# SHORT COMMUNICATIONS

## TEMPERATURE PROFILES IN SOME HEAT CONDUCTION PROBLEMS WITH SPHERICAL SYMMETRY*

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Temperature profiles are known for several composite systems involving a sphere losing heat into an infinite surrounding medium. Lovering (1935, 1936) considered the diffusion of heat from a sphere into a cooler medium with the same thermal properties, i.e. the same conductivity $K$ and diffusivity $k$. The initial temperatures of the two bodies were assumed constant. Further, Philip (1964, 1965) and Brown (1965) have obtained results for the system in which $K_1 \neq K_2$ and $k_1 \neq k_2$, where the suffixes 1 and 2 denote sphere and surrounding medium respectively, again the temperatures being constant initially. Recently Brown (1969) has extended this work to systems in which the temperature within the core at $t = 0$ is a function of $r$, the distance from the centre of the sphere. The present paper evaluates several temperature profiles, given as formal integral solutions in this last paper by Brown.

The functions which Brown (1969) considered were of the forms $r^n$ and $r^{-1} \sin kr$ for various $n$ and $k$. As it is possible to approximate any continuous function, as closely as we wish, with a polynomial or a Fourier series, then once we have solutions for the above forms we can solve the conduction problem for any continuous initial temperature distribution by superposition of the results already obtained. In this paper we tabulate results for $n = 0$, 1, and 2 and $k = \pi/a$ and $\pi/2a$, where $a$ is the radius of the sphere.

Although the problem has many applications (Philip 1964), the geophysical problem of a hot igneous body suddenly intruding into a cooler deposit is most widely known. As a consequence the conductivity and diffusivity, which are assumed constant, have been chosen to represent the intrusion of granite into limestone.

### *Temperature Integrals*

The notation used in this paper is: $a$, radius of sphere; $r$, distance from centre of sphere; $K_1$ and $K_2$, conductivity in core and outer medium respectively; $k_1$ and $k_2$, diffusivity in core and medium respectively; $T_1$ and $T_2$, temperature in core and medium respectively; and $T_0$, a constant temperature. We define also

$$R = r/a, \qquad \sigma = (k_2/k_1)^{\frac{1}{2}}, \qquad \tau = a^2/k_1,$$

$$L = (K_2 - K_1)/K_1, \qquad T(R,t) = T_1(R,t)/T_0 \quad \text{for} \quad R \leqslant 1,$$

$$Q = K_2/K_1\sigma, \qquad \phantom{T(R,t)} = T_2(R,t)/T_0 \quad \text{for} \quad R > 1.$$

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The notation differs slightly from that of Brown (1969), as nondimensional quantities are used.

Brown found the solutions were of the form

$$T(R,t) = \int_0^\infty A(u)\,F_1(u,R,t)\,\mathrm{d}u\,, \qquad R \leqslant 1\,,$$

$$= \int_0^\infty A(u)\,F_2(u,R,t)\,\mathrm{d}u\,, \qquad R > 1\,,$$

where the functions $F_1$ and $F_2$ were independent of the initial conditions and were found to be given by

$$F_1(u,R,t) = (Q/R)\sin uR\,\exp(-u^2t/\tau)\,,$$

$$F_2(u,R,t) = (1/Ru)[(u\cos u + L\sin u)\cos\{u(R-1)/\sigma\}$$

$$+Qu\sin u\sin\{u(R-1)/\sigma\}]\exp(-u^2t/\tau)\,.$$

It remains only to specify the "amplitude factor" $A(u)$ for each initial temperature distribution. Thus if:

(1) $$T(R,0) = R^n\,, \qquad R \leqslant 1\,,$$

$$= 0\,, \qquad R > 1\,,$$

then

$$A(u) = (2/\pi)\{\mathscr{C}_n(u)\}/D^2\,,$$

where

$$D^2 = (u\cos u + L\sin u)^2 + (Qu\sin u)^2$$

and the $\mathscr{C}_n(u)$ are defined by

$$\mathscr{C}_0(u) = \sin u - u\cos u\,,$$

$$\mathscr{C}_1(u) = 2\sin u - u\cos u - 2u^{-1}(1-\cos u)\,,$$

and for $n \geqslant 0$

$$\mathscr{C}_{n+2}(u) = (n+3)\sin u - u\cos u - u^{-2}(n+2)(n+3)\mathscr{C}_n(u)\,.$$

(2) $$T(R,0) = (\pi Rh)^{-1}\sin\pi Rh\,, \qquad R \leqslant 1\,,$$

$$= 0\,, \qquad R > 1\,,$$

then

$$A(u) = 2u^2\xi(h,u)/\pi(\pi^2h^2-u^2)D^2\,,$$

where $D^2$ is already defined and

$$\xi(h,u) = (u\cos u\sin\pi h - \pi h\sin u\cos\pi h)/\pi h\,.$$

We also use the relations

$$H(t) = -4\pi K_2 a \int_0^t (\partial T/\partial R)_{R=1}\,\mathrm{d}t$$

for the total heat loss from the core after time $t$, and

$$H_0 = 4\pi a^3 (K_1/k_1) \int_0^1 R^3\, T(R,0)\,\mathrm{d}R$$

for the total amount of heat initially in the core.

Brown (1969) found an asymptotic expansion for the fraction of the total heat content of the system still contained in the core, for large values of the time $t$. The expansion was

$$\{H_0 - H(t)\}/H_0 = (a^3/6\pi^{\frac{1}{2}})(K_1/K_2)(1/k_1 k_2^{\frac{1}{2}})t^{-3/2} + O(t^{-5/2})$$

$$= (1/6\pi^{\frac{1}{2}})(K_1/K_2\,\sigma)(t/\tau)^{-3/2} + O(t^{-5/2}).$$

The leading term of this expression is independent of the initial conditions considered.

## *Evaluation of Temperature Integrals*

The numerical integration was carried out on an IBM 360/50 computer using a program based on Simpson's rule. The approximation used $2^m+1$ points, $m$ increasing from 2 to a maximum of 15, until an accuracy of five significant figures was obtained; the results have been tabulated to four figures only. The upper limit of the integration was reduced from infinity to $x$, the value of $x$ being taken so that integration from $x$ to infinity gave a result $< 10^{-5}$.

Some difficulty was encountered in finding results for the cases $T(R,0) = R$ or $R^2$; however, immediate solutions were obtained when the forms $2+R$ and $1+R-R^2$ were used.

In order that results could be compared directly, a restriction was placed on the initial temperature distributions. We required that the heat content $H_0$ was the same for each distribution. The five cases evaluated were for $R \leqslant 1$:

(1) $T(R,0) = 1$, (2) $T(R,0) = (4/3)R$, (3) $T(R,0) = (5/3)R^2$,

(4) $T(R,0) = (\pi/3R)\sin(\pi R)$, (5) $T(R,0) = (\pi^2/12R)\sin(\frac{1}{2}\pi R)$.

Conductivities and diffusivities used were (in CGS units): $K_1 = 0{\cdot}008$, $k_1 = 0{\cdot}016$ for granite and $K_2 = 0{\cdot}005$, $k_2 = 0{\cdot}008$ for limestone and $a$ was taken to be 1000 m.

Lovering's (1935, 1936) results were also recalculated and tabulated for comparison (case (6), Table 1). For this problem he took $K_1 = K_2 = 0{\cdot}008$, $k_1 = k_2 = 0{\cdot}016$, and again $a = 1000$ m.

## *Evaluations at Different $K, k$*

It is clear from the general forms of the integrals that each depends on the ratios of $K_1/K_2$, $k_1/k_2$, and $t/\tau$ only. It follows that the numerical results we have

TABLE 1

TEMPERATURE DISTRIBUTIONS $T(R, \alpha)$ FOR DIFFERENT INITIAL CONDITIONS (FOR $R \leqslant 1$)

For the conditions (1) to (5) $K_1/K_2 = 1{\cdot}6$ and $k_1/k_2 = 2{\cdot}0$, while for Lovering's problem (6) $K_1/K_2 = k_1/k_2 = 1$

| $r$ (m) | $t$ ($10^3$ yr) =<br>$R$ $\alpha$ = | 5<br>0·25 | 10<br>0·50 | 15<br>0·75 | 20<br>1·0 | 40<br>2·0 | 60<br>3·0 | 80<br>4·0 | 100<br>5·0 | 120<br>6·0 |
|---|---|---|---|---|---|---|---|---|---|---|
| (1) $T(R, 0) = 1$ | | | | | | | | | | |
| 500 | 0·5 | 0·4045 | 0·2250 | 0·1513 | 0·1118 | 0·0505 | 0·0305 | 0·0210 | 0·0156 | 0·0122 |
| 1000 | 1·0 | 0·2698 | 0·1764 | 0·1270 | 0·0974 | 0·0468 | 0·0289 | 0·0202 | 0·0151 | 0·0119 |
| 1500 | 1·5 | 0·0770 | 0·0840 | 0·0745 | 0·0640 | 0·0372 | 0·0246 | 0·0178 | 0·0137 | 0·0109 |
| 2000 | 2·0 | 0·0096 | 0·0258 | 0·0321 | 0·0332 | 0·0261 | 0·0193 | 0·0148 | 0·0117 | 0·0096 |
| 3000 | 3·0 | 0 | 0·0006 | 0·0024 | 0·0045 | 0·0090 | 0·0093 | 0·0084 | 0·0075 | 0·0065 |
| 4000 | 4·0 | 0 | 0 | 0·0001 | 0·0003 | 0·0019 | 0·0033 | 0·0038 | 0·0039 | 0·0038 |
| 5000 | 5·0 | 0 | 0 | 0 | 0 | 0·0003 | 0·0008 | 0·0014 | 0·0017 | 0·0019 |
| 6000 | 6·0 | 0 | 0 | 0 | 0 | 0 | 0·0002 | 0·0004 | 0·0006 | 0·0008 |
| (2) $T(R, 0) = (4/3)R$ | | | | | | | | | | |
| | 0·5 | 0·3847 | 0·2194 | 0·1487 | 0·1103 | 0·0501 | 0·0303 | 0·0209 | 0·0156 | 0·0122 |
| | 1·0 | 0·2638 | 0·1732 | 0·1252 | 0·0963 | 0·0463 | 0·0288 | 0·0201 | 0·0151 | 0·0118 |
| | 1·5 | 0·0793 | 0·0839 | 0·0740 | 0·0636 | 0·0370 | 0·0245 | 0·0178 | 0·0136 | 0·0109 |
| | 2·0 | 0·0104 | 0·0263 | 0·0323 | 0·0333 | 0·0260 | 0·0192 | 0·0147 | 0·0117 | 0·0096 |
| | 3·0 | 0 | 0·0006 | 0·0025 | 0·0046 | 0·0090 | 0·0093 | 0·0084 | 0·0075 | 0·0065 |
| | 4·0 | 0 | 0 | 0·0001 | 0·0003 | 0·0020 | 0·0033 | 0·0038 | 0·0039 | 0·0038 |
| | 5·0 | 0 | 0 | 0 | 0 | 0·0003 | 0·0008 | 0·0014 | 0·0017 | 0·0019 |
| | 6·0 | 0 | 0 | 0 | 0 | 0 | 0·0002 | 0·0004 | 0·0006 | 0·0008 |
| (3) $T(R, 0) = (5/3)R^2$ | | | | | | | | | | |
| | 0·5 | 0·3713 | 0·2156 | 0·1470 | 0·1094 | 0·0500 | 0·0304 | 0·0209 | 0·0156 | 0·0122 |
| | 1·0 | 0·2596 | 0·1709 | 0·1247 | 0·0963 | 0·0464 | 0·0295 | 0·0201 | 0·0151 | 0·0119 |
| | 1·5 | 0·0818 | 0·0847 | 0·0745 | 0·0641 | 0·0370 | 0·0253 | 0·0178 | 0·0137 | 0·0109 |
| | 2·0 | 0·0118 | 0·0276 | 0·0333 | 0·0341 | 0·0268 | 0·0200 | 0·0156 | 0·0117 | 0·0096 |
| | 3·0 | 0 | 0·0015 | 0·0032 | 0·0055 | 0·0099 | 0·0102 | 0·0093 | 0·0075 | 0·0065 |
| | 4·0 | 0 | 0·0001 | 0·0002 | 0·0004 | 0·0028 | 0·0040 | 0·0038 | 0·0039 | 0·0038 |
| | 5·0 | 0 | 0 | 0 | 0 | 0·0003 | 0·0010 | 0·0014 | 0·0017 | 0·0019 |
| | 6·0 | 0 | 0 | 0 | 0 | 0 | 0·0002 | 0·0004 | 0·0006 | 0·0008 |
| (4) $T(R, 0) = (\pi/3R)\sin(\pi R)$ | | | | | | | | | | |
| | 0·5 | 0·4662 | 0·2425 | 0·1592 | 0·1165 | 0·0516 | 0·0309 | 0·0212 | 0·0158 | 0·0123 |
| | 1·0 | 0·2889 | 0·1865 | 0·1326 | 0·1010 | 0·0477 | 0·0293 | 0·0204 | 0·0152 | 0·0119 |
| | 1·5 | 0·0697 | 0·0842 | 0·0760 | 0·0655 | 0·0378 | 0·0249 | 0·0180 | 0·0138 | 0·0110 |
| | 2·0 | 0·0072 | 0·0240 | 0·0316 | 0·0332 | 0·0263 | 0·0195 | 0·0149 | 0·0118 | 0·0096 |
| | 3·0 | 0 | 0·0006 | 0·0020 | 0·0043 | 0·0089 | 0·0093 | 0·0085 | 0·0075 | 0·0066 |
| | 4·0 | 0 | 0 | 0·0001 | 0·0003 | 0·0019 | 0·0032 | 0·0038 | 0·0039 | 0·0038 |
| | 5·0 | 0 | 0 | 0 | 0 | 0·0003 | 0·0008 | 0·0013 | 0·0017 | 0·0019 |
| | 6·0 | 0 | 0 | 0 | 0 | 0 | 0·0002 | 0·0004 | 0·0006 | 0·0008 |
| (5) $T(R, 0) = (\pi^2/12R)\sin(\frac{1}{2}\pi R)$ | | | | | | | | | | |
| | 0·5 | 0·4137 | 0·2276 | 0·1525 | 0·1125 | 0·0506 | 0·0306 | 0·0210 | 0·0156 | 0·0122 |
| | 1·0 | 0·2727 | 0·1780 | 0·1278 | 0·0979 | 0·0469 | 0·0290 | 0·0202 | 0·0151 | 0·0119 |
| | 1·5 | 0·0759 | 0·0841 | 0·0747 | 0·0642 | 0·0372 | 0·0247 | 0·0178 | 0·0137 | 0·0109 |
| | 2·0 | 0·0092 | 0·0255 | 0·0321 | 0·0332 | 0·0261 | 0·0193 | 0·0148 | 0·0117 | 0·0096 |
| | 3·0 | 0 | 0·0006 | 0·0024 | 0·0045 | 0·0090 | 0·0093 | 0·0084 | 0·0075 | 0·0065 |
| | 4·0 | 0 | 0 | 0·0001 | 0·0002 | 0·0019 | 0·0032 | 0·0038 | 0·0039 | 0·0038 |
| | 5·0 | 0 | 0 | 0 | 0 | 0·0003 | 0·0008 | 0·0013 | 0·0017 | 0·0019 |
| | 6·0 | 0 | 0 | 0 | 0 | 0 | 0·0002 | 0·0004 | 0·0006 | 0·0008 |
| (6) $T(R, 0) = 1$ (Lovering's problem) | | | | | | | | | | |
| | 0·5 | 0·3605 | 0·1777 | 0·1094 | 0·0758 | 0·0296 | 0·0167 | 0·0110 | 0·0080 | 0·0061 |
| | 1·0 | 0·2196 | 0·1312 | 0·0881 | 0·0640 | 0·0271 | 0·0157 | 0·0105 | 0·0077 | 0·0059 |
| | 1·5 | 0·0935 | 0·0790 | 0·0614 | 0·0484 | 0·0234 | 0·0142 | 0·0098 | 0·0072 | 0·0056 |
| | 2·0 | 0·0270 | 0·0386 | 0·0370 | 0·0327 | 0·0190 | 0·0123 | 0·0088 | 0·0066 | 0·0052 |
| | 3·0 | 0·0006 | 0·0048 | 0·0086 | 0·0106 | 0·0105 | 0·0083 | 0·0065 | 0·0052 | 0·0043 |
| | 4·0 | 0 | 0·0003 | 0·0011 | 0·0022 | 0·0046 | 0·0047 | 0·0043 | 0·0037 | 0·0032 |
| | 5·0 | 0 | 0 | 0·0001 | 0·0003 | 0·0016 | 0·0023 | 0·0025 | 0·0024 | 0·0022 |
| | 6·0 | 0 | 0 | 0 | 0 | 0·0004 | 0·0010 | 0·0013 | 0·0014 | 0·0014 |

obtained are applicable to all systems in which $K_1/K_2 = 1\cdot6$ and $k_1/k_2 = 2\cdot0$, and for which $t/\tau$ $(= tk_1/a^2)$ takes the given set of values, defined by $t/\tau = 1\cdot0009152\,\alpha$, for $\alpha = 0\cdot25, 0\cdot5, 0\cdot75, 1, 2, \ldots, 6$. For the $k_1$ and $a$ we have used in calculations, this results in values of the time $t$ at which temperature distributions are determined of exactly 5000, 10 000, 15 000 yr . . ..

*Numerical Results*

Results of the calculations are given in Table 1. Profiles for times corresponding to $\alpha = 0\cdot125, 0\cdot25, \ldots, 2\cdot0$ are shown in Figure 1(*a*) for the case in which the initial temperature distribution is uniform. Profiles for $T(R,0)$ $(R \leqslant 1)$ of the forms $R$, $R^2$, $(1/\pi R)\sin(\pi R)$, and $(2/\pi R)\sin(\frac{1}{2}\pi R)$ are very similar and for $R > 3$ or $\alpha > 3$ are indistinguishable within 2% from the case $T(R,0) = 1$. Results for Lovering's problem are also shown (Fig. 1(*b*)) for comparison, and as expected it is seen that the temperatures inside the core fall more rapidly than in the problems considered here, as the conductivity and diffusivity in the outer regions have been increased.

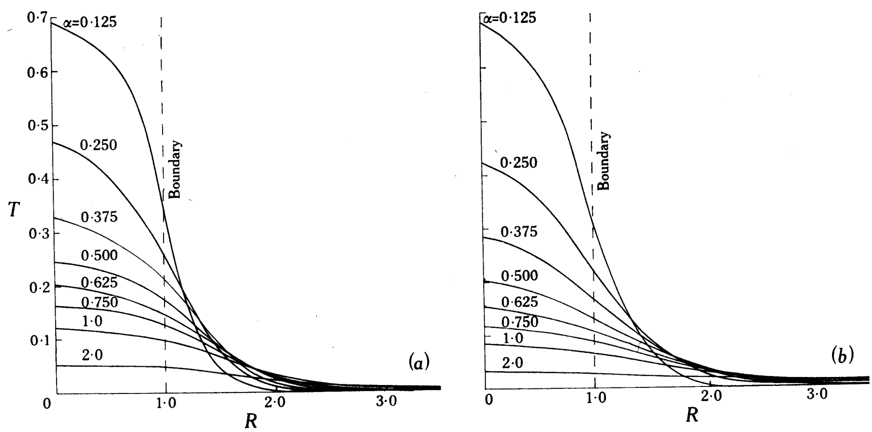


Fig. 1.—Temperature profiles for constant initial temperature at times corresponding to the indicated values of $\alpha$ for (*a*) the present work ($K_1/K_2 = 1\cdot6$, $k_1/k_2 = 2\cdot0$) and (*b*) Lovering's problem ($K_1/K_2 = k_1/k_2 = 1$).

The fraction of the total heat which remains in the core after various times $t$ (or $\alpha$) was estimated, using Simpson's rule over 10 points in the core, for comparison with results obtained using the asymptotic expansion for large $t$. It would seem that this approximation is valid for $\alpha \geqslant 4$.

*Geophysical Problem: The Laccolith*

The present diffusion problem has been considered with the geophysical application in mind; however, by necessity, the conditions have been idealized to produce a system which is mathematically manageable while retaining some degree of applicability. In problems which involve a reasonably low temperature difference the difficulties may not arise. The geophysical case requires temperature differences of perhaps 1000°C, and as a result it must be taken into account that the core may be molten. The effects of the latent heat of fusion released when the molten material solidifies, the changes in diffusivity (probably small), and the conductivity (large) with temperature must be averaged out.

Lovering (1935) states that the latent heat of fusion can be as much as 25% of the total heat of the molten core, though in most cases it will be considerably less. Further, he suggests that the error involved in neglecting this quantity can be minimized by increasing the radius of the intrusive material, or by increasing the initial temperature, proportional to the amount of heat released in the form of latent heat of fusion. The latter method is better for large values of $\alpha$. Errors in using constant conductivity and diffusivity are minimized by averaging over the range of temperatures considered.

It would appear from Table 1, taking an initial temperature difference of 1000°C, that the heat effects do not extend very far; for $R > 3$, where there is at most 10°C rise in temperature, effects are negligible.

*References*

Brown, A. (1965).—*Aust. J. Phys.* **18**, 393.
Brown, A. (1969).—*J. Aust. math. Soc.* **10**, 403.
Lovering, T. S. (1935).—*Bull. geol. Soc. Am.* **46**, 69.
Lovering, T. S. (1936).—*Bull. geol. Soc. Am.* **47**, 87.
Philip, J. R. (1964).—*Aust. J. Phys.* **17**, 423.
Philip, J. R. (1965).—*Aust. J. Phys.* **18**, 393.