}, the wavefunction

$$\sum_j (3z_j^2 - r_j^2)\, f(r_j)\, \Phi\,. \tag{66}$$

This wavefunction is also an eigenstate of angular momentum with $L = 2$. We can calculate the expectation value of the energy in this state in the weak correlation approximation, as we did before with $f = 1$. We find

$$\Delta E_2 \leqslant \frac{\hbar^2}{m}\,\frac{5\langle r^2 f^2\rangle + 2\langle r^3 f f'\rangle + \frac{1}{2}\langle r^4 f'^2\rangle}{\langle r^4 f^2\rangle}, \tag{67}$$

where f' denotes $\mathrm{d}f/\mathrm{d}r$ and the expectation values are calculated in the spherical state Φ as before, with r the distance from the centre of mass. We now optimize with respect to f. A short variational calculation gives the following differential equation to be satisfied by the best f:

$$f'' + (6r^{-1} + n'n^{-1})f' + (q^2 + 2r^{-1}n'n^{-1})f = 0\,. \tag{68}$$

Here n is again the number density, and $\hbar^2 q^2/2m = \Delta E_2$. For n constant, the regular solution is $f = r^{-2}\,\mathrm{j}_2(qr)$ and so $f =$ const. is a good approximation while qr is small, i.e. r/R small from the result (42). Thus our trial wavefunction with $f = 1$ is good inside the nucleus.

We can turn the problem around and ask: what density $n(r)$ has $f = 1$ as the optimum solution? From equation (68) we find

$$n'n^{-1} = -\tfrac{1}{2} r q^2\,,$$

so that

$$n(r) = n(0)\exp(-\tfrac{1}{4}q^2r^2) \approx n(0)\exp(-7r^2/2R^2). \tag{69}$$

This describes a typical nuclear density variation fairly well, and gives $\langle r^2\rangle = 3R^2/7$, whereas a sharp boundary has $\langle r^2\rangle = 3R^2/5$. We thus see that, on both counts, $f = 1$ gives a suitable trial wavefunction, and it is unlikely that the considerable extra mathematical complexity of a general $f(r)$ is warranted.

We conclude then that the collective rotational states studied here are unlikely to be the lowest 2^+ excitations of even–even spherical nuclei (except perhaps for ^{208}Pb, where the energy is of the right order; from equation (56), the width comes to about $38\Gamma_W$ whereas experiment gives approximately $8\Gamma_W$ (Lewis 1971, p. 266)). This conclusion has been reached from first principles.

7. Physical Significance of Proposed States

It was conjectured in Paper I that these new states represented surface oscillations, on the basis of the similarity between the effective moment of inertia,

$$I_2 \geqslant \tfrac{3}{4}Am\langle(x_{12}^2-y_{12}^2)(x_{34}^2-y_{34}^2)\rangle/\langle x_{12}^2+y_{12}^2\rangle, \tag{70}$$

and the irrotational moment of inertia of an ellipsoid of constant density deformed along the x axis, rotating about the z axis (Gustafson 1955; Katz 1962), namely

$$I_2 = Am\langle x^2-y^2\rangle^2/\langle x^2+y^2\rangle \tag{71}$$

(in equation (71) the expectation values are to be taken in the rotating state). However, the similarity of equation (70) to (71) is misleading, for two reasons: firstly because (70) is zero in the weak correlation approximation, and secondly because the considerations of the next paragraph point to a different physical interpretation.

We have seen that on breaking translational invariance of the Ψ_{20} state we get the wavefunction

$$\Psi'_{20} = \sum_j (3z_j^2 - r_j^2)\Phi.$$

This wavefunction has the same form that Feynman (1954, 1972) proposed for excitations in liquid helium, namely $\Psi = F\Phi$ with $F = \sum_j f(\boldsymbol{r}_j)$. The analogy is closer than this similar form however. Feynman showed that, in the bulk, the optimum f is a plane wave $\exp(\mathrm{i}\boldsymbol{k}.\boldsymbol{r})$. Now the plane wave can be expanded in angular momentum eigenfunctions as

$$\exp(\mathrm{i}kr\cos\theta) = \sum_{L=0}^{\infty} (2L+1)\mathrm{i}^L \mathrm{j}_L(kr)\,\mathrm{P}_L(\cos\theta)$$

and we thus see that the $L = 2$ component of Feynman's wavefunction is

$$\sum_j \mathrm{j}_2(kr_j)\,\mathrm{P}_2(\cos\theta_j)\Phi, \tag{72}$$

which we showed to be the optimum wavefunction of the type (66). For small k, the form (72) is just our Ψ'_{20}. Thus the proposed states are finite-system, angular momentum projections of Feynman's excitations.

References

Aviles, J. B. (1968). *Ann. Phys. (NY)* **50**, 393.
Blatt, J. M., and Weisskopf, V. F. (1952). 'Theoretical Nuclear Physics' (Wiley: New York).
Castilho Alcarás, J. A., and Leal Ferreira, J. (1971). *Rev. Bras. Fis.* **1**, 63.
Elliott, J. P., and Skyrme, T. H. R. (1955). *Proc. R. Soc. London* A **232**, 561.
Feynman, R. P. (1954). *Phys. Rev.* **94**, 262.
Feynman, R. P. (1972). 'Statistical Mechanics' (Benjamin: Reading, Mass.).
Gustafson, T. (1955). *Dan. Mat. Fys. Medd.* **30**, No. 5.
Hill, T. L. (1956). 'Statistical Mechanics' (McGraw-Hill: New York).
Irvine, J. M. (1972). 'Nuclear Structure Theory' (Pergamon: Oxford).
Karl, G., and Obryk, E. (1968). *Nucl. Phys.* B **8**, 609.
Katz, A. (1962). *In* 'The Many-body Problem' (Ed. C. Fronsdal), p. 246 (Benjamin: New York).
Lekner, J. (1974). *Mol. Phys.* **28**, 431.
Lewis, M. B. (1971). *Nucl. Data* B **5**, 243.
Lipkin, H. J. (1958). *Phys. Rev.* **110**, 1395.
de Shalit, A., and Feshbach, H. (1974). 'Theoretical Nuclear Physics', Vol. 1 (Wiley: New York).

Manuscript received 12 December 1975