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# Ising Model on a Triangular Lattice with Three-spin Interactions. I The Eigenvalue Equation

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*Abstract*

It is shown that the Ising model with three-spin interactions on a triangular lattice is equivalent to a site-colouring problem on a hexagonal lattice. The transfer matrix method is then used to solve the colouring problem. The colouring of two neighbouring rows of sites is described by the positions of dislocations in an otherwise regular sequence of colours. This permits the use of a generalized Bethe's ansatz for the eigenvectors of the transfer matrix; the eigenvalues are found to be given by the solution of a set of equations.

## 1. Introduction

The Ising model plays a central role in the study of phase transitions in lattice systems. For many years, the two-dimensional system with nearest pair interactions (Onsager 1944) stood as the only model of a phase transition that yielded to mathematical analysis. More recently, the exact solution of a certain Ising model with pair and four-spin interactions (Kadanoff and Wegner 1971; Wu 1971), which is equivalent to an eight-vertex model (Baxter 1972), has been found. Since its critical behaviour is quite different from that of the nearest-neighbor model, there has been considerable recent interest in studying other Ising models with multiple-spin interactions.

One multiple-spin system that has been considered is the Ising model with three-spin interactions on a triangular lattice (Wood and Griffiths 1972; Griffiths and Wood 1973; Merlini *et al.* 1973; Merlini 1973). It has been suggested (Griffiths 1971; Gallavotti 1972) that interesting properties may be found in such systems which do not possess the up–down spin-reversal symmetry. We remark that while the three-spin model on a 'Union Jack' lattice is soluble (Hintermann and Merlini 1972), it is natural to consider a three-spin model on a triangular lattice which has the symmetry of the interactions. This model is self-dual (Merlini and Gruber 1972; Wood and Griffiths 1972) so that its transition temperature can be conjectured using the Kramers and Wannier (1941) argument. The critical exponents $\alpha'$, $\beta$ and $\gamma'$ have also been estimated from the low-temperature series analysis (Griffiths and Wood 1973). No exact result was hitherto known.

In a previous Letter (Baxter and Wu 1973) we have reported the exact solution of this model. Details of the analysis are now presented. In this paper the equations leading to the solution are derived, while the solution of these equations and the determination of the exponents $\alpha$, $\alpha'$, $\nu$ and $\nu'$ are given in the subsequent Part II (Baxter 1974; present issue pp. 369–81).

## 2. Definition of Model

Consider a system of $N$ spins $\sigma_i = \pm 1$ located at the vertices of a triangular lattice $L$ (Fig. 1). The three spins surrounding each face interact with a three-body interaction of strength $-J$ so that the Hamiltonian reads

$$\mathscr{H} = -J \sum \sigma_i \sigma_j \sigma_k, \tag{1}$$

with the summation extending over all faces of $L$. We wish to evaluate the partition function

$$Z = \sum_{\sigma_i = \pm 1} \exp(-\mathscr{H}/kT). \tag{2}$$

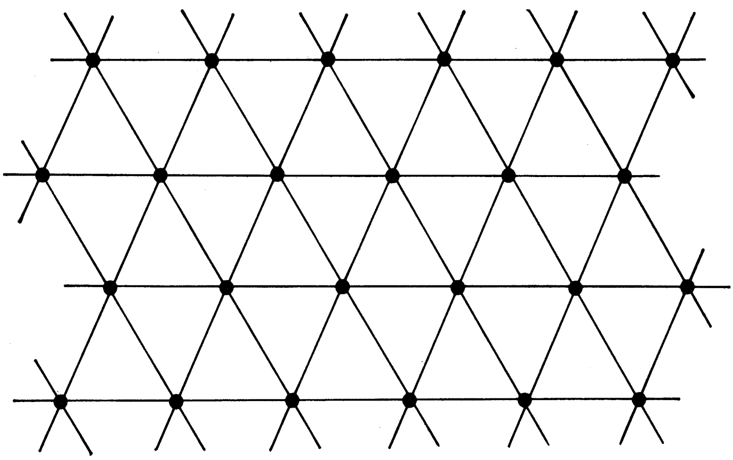

Fig. 1. Triangular lattice $L$, the dots denoting the spins.

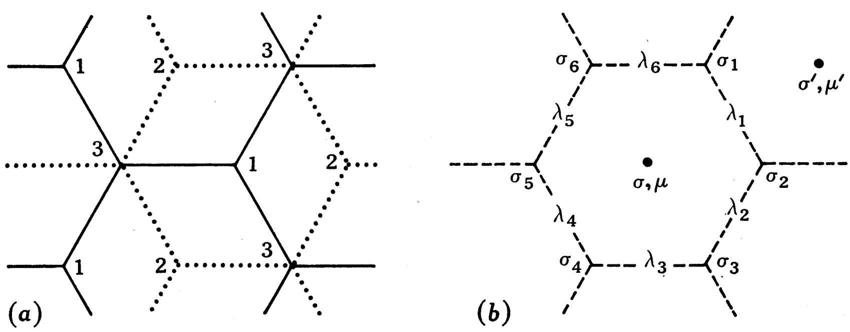


Fig. 2. (a) Decomposition of $L$ into triangular sublattices $L_1$, $L_2$ and $L_3$ consisting of sites numbered 1, 2, and 3 respectively. The full lines form the hexagonal lattice $L_{13}$ while the dotted lines form $L_{23}$; the lattice $L_2$ is the dual of $L_{13}$.
(b) Spin labelling of a typical face of $L_{23}$.

The lattice $L$ can be decomposed into three triangular sublattices $L_1$, $L_2$ and $L_3$. The sites of $L_i$ and $L_j$ form a hexagonal lattice $L_{ij}$ such that $L_{12}$ is the dual of $L_3$, etc. Two such lattices, $L_{13}$ and $L_{23}$, are shown in Fig. 2a. It is clear that replacing $J$ by $-J$ is equivalent to reversing all spins on one of the sublattices $L_i$. Since such reversal leaves $Z$ unchanged, we shall hereinafter take $J > 0$ without loss of generality.

Our first step is to eliminate the spins on $L_{23}$, denoted by the dotted lines in Fig. 2*a*. One way to accomplish this (Wegner 1973) is as follows. Consider a typical face of $L_{23}$, as shown in Fig. 2*b*, enclosing a site of $L_1$. Labelling the spins as indicated and defining

$$\lambda_1 = \sigma_1\sigma_2, \quad \lambda_2 = \sigma_2\sigma_3, \quad \dots, \quad \lambda_6 = \sigma_6\sigma_1, \tag{3}$$

the Hamiltonian (1) can then be written as

$$\mathscr{H} = -J\sum{}^{(23)}\sigma(\lambda_1+\lambda_2+\dots+\lambda_6), \tag{4}$$

where the superscript (23) on the summation indicates that it is taken over all hexagonal faces of $L_{23}$. Thus we can eliminate the spins $\sigma_i$ on $L_{23}$ in favour of the variables $\lambda = \pm 1$, provided that we ensure that $\lambda_1 \dots \lambda_6 = +1$ for each hexagon. This can be done by associating with each face of $L_{23}$ a factor $\frac{1}{2}(1+\lambda_1\dots\lambda_6)$. The partition function then takes the form

$$Z = \sum_{\sigma,\lambda}\prod{}^{(23)}[\exp\{K\sigma(\lambda_1+\dots+\lambda_6)\}\tfrac{1}{2}(1+\lambda_1\dots\lambda_6)], \tag{5}$$

where $K \equiv J/kT$ and the summation is over all values ($+1$ or $-1$) of the $\sigma$'s on $L_1$ and the $\lambda$'s on the edges of $L_{23}$.

If we define, for $\lambda, \mu = \pm 1$, a function $g(\lambda, \mu)$ by

$$g(\lambda, 1) = 2^{-1/6}, \qquad g(\lambda, -1) = \lambda 2^{-1/6}, \tag{6a}$$

or equivalently

$$g(1,\mu) = 2^{-1/6}, \qquad g(-1,\mu) = \mu 2^{-1/6}, \tag{6b}$$

then the factor $\frac{1}{2}(1+\lambda_1\dots\lambda_6)$ in equation (5) can be written as

$$\sum_{\mu} g(\lambda_1,\mu)\,g(\lambda_2,\mu)\dots g(\lambda_6,\mu). \tag{7}$$

Substituting the expression (7) into equation (5), there will be one such $\mu$ variable for each face of $L_{23}$ or equivalently for each site of $L_1$. Thus we may consider the sites of $L_1$ to be described by the four-valued variable $(\sigma,\mu)$. Collecting terms in $Z$ according to the edges of $L_{23}$ or equivalently the nearest neighbours of $L_1$, the $\lambda$-summations can be performed to give

$$Z = \sum_{\sigma,\mu}\prod \omega(\sigma,\mu;\sigma',\mu'), \tag{8}$$

where

$$\begin{aligned}\omega(\sigma,\mu;\sigma',\mu') &= \sum_{\lambda=\pm 1}\exp\{K(\sigma+\sigma')\lambda\}\, g(\lambda,\mu)\,g(\lambda,\mu') \\ &= 2^{-1/3}[\exp\{K(\sigma+\sigma')\}+\mu\mu'\exp\{-K(\sigma+\sigma')\}]. \end{aligned}\tag{9}$$

In equation (8) the summation is over all states $(\sigma,\mu)$ of $L_1$ and the product is over all nearest neighbours of $L_1$.

## 3. Equivalent Colouring Problem

We now convert $Z$ into a colouring generating function for $L_{13}$ (the full lines in Fig. 2*a*). It is seen from equations (8) and (9) that $Z$ does not contain terms with $\sigma\sigma' = \mu\mu' = -1$. Therefore if for each term in (8) we associate colours 1, 3, 5, 7 with the sites of $L_1$ according to the rule

$$\left.\begin{aligned} (+,+) &= \text{colour } 1\,, \qquad (-,+) = \text{colour } 3\,, \\ (-,-) &= \text{colour } 5\,, \qquad (+,-) = \text{colour } 7\,, \end{aligned}\right\} \tag{10}$$

then two neighbouring sites of $L_1$ cannot be coloured $\{1,5\}$ or $\{3,7\}$. We can then complete the colouring of $L_{13}$ by associating colours 2, 4, 6, 8 with the sites of $L_3$ under the restriction that the colours of neighboring sites on $L_{13}$ differ by exactly 1 (to modulus 8, i.e. colour $m$ = colour $m+8$). In this way the colour of a given site $\hat{s}$ of $L_3$ is uniquely determined unless the three sites of $L_1$ surrounding $\hat{s}$ have the same colour $m$; in the latter case $\hat{s}$ can be coloured either $m-1$ or $m+1$.

We next introduce colour activities $z_m \equiv z(m)$ so that the generating function for the site-colouring of $L_{13}$,

$$Z_c = \sum z_1^{n_1} z_2^{n_2} \ldots z_8^{n_8}\,, \tag{11}$$

is proportional to $Z$. The simplest choice (Baxter and Wu 1973) is

$$z_1 = z_3 = z_5 = z_7 = 1\,, \qquad z_2 = z_4^{-1} = z_6 = z_8^{-1} = \sinh 2K \equiv t\,,$$

for which a site $\hat{s}$ of $L_3$ has the weight $w = t, t^{-1}$ or $t+t^{-1}$. It is easily verified that $2w \sinh 4K$ generates precisely the factor in equation (8) arising from the three sites of $L_1$ surrounding $\hat{s}$. Thus we have

$$Z = (2\sinh 4K)^{N/3} Z_c\,. \tag{12}$$

It can be shown that equation (12) is valid more generally provided that the activities satisfy

$$(z_m + z_m^{-1})(z_{m+1} + z_{m+1}^{-1}) = \varDelta \equiv 2(t+t^{-1}) \tag{13}$$

for all $m$ and the solutions are chosen so that $z_m z_{m+2} = 1$. Note that we can choose $z_1$ arbitrary, in which case $z_2, \ldots, z_8$ are determined by equation (13). We do not expect $Z_c$ to depend on the choice of $z_1$. In the following we shall use any activities

$$z_1, \ldots, z_8 = z_1, z_2, z_1^{-1}, z_2^{-1}, z_1, z_2, z_1^{-1}, z_2^{-1}\,, \tag{14}$$

which satisfy (13). We have in particular

$$z_m = z_{m+2}^{-1} = z_{m+4}\,. \tag{15}$$

Also, equation (13) is unaffected by the transformation $t \to t^{-1}$. This is the duality relation which predicts the critical point to occur at $t = 1$ (Merlini and Gruber 1972; Wood and Griffiths 1972).

## 4. Transfer Matrix

We proceed to evaluate the colouring generating function $Z_c$ using the method of the transfer matrix. The sites in each row of $L_{13}$ are numbered as in Fig. 3, where we assume cyclic boundary conditions in both directions. Let there be $M$ (even) sites in a row and $r \equiv 2N/3M$ rows. Let $C = \{c_1, ..., c_M\}$ be the colouring of the upper row and $C' = \{c'_1, ..., c'_M\}$ the colouring of the lower row. Introduce the transfer matrix

$$A(C, C') = \prod_{i=1}^{M} [z(c_i)\, z(c'_i)]^{\frac{1}{2}}, \tag{16}$$

if the colours of the adjacent sites along the bold line in Fig. 3 differ by 1, and $A(C, C') = 0$ otherwise. Then in the usual manner we have

$$Z_c = \sum_{C, C', ...} A(C, C')\, A(C', C'') \dots A(C^{(r-1)}, C) = \operatorname{tr} A^r. \tag{17}$$

For a large lattice, $N, M, r \to \infty$ and we then have

$$Z_c^{3/2N} \sim \Lambda_0^{1/M}, \tag{18}$$

where $\Lambda_0$ is the largest eigenvalue of $A$.

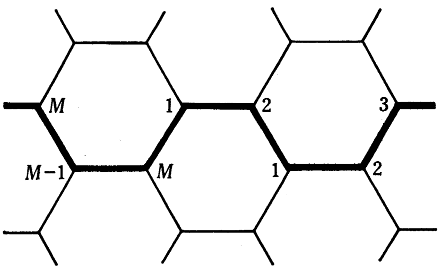


**Fig. 3.** Numbering (left to right) of the sites in each of two adjacent horizontal rows of the lattice $L_{13}$. The bold line joins the sites in both rows.

To compute the eigenvalues of $A$, a more convenient expression of $A(C, C')$ is needed. We observe that a basic sequence of colours along the bold line in Fig. 3, starting from $c_1 = m$, is

$$\{m, m+1, m+2, ..., m+M-1\}. \tag{19}$$

Using equations (15) and assuming $M = 4\mathscr{I}$, where $\mathscr{I}$ is an integer, we obtain for this sequence $A(C, C') = 1$. The general matrix element $A(C, C')$ can now be described by introducing dislocations in this otherwise increasing sequence. For this purpose it is convenient to consider $L_2$, the dual of $L_{13}$. The site-colouring of $L_{13}$ now becomes the face-colouring of $L_2$. A typical dislocated colouring of $L_2$ is shown in Fig. 4, where the faces of $C$ and $C'$ have been shaded for easy identification.

Since the colours of adjacent faces of $L_2$ differ by 1, we may draw arrows on the edges of $L_2$ such that each points to an observer's left (right) if the colours increase (decrease) as he crosses the arrow. Then the basic sequence of colouring (19) corresponds to there being up arrows on the edges bordering the shaded faces (the bold lines in Fig. 4). A dislocation of the sequence is therefore denoted by a down arrow along these edges. Before we proceed further, we remark that there are always three arrows in and three out at each vertex of $L_2$. Hence our colouring problem bears the same relation to the 'triangular ice' model (Baxter 1969) as the three-colourings of the square lattice (Baxter 1970) does to the 'square ice' model (Lieb 1967).

Starting from the basic sequence (19), the effect of a down arrow is to repeat the activities $z_{m-1}$ and $z_m$, where $m$ is the colour immediately preceding the down arrow. This introduces a factor

$$w_{m-1} \equiv w(m-1) = (z_m z_{m-1})^{\frac{1}{2}} \tag{20}$$

to $A(C, C')$. In general, if there are $n$ down arrows bordering the shaded faces in Fig. 4 then

$$A(C, C') = \prod_{j=1}^{n} w(m_j - 1), \tag{21}$$

where $m_j$ is the colour immediately preceding the $j$th down arrow.

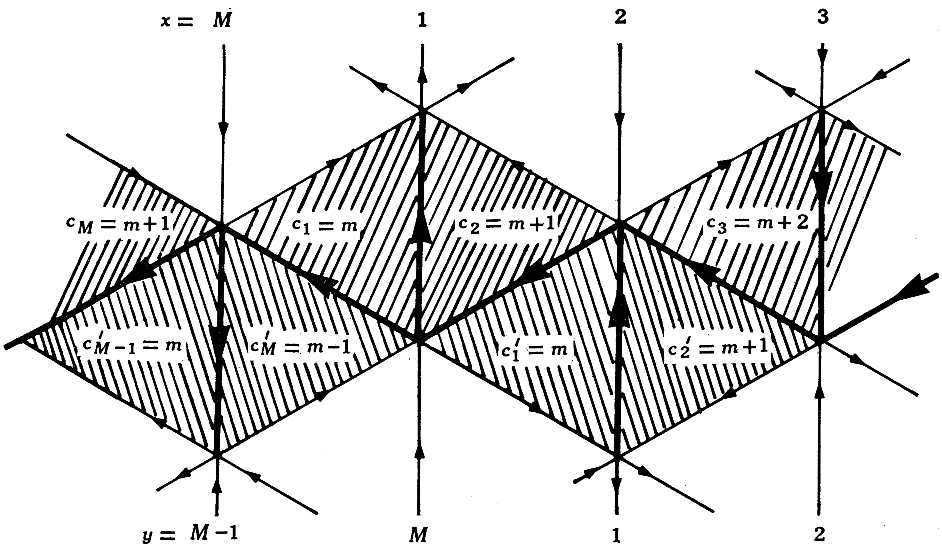


**Fig. 4.** Typical face-colouring of $L_2$. Dislocations in the colours $C$ and $C'$ are denoted by down arrows along the intervening edges.

Let the position between $c_x$ and $c_{x+1}$ in $C$ be numbered by $x$. Then the colours $C$ can be specified by the positions of the down arrows

$$X = \{x_1, x_2, ..., x_n\}. \tag{22}$$

Since at $x$ odd there is only one arrow separating two faces of row $C$, there can be at most one down arrow there, whilst at $x$ even there are three arrows into the vertex separating faces of row $C$, and so up to three down arrows can be crossed on moving from the face to the left of $x$ to the face on the right *via the faces of the next row* $C'$. Hence $X$ must lie in the domain $D$ specified by

$$1 \leqslant x_1 \leqslant x_2 \leqslant ... \leqslant x_n \leqslant M \tag{23}$$

with no odd $x$'s equal and at most three even $x$'s equal.

Similarly for $C'$ let the position between $c'_y$ and $c'_{y+1}$ be $y$ and the down arrows be at $Y = \{y_1, ..., y_n\}$. The possible values of $y_j$ are

$$y_j = x_j - 1, \qquad x_j \text{ odd}, \tag{24a}$$

$$= x_j - 2, x_j - 1, x_j, \qquad x_j \text{ even}. \tag{24b}$$

Furthermore, if $x_j = x_{j+1}$ then $y_j \neq y_{j+1}$. As an example, the down arrows in Fig. 4 are at

$$X = \{2, 3, ..., M, M\}, \qquad Y = \{2, 2, ..., M-1, M\}.$$

Note that the number $n$ of down arrows is the same in each row.

We can now write down $m_j$ and hence $A(C, C')$ in these notations. If there is only one down arrow which goes through the position $x$ in row $C$ and the position $y$ in row $C'$, it can be readily verified that in all cases the colour preceding this down arrow is $m+x+y-1$, if $c_1 = m$. If there is more than one down arrow then $m_j = m+x_j+y_j-1-2(j-1)$, because each down arrow repeats two colours. Consequently, we obtain from equation (21)

$$A(C, C') = \prod_{j=1}^{n} w(m+x_j+y_j-2j), \qquad X, Y \in D \text{ and (24)}, \tag{25a}$$

$$= 0, \qquad \text{otherwise}. \tag{25b}$$

Also, to fulfil the cyclic boundary condition in the horizontal direction, we require

$$2M - 2n = 8\mathscr{I}, \qquad \mathscr{I} \text{ integer}. \tag{26}$$

Since $M$ is even, so must be $n$. The diagonalization of the transfer matrix whose elements are given by equations (25) will be taken up in the next section.

## 5. Bethe's Ansatz

Since the number $n$ of down arrows is the same for each row, the transfer matrix $A(C, C')$ breaks up into diagonal blocks connecting states with the same number $n$. We can then look at one particular value of $n$ $(= 0, 2, ..., 2M)$ subject to the condition (26). Let

$$f_m(X) \equiv f_m(x_1, x_2, ..., x_n) \tag{27}$$

be the element of the eigenvector of $A$ corresponding to the state with $c_1 = m$ and down arrows at $X$. The eigenvalue equation for $A(C, C')$ can be written as

$$\sum_Y{}^{*} \left( \prod_{j=1}^{n} w(m+x_j+y_j-2j) \right) f_{m+2}(Y) = \Lambda f_m(X), \tag{28}$$

where $\Lambda$ is the eigenvalue, $X$ and $Y$ are contained in the domain $D$, and the asterisk indicates that the summations are restricted to the conditions (24). Also in equation (28) the terms with $y_1 = 0$ should be replaced by the boundary condition

$$f_{m+2}(0, y_2, ..., y_n) = f_m(y_2, ..., y_n, M). \tag{29}$$

To solve equation (28), we try the generalized Bethe's (1931) ansatz

$$f_m(X) = \sum_P a(P)\,\phi_{P1}(m-2, x_1) \dots \phi_{Pn}(m-2n, x_n), \tag{30}$$

where the summation is over all $n!$ permutations $P = \{P1, \dots, Pn\}$ of the integers $\{1, \dots, n\}$. The coefficients $a(P)$ and the $n$ functions $\phi_1(m,x), \dots, \phi_n(m,x)$ are at our disposal. We require that there exist $n$ wave numbers $k_1, \dots, k_n$ such that

$$\phi_j(m,x) = \phi_j(m+4,x) = a_{jm} \exp(\mathrm{i}k_j x), \qquad x \text{ odd}, \tag{31a}$$

$$= b_{jm} \exp(\mathrm{i}k_j x), \qquad x \text{ even}. \tag{31b}$$

We now proceed to determine $\Lambda$ by considering various cases.

*All down arrows distinct.* First consider the case when all $x_j$'s in equation (28) are distinct. Then each $y_j$ summation in this equation is independent and we may satisfy (28) simply by solving

$$\sum_y{}^{*}\, w(m+x+y)\,\phi_j(m+2,y) = \lambda_j\,\phi_j(m,x) \tag{32}$$

for $j = 1, \dots, n$ and equate

$$\Lambda = \lambda_1 \lambda_2 \dots \lambda_n. \tag{33}$$

Equation (32) represents two equations corresponding to $x$ odd or even. Let

$$T_{j,m+2} = \begin{bmatrix} 0 & w_{m+1}\exp(-\mathrm{i}k_j) \\ w_{m-1}\exp(-\mathrm{i}k_j) & A_{jm} \end{bmatrix}, \qquad V_{jm} = \begin{bmatrix} a_{jm} \\ b_{jm} \end{bmatrix}, \tag{34}$$

where $A_{jm} \equiv w_m + w_{m-2}\exp(-2\mathrm{i}k_j)$. Then equation (32) can be written as

$$T_{j,m+2}\,V_{j,m+2} = \lambda_j\,V_{jm}. \tag{35}$$

Operating on equation (35) by $T_{jm}$ and using the fact that $V_{j,m-2} = V_{j,m+2}$, we see that $\lambda_j^2$ is the eigenvalue of $T_{jm}T_{j,m+2}$. We then find, using (13),

$$\lambda_j^4 - \lambda_j^2[\exp(-4\mathrm{i}k_j) + \Delta\exp(-2\mathrm{i}k_j) + 1] + \exp(-4\mathrm{i}k_j) = 0. \tag{36}$$

Here, use has also been made of equations (15), or the relations

$$w_m = w_{m+2}^{-1} = w_{m+4}. \tag{37}$$

We are gratified to see that $\Lambda$ is independent of $m$ as desired. The solution of equation (35), which is given here for later use, is:

$$a_{j,m+2} = b_{jm}\,w_{m-1}\,\lambda_j^{-1}\exp(-\mathrm{i}k_j), \tag{38a}$$

$$b_{j,m+2} = b_{jm}\,A_{jm}^{-1}[\lambda_j - \lambda_j^{-1}\,w_{m-1}^2\exp(-2\mathrm{i}k_j)], \tag{38b}$$

$$a_{jm} = b_{jm}\,A_{jm}^{-1}\,w_{m+1}\exp(-\mathrm{i}k_j)[1 - \lambda_j^{-2}\,w_{m-1}^2\exp(-2\mathrm{i}k_j)]. \tag{38c}$$

Finally, if we let

$$\lambda_j = \exp(E_j - \mathrm{i}k_j), \tag{39}$$

equation (36) then becomes

$$\cosh 2E_j = \cos 2k_j + t + t^{-1}. \tag{40}$$

It follows that $E_j$ is real if $k_j$ is real.

*Two down arrows together.* Next consider the case when two even $x$'s coincide. We now choose $a(P)$ to make the ansatz (30) satisfy equation (28) with the eigenvalue (33) unchanged. For convenience consider $n = 2$ and $x_1 = x_2 = x$ (even). Then we require

$$\sum_{y_1 y_2}^{*} w(m+x+y_1-2)\, w(m+x+y_2-4)\, f_{m+2}(y_1, y_2) = \lambda_1 \lambda_2 f_m(x, x), \tag{41}$$

where

$$f_{m+2}(x_1, x_2) = a(1,2)\,\phi_1(m, x_1)\,\phi_2(m-2, x_2) + a(2,1)\,\phi_2(m, x_1)\,\phi_1(m-2, x_2) \tag{42}$$

and the asterisk in equation (41) indicates summation over the three possible $Y$ configurations

$$(y_1, y_2) = (x-2, x-1), \quad (x-2, x), \quad (x-1, x).$$

Straightforward substitution of the relation (42) into (41) and use of equations (31) and (38) then lead to

$$B_{12} \equiv a(1,2)/a(2,1) = -S_{21}/S_{12}, \tag{43}$$

where

$$\begin{aligned} S_{12} \equiv{}& \lambda_2(\lambda_1^2 - w_{m-1}^2 e_1)A_{2m} - \lambda_2^{-1} w_{m-1}^2 e_1 e_2 A_{1m} A_{2m} w_m - w_m^2 e_1(\lambda_2 - \lambda_2^{-1} w_{m-1}^2 e_2)A_{1m} \\ & - w_m w_{m+1}^2 e_1(1 - \lambda_1^{-2} w_{m-1}^2 e_1)(\lambda_2 - \lambda_2^{-1} w_{m-1}^2 e_2) \end{aligned} \tag{44}$$

and

$$e_j \equiv \exp(-2ik_j).$$

For the ansatz to work we need the ratio $B_{12}$ to be independent of $m$. Fortunately this is *indeed* the case, as we can simplify equation (43) to

$$B_{12} = -(\lambda_1 + \lambda_1^{-1} e_1 e_2)/(\lambda_2 + \lambda_2^{-1} e_1 e_2). \tag{45}$$

The simplest way to prove equation (45) is to see by direct multiplication that

$$\begin{aligned} S_{12}(\lambda_1 + \lambda_1^{-1} e_1 e_2) ={}& S_{21}(\lambda_2 + \lambda_2^{-1} e_1 e_2) \\ ={}& \lambda_1 \lambda_2 w_{m-2}[(e_1+e_2)(1 + e_1 e_2) + e_1 e_2 - (\Delta - w_{m-1}^2) + w_m^2] \\ & - w_m w_{m-1}^2 (\lambda_1 \lambda_2)^{-1}[w_{m-2}^2 e_1^3 e_2^3 + e_1 e_2(e_1+e_2)(1 + e_1 e_2) + e_1^2 e_2^2(\Delta - w_{m+1}^2)] \\ & + (\lambda_1 \lambda_2^{-1} + \lambda_1^{-1} \lambda_2) e_1 e_2 w_m (1 - w_{m-1}^2 w_{m-2}^2). \end{aligned} \tag{46}$$

Here, equation (36) has been used to eliminate $\lambda_1^3$ and $\lambda_1^{-3}$ occurring in the product.

More generally we require

$$B_{jl} \equiv \frac{a(\ldots, j, l, \ldots)}{a(\ldots, l, j, \ldots)} = -\frac{\lambda_j + \lambda_j^{-1} \exp[-2i(k_l + k_j)]}{\lambda_l + \lambda_l^{-1} \exp[-2i(k_l + k_j)]} = -\frac{\cosh(E_j + ik_l)}{\cosh(E_l + ik_j)} \tag{47}$$

for all permutations of adjacent elements in $a(P)$. It follows then, within an overall constant, that

$$a(P1, \ldots, Pn) = \prod_{1 \leqslant i < j \leqslant n} B^{\frac{1}{2}}_{Pi,Pj}\,. \tag{48}$$

*Three down arrows together.* It remains to see that the ansatz (30) leads to the same eigenvalue (33) even if three (even) $x$'s are equal. It suffices to consider $n = 3$ for which one needs to show

$$w_{m-2}\, w_{m-1}\, w_m f_{m+2}(x-2, x-1, x) = \lambda_1 \lambda_2 \lambda_3 f_m(x, x, x)\,. \tag{49}$$

It is far from obvious that the relation (49) should hold. But again we are fortunate to find that (49) is *indeed* an identity ensured by equation (47). Details of the proof are tedious, although straightforward, and will be omitted.

Thus we have shown that the ansatz (30) will satisfy equation (28) with $\Lambda = \lambda_1 \ldots \lambda_n$ provided that the coefficients $a(P)$ are given by equation (48).

## 6. Boundary Condition

We now fix $k_1, \ldots, k_n$ to satisfy the boundary condition (29). Substituting equation (30) into (29) and making a cyclic shift of $P$ on the right-hand side, we obtain

$$\sum_P a(P1, \ldots, Pn)\, \phi_{P1}(m, 0)\, [\phi_{P2}(m-2, x_2) \ldots \phi_{Pn}(m-2n+2, x_n)]$$
$$= \sum_P a(P2, \ldots, Pn, P1)\, [\phi_{P2}(m-2, x_2) \ldots \phi_{Pn}(m-2n+2, x_n)]\, \phi_{P1}(m-2n, M)\,. \tag{50}$$

This can be made an identity if we require, for all $P$ and $m$,

$$a(P1, \ldots, Pn)\, \phi_{P1}(m, 0) = a(P2, \ldots, Pn, P1)\, \phi_{P1}(m-2n, M)\,. \tag{51}$$

Now $n$ is even. Using the conditions (31) and (47), equation (51) then becomes

$$\exp(\mathrm{i}Mk_{P1}) = a(P1, \ldots, Pn)/a(P2, \ldots, Pn, P1) = B_{P1,P2}\, B_{P1,P3} \ldots B_{P1,Pn}\,,$$

or

$$\exp(\mathrm{i}Mk_j) = -\prod_{l=1}^{n} B_{jl}\,, \qquad j = 1, 2, \ldots, n\,. \tag{52}$$

Equation (52) is a set of $n$ equations for determining the $n$ wave numbers $k_1, \ldots, k_n$. Here, it must be remembered that $\lambda_j$ depends on $k_j$ through equation (36).

It is clear that $k_j$ is real if $B_{jl}$ is unimodular. Since $B_{jl}$ also depends on $k_j$ and $k_l$, to be consistent we need to show that $B_{jl}$ is unimodular if $k_j$ and $k_l$ are real. This is indeed the case since

$$B_{jl} B^*_{jl} = \cosh(E_j + \mathrm{i}k_l) \cosh(E_j - \mathrm{i}k_l)/\cosh(E_l + \mathrm{i}k_j) \cosh(E_l - \mathrm{i}k_j)$$
$$= (\cosh 2E_j + \cos 2k_l)/(\cosh 2E_l + \cos 2k_j) = 1\,. \tag{53}$$

The last step follows from equation (40), and we have used the fact that $E_j$ and $E_l$ are real if $k_j$ and $k_l$ are real.

## 7. Conclusions

We have shown that the eigenvalues of the transfer matrix (25) can be obtained from equation (28) using the ansatz (30). The procedure for computing the partition function $Z$ may be summarized as follows.

(i) Solve equation (52) for the $n$ *real* wave numbers $k_1, ..., k_n$, where $B_{jl}$ is given by equation (47) and $\lambda_j$ depends on $k_j$ through equation (36).

(ii) Put $\Lambda = \lambda_1 \lambda_2 ... \lambda_n$.

(iii) Let

$$v = \lim_{M\to\infty} \Lambda_0^{1/M}, \tag{54}$$

where $\Lambda_0$ is the largest $\Lambda$ in (ii) for all $n$.

(iv) Then

$$\lim_{N\to\infty} Z^{1/N} = (2\sinh 4K)^{1/3} v^{2/3}. \tag{55}$$

(v) The eigenvectors of the transfer matrix are given by equations (30) and (48).

It should be noted that the equations for $k_1, ..., k_n$ and $\lambda_1, ..., \lambda_n$ are indeed independent of the choice of $z_1$ in equation (13), as has been stated. Thus there is a class of non-trivially related activities for which the generating function of the colouring problem is a constant. A similar property is possessed by the three-colourings of the square lattice (Baxter 1970).

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