PROPAGATION OF SHOCK WAVES IN A POLYTROPE WITH A TOROIDAL MAGNETIC FIELD

I. SIMPLIFIED SOLUTION OF DIFFERENTIAL EQUATIONS

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Summary

The propagation of an initially spherical shock wave in a polytrope with a magnetic field has been studied. The model chosen for the purpose was that of a polytrope with a toroidal magnetic field given previously by Sinha. Butler's method has been extended to transform the set of governing partial differential equations into a set of ordinary differential equations involving derivatives in the direction of propagation of the shock element at any point. An approximate solution is obtained and the effect of the toroidal magnetic field on the geometry of the front as well as on the effects brought about by the shock is discussed.

I. INTRODUCTION

The propagation of shock waves in stellar media has been considered by a number of authors recently, but very few investigations have been made on shock propagation in a stellar model with a magnetic field. Ôno, Sakashita, and Yamazaki (1960) considered the propagation of a magnetohydrodynamic normal shock wave in inhomogeneous gases and then applied it to certain astrophysical problems. However, they started with a one-dimensional formulation. This imposes a restriction on the choice of the magnetic field and, furthermore, leaves out the geometrical effects of the medium. Sinha (1968*a*) has discussed the structure of polytropes with a toroidal magnetic field. In the present paper this model has been chosen to study the propagation of a shock wave.

The magnetic field under consideration is nonuniform and axisymmetric. Consequently the flow variables become functions of three independent variables. Butler (1960) has discussed the numerical solution of hyperbolic systems of partial differential equations in three independent variables and has also described a method to determine the propagation of a point on the shock front. His method consists in reducing the set of governing quasi-linear hyperbolic partial differential equations into a set of ordinary differential equations along the shock ray for the determination of two shock parameters. A shock ray is the trajectory along which a shock element at any point propagates under given conditions in the space of the independent variables. The reduced set of equations, however, contains spatial derivatives of velocity components which are yet unknown and a further analysis is required in order to eliminate them. These equations have then to be solved in conjunction with the equations of the shock ray to obtain the complete run of variation of these

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two shock parameters. Knowledge of these parameters then enables us to determine the flow behind the shock completely.

Stellar interiors, where the density is always decreasing towards the surface, furnish favourable conditions for the generation of shock waves. For example, even a moderate disturbance in the medium in the form of a sound wave will increase in amplitude during its passage outwards and may sooner or later develop into a shock wave. Moreover, Jeffrey (1967) has discussed the formation of a shock by the steepening of a continuous wave in atmospheres with exponentially decreasing density. Amongst other possible circumstances for the formation of a shock there is the possibility of the sudden release of a huge amount of energy in the inner core producing a steep pressure gradient which develops and propagates outwards as a shock wave. Ledoux and Walraven (1958) mention that this may be due to a sudden release of energy by special nuclear reactions that generate in a short time a quantity of energy at least of the same order as the internal energy of the region where they take place. Following these authors, a search has been made for a plausible value of the shock Mach number (Section VI) at the time when the shock is formed.

II. Basic Equations

In cylindrical coordinates (r, ϕ, z), the magnetic induction $\boldsymbol{B}$ may be expressed as $\boldsymbol{B} \equiv (0, B, 0)$. The origin of coordinates lies at the centre and the z axis lies along the axis of symmetry of the polytrope. The solenoidal condition for $\boldsymbol{B}$ establishes the independence of B from ϕ. The fluid velocity $\boldsymbol{V}$, the pressure p, and the density ρ will now depend only upon r, z, and t. The velocity $\boldsymbol{V}$ may be written as $\boldsymbol{V} \equiv (u, 0, v)$. Then assuming that the viscosity is negligible, the conductivity infinite, and the medium a perfect fluid, the Lundquist (1952) equations for a self-gravitating fluid reduce to the following system of quasi-linear hyperbolic partial differential equations:

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial r}+v\frac{\partial \rho}{\partial z}+\rho\frac{\partial u}{\partial r}+\rho\frac{\partial v}{\partial z}+\rho\frac{u}{r}=0, \tag{1a}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+v\frac{\partial u}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{B}{\mu\rho}\frac{\partial B}{\partial r}+\frac{B^2}{\mu\rho r}=-\frac{\partial \Phi}{\partial r}, \tag{1b}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+v\frac{\partial v}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{B}{\mu\rho}\frac{\partial B}{\partial z}=-\frac{\partial \Phi}{\partial z}, \tag{1c}$$

$$\frac{\partial B}{\partial t}+u\frac{\partial B}{\partial r}+v\frac{\partial B}{\partial z}+B\frac{\partial u}{\partial r}+B\frac{\partial v}{\partial z}=0, \tag{1d}$$

$$\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial r}+v\frac{\partial p}{\partial z}-c^2\frac{\partial \rho}{\partial t}-c^2u\frac{\partial \rho}{\partial r}-c^2v\frac{\partial \rho}{\partial z}=0, \tag{1e}$$

where μ is the permeability of the medium and $c^2 = \gamma p/\rho$ is the square of the sound speed. The entropy relation $s = s_0+C_V\ln(p\rho^{-\gamma})$ has been assumed. Further, the gravitational potential Φ satisfies Poisson's equation

$$\frac{\partial^2 \Phi}{\partial r^2}+\frac{1}{r}\frac{\partial \Phi}{\partial r}+\frac{\partial^2 \Phi}{\partial z^2}=4\pi G\rho. \tag{2}$$

Let

$$P = p + B^2/2\mu$$

be the sum of the fluid pressure and the magnetic pressure and

$$\boldsymbol{b} = \boldsymbol{B}(\mu\rho)^{-\frac{1}{2}}$$

be the Alfvén velocity vector. Then the system of equations (1) may be simplified and rewritten as

$$\frac{\mathrm{D}P}{\mathrm{D}t} + \rho\left(c^2 + b^2\right)\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right) + \rho c^2 \frac{u}{r} = 0\,, \tag{3}$$

$$\frac{\mathrm{D}u}{\mathrm{D}t} + \frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{\partial \Phi}{\partial r} + \frac{b^2}{r} = 0\,, \tag{4}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + \frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{\partial \Phi}{\partial z} = 0\,, \tag{5}$$

where

$$\mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + \boldsymbol{V}.\nabla\,.$$

Since the motion is axisymmetrical, we discuss the motion in an azimuthal plane only.

III. Equilibrium Solutions

For the equilibrium configuration we take here the solution given by Sinha (1968*a*). The equilibrium distributions of density ρ_0, pressure p_0, and magnetic field B_0 are given as

$$\rho_0 = \rho_c\,\Theta^n\,, \qquad p_0 = k\rho_0^{1+n^{-1}}\,, \qquad \text{and} \qquad B_0 = Lr\rho_0\,,$$

where ρ_c is the central density, n is the polytropic index, and L is a constant. Further, using the notations

$$\alpha^2 = \{(n+1)/4\pi G\}k\rho_c^{n^{-1}-1}$$

and

$$(r^2+z^2)^{\frac{1}{2}} = \alpha\xi\,,$$

Θ is defined as

$$\Theta = \theta_0(\xi) + \beta^2\{\Psi_0(\xi) + \Psi_2(\xi)\,\mathrm{P}_2(\cos\theta)\}\,,$$

where θ is the polar angle and P_2 is the Legendre polynomial of the second order. The Emden function θ_0 and functions Ψ_0 and Ψ_2 satisfy the following differential equations

$$\mathrm{D}(\theta_0) = -\theta_0^n\,, \tag{6}$$

$$\mathrm{D}(\Psi_0) = -n\theta_0^{n-1}\Psi_0 - \tfrac{2}{3}\mathrm{D}(\xi^2\theta_0^n)\,, \tag{7}$$

and

$$\mathrm{D}(\Psi_2) = (6\xi^{-2} - n\theta_0^{n-1})\Psi_2 + \tfrac{2}{3}\mathrm{D}(\xi^2\theta_0^n) - 4\theta_0^n\,, \tag{8}$$

where D is the operator

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}}{\mathrm{d}\xi}\right).$$

The constant β^2 is small and is a measure of the ratio of the magnetic force to the gravitational force. The quantities L and β^2 are connected by the relation

$$L^2 = 4\pi G\mu\beta^2\,.$$

The axisymmetrical magnetic field is also symmetrical about the r axis in an azimuthal plane. The magnetic field vanishes on the z axis and on the boundary. Further, along a line $z =$ constant the magnitude of the field strength continuously increases, reaches a maximum, and then continuously falls to zero. On $\bar{\xi} =$ constant, however, the magnitude of the field (Fig. 1) increases as θ increases, where $\bar{\xi}$ is defined as

$$\bar{\xi} = \xi/\xi_0$$

and $\xi = \xi_0$ defines the boundary of the polytrope.

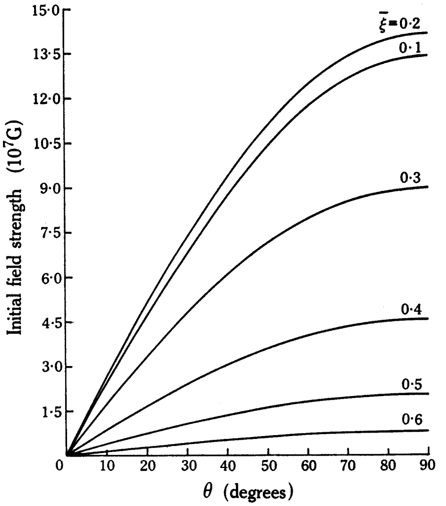

Fig. 1.—Initial distribution of field strength on curves $\bar{\xi} =$ constant in the upper half of the azimuthal plane for $\beta^2 = 0\cdot01$.

IV. Shock Relations

The motion is discontinuous across a shock front and the physical variables suffer a jump across it. The shock relations are equations connecting the values of the physical variables immediately on each side of an element of the shock front moving with local normal velocity U. If the strength of the shock is λ and if the angle its normal makes with the r axis is ω, the shock relations give

$$u = V\cos\omega\,, \tag{9a}$$

$$v = V\sin\omega\,, \tag{9b}$$

$$\rho = \lambda\rho_0\,, \tag{9c}$$

$$B = \lambda B_0\,, \tag{9d}$$

$$V = (1-1/\lambda)U\,, \tag{9e}$$

and

$$P = P_0+(1-1/\lambda)U^2\rho_0\,, \tag{9f}$$

where the velocity U itself of the fast shock is given as

$$U^2 = \frac{2\lambda c_0^2+\{(2-\gamma)\lambda+\gamma\}\lambda b_0^2}{(\gamma+1)-(\gamma-1)\lambda}. \tag{9g}$$

The other possible value $U = 0$ is not acceptable since it corresponds to a contact discontinuity in the medium. In equations (9), the subscript 0 refers to the region at rest ahead of the shock. The unsubscripted variables refer to the flow in the shocked region and are made to depend on the two shock parameters λ and ω only.

In the present case we find that the fluid velocity V is normal to the front and that the magnetic field increases in the same ratio across the front as the material density. Further, λ must satisfy the inequality

$$1 < \lambda < (\gamma+1)/(\gamma-1), \tag{10}$$

in order that the shock may be physically relevant and the shock velocity finite and positive.

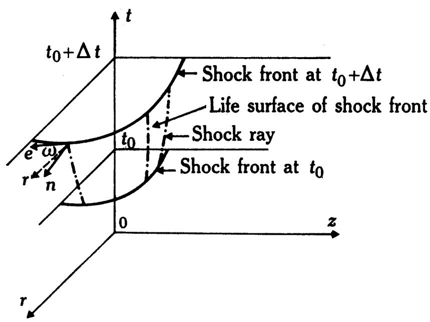

Fig. 2.—Diagram showing shock propagation in (t, r, z) space.

V. Shock Ray and Equations Determining λ and ω

The equations $dr = U\cos\omega\, dt$ and $dz = U\sin\omega\, dt$ together define shock rays along which an element of the shock front propagates. We write

$$\frac{\delta}{\delta t} \equiv \frac{\partial}{\partial t}+U\cos\omega\frac{\partial}{\partial r}+U\sin\omega\frac{\partial}{\partial z}, \tag{11a}$$

$$\frac{\delta}{\delta n} \equiv \cos\omega\frac{\partial}{\partial r}+\sin\omega\frac{\partial}{\partial z}, \tag{11b}$$

and

$$\frac{\delta}{\delta e} \equiv -\sin\omega\frac{\partial}{\partial r}+\cos\omega\frac{\partial}{\partial z}, \tag{11c}$$

and we find (Fig. 2) that $\delta/\delta t$ denotes differentiation in the direction of a shock ray at the shock surface, while $\delta/\delta e$ and $\delta/\delta n$ respectively denote, in a plane t = constant, differentiation along the shock front and in a direction normal to the shock front. This means that $\delta/\delta e$ can operate compatibly on any of the equations (9) which hold everywhere on the shock front. Thus $\delta F/\delta e$ can be expressed in terms of the known equilibrium values. Clearly, this advantage holds for $\delta F/\delta t$ also, but not for $\delta F/\delta n$. We shall now obtain equations for λ and ω along the shock rays.

Since the gravitational potential Φ satisfies the elliptic equation (2), Φ and its first partial derivatives are continuous everywhere. Further, from the equilibrium conditions we have

$$\frac{\partial \Phi}{\partial r} = -\frac{1}{\rho_0}\frac{\partial P_0}{\partial r} - \frac{b_0^2}{r} \qquad \text{and} \qquad \frac{\partial \Phi}{\partial z} = -\frac{1}{\rho_0}\frac{\partial P_0}{\partial z}.$$

After substituting these values, the equations (3)–(5) are rewritten in terms of the notations (11):

$$\frac{\delta P}{\delta t} = \left(U-V\right)\frac{\delta P}{\delta n} - \rho A^2\left(\cos\omega\frac{\delta u}{\delta n} - \sin\omega\frac{\delta u}{\delta e} + \sin\omega\frac{\delta v}{\delta n} + \cos\omega\frac{\delta v}{\delta e}\right) - \frac{\rho c^2 u}{r},$$

$$\rho\frac{\delta u}{\delta t} = \rho\left(U-V\right)\frac{\delta u}{\delta n} - \left(\cos\omega\frac{\delta P}{\delta n} - \sin\omega\frac{\delta P}{\delta e}\right) + \frac{\rho}{\rho_0}\frac{\partial P_0}{\partial r} - \frac{\rho}{r}\left(b^2 - b_0^2\right),$$

and

$$\rho\frac{\delta v}{\delta t} = \rho\left(U-V\right)\frac{\delta v}{\delta n} - \left(\sin\omega\frac{\delta P}{\delta n} + \cos\omega\frac{\delta P}{\delta e}\right) + \frac{\rho}{\rho_0}\frac{\partial P_0}{\partial z},$$

where

$$A^2 = b^2 + c^2.$$

Now the right-hand sides in the above equations contain spatial derivatives of P, u, and v. Elimination of $\delta P/\delta n$ between them yields the two equations

$$\frac{\delta P}{\delta t} + \rho A\cos\chi\left(\cos\omega\frac{\delta u}{\delta t} + \sin\omega\frac{\delta v}{\delta t}\right)$$
$$= -\rho A^2\left((1-\cos^2\chi)\cos\omega\frac{\delta u}{\delta n} + (1-\cos^2\chi)\sin\omega\frac{\delta v}{\delta n} - \sin\omega\frac{\delta u}{\delta e} + \cos\omega\frac{\delta v}{\delta e}\right)$$
$$+\rho A\cos\chi\left(\frac{\cos\omega}{\rho_0}\frac{\partial P_0}{\partial r} + \frac{\sin\omega}{\rho_0}\frac{\partial P_0}{\partial z} - (\lambda-1)\cos\omega\frac{b_0^2}{r}\right) - \rho c^2\frac{u}{r}, \tag{12a}$$

$$-\rho\sin\omega\frac{\delta u}{\delta t} + \rho\cos\omega\frac{\delta v}{\delta t} = \rho A\cos\chi\left(-\sin\omega\frac{\delta u}{\delta n} + \cos\omega\frac{\delta v}{\delta n}\right)$$
$$-\rho\left(\frac{\sin\omega}{\rho_0}\frac{\partial P_0}{\partial r} - \frac{\cos\omega}{\rho_0}\frac{\partial P_0}{\partial z} - (\lambda-1)\sin\omega\frac{b_0^2}{r}\right) - \frac{\partial P}{\partial e}, \tag{12b}$$

where $\cos\chi = (U-V)/A$. From the shock relations (9) we get

$$\cos\omega\frac{\delta u}{\delta t} + \sin\omega\frac{\delta v}{\delta t} = \frac{\delta V}{\delta t}, \qquad -\sin\omega\frac{\delta u}{\delta t} + \cos\omega\frac{\delta v}{\delta t} = V\frac{\delta\omega}{\delta t},$$

and

$$\frac{\delta P}{\delta e} = \left(2 - \frac{1}{\lambda(\lambda-1)E}\right)\rho_0 U\left(\cos\omega\frac{\delta u}{\delta e} + \sin\omega\frac{\delta v}{\delta e}\right)$$
$$+\frac{\rho_0 U^2}{E\lambda^2}\left\{F\frac{\delta}{\delta e}\left(b_0^2\right) + H\frac{\delta}{\delta e}\left(c_0^2\right)\right\} + \frac{\delta P_0}{\delta e} + \left(1-\frac{1}{\lambda}\right)U^2\frac{\delta\rho_0}{\delta e},$$

where

$$H = \{(2c_0^2+\gamma b_0^2)+(2-\gamma)b_0^2\lambda\}^{-1},$$

$$F = \tfrac{1}{2}\{\gamma+(2-\gamma)\lambda\}H,$$

and

$$E = \tfrac{1}{2}\left(\frac{2}{\lambda-1}-\frac{1}{\lambda}+\frac{\gamma-1}{(\gamma+1)-(\gamma-1)\lambda}+(2-\gamma)b_0^2 H\right).$$

It may be noted that

$$(\lambda-1)\lambda E > 1.$$

Using these results and after some simplifications, equations (12) become

$$\frac{\delta P}{\delta t}+\rho A\cos\chi\frac{\delta V}{\delta t}$$

$$= -\rho A^2\left\{\left(1-\cos^2\chi\cos^2\omega\right)\frac{\partial u}{\partial r}-\cos^2\chi\sin\omega\cos\omega\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}\right)+\left(1-\cos^2\chi\sin^2\omega\right)\frac{\partial v}{\partial z}\right\}$$

$$+\rho A\cos\chi\left(\frac{\cos\omega}{\rho_0}\frac{\partial P_0}{\partial r}+\frac{\sin\omega}{\rho_0}\frac{\partial P_0}{\partial z}-(\lambda-1)\cos\omega\frac{b_0^2}{r}\right)-\rho c^2\frac{u}{r} \tag{13a}$$

and

$$V\frac{\delta\omega}{\delta t} = \tfrac{1}{2}A\cos\chi\left[\left(\frac{1}{\lambda(\lambda-1)E}-1\right)\left\{-\sin 2\omega\frac{\partial u}{\partial r}+\cos 2\omega\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}\right)+\sin 2\omega\frac{\partial v}{\partial z}\right\}\right.$$

$$\left.+\left(\frac{1}{\lambda(\lambda-1)E}-3\right)\left(\frac{\partial u}{\partial z}-\frac{\partial v}{\partial r}\right)\right]+\frac{\lambda-1}{\lambda\rho_0}\frac{\delta P_0}{\delta e}-\frac{\lambda-1}{\lambda^2\rho_0}U^2\frac{\delta\rho_0}{\delta e}$$

$$-\frac{U^2}{E\lambda^3}\left\{F\frac{\delta}{\delta e}\left(b_0^2\right)+H\frac{\delta}{\delta e}\left(c_0^2\right)\right\}+(\lambda-1)\sin\omega\frac{b_0^2}{r}. \tag{13b}$$

The vorticity term $(\partial u/\partial z-\partial v/\partial r)$ is eliminated from (13b) with the help of an equation which results from the geometrical condition, namely

$$\delta\omega/\delta t = -\delta U/\delta e. \tag{14}$$

This relation can be proved by remembering that at any point on the shock $h(r,z,t) = 0$, the spatial normal $\boldsymbol{n}$ and the local normal shock velocity U are given by

$$\boldsymbol{n} = \frac{\nabla h}{|\nabla h|} \qquad \text{and} \qquad U = -\frac{\partial h/\partial t}{|\nabla h|}.$$

Then using (11a) we can get the equation (14), which may be rewritten as

$$\frac{\delta\omega}{\delta t} = \frac{\lambda}{2(\lambda-1)}\left(\frac{1}{\lambda(\lambda-1)E}-1\right)\left\{-\sin 2\omega\frac{\partial u}{\partial r}+\cos 2\omega\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}\right)+\sin 2\omega\frac{\partial v}{\partial z}+\left(\frac{\partial u}{\partial z}-\frac{\partial v}{\partial r}\right)\right\}$$

$$-\frac{U}{\lambda(\lambda-1)E}\left\{F\frac{\delta}{\delta e}\left(b_0^2\right)+H\frac{\delta}{\delta e}\left(c_0^2\right)\right\}. \tag{15}$$

With $(\partial u/\partial z-\partial v/\partial r)$ replaced through equation (15), we apply shock relations to equations (13) and get the following equations along the shock ray for the shock parameters λ and ω:

$$\frac{\delta\lambda}{\delta t}=\frac{\lambda^3A^2}{\eta_1U^2}\left\{\left(1-\cos^2\chi\cos^2\omega\right)\frac{\partial u}{\partial r}-\cos^2\chi\sin\omega\cos\omega\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}\right)\right.$$
$$\left.+\left(1-\cos^2\chi\sin^2\omega\right)\frac{\partial v}{\partial z}\right\}$$
$$-3\eta_2U\left\{\left(\frac{2B_0F}{\mu}\frac{\partial B_0}{\partial r}-\eta_3\frac{\partial\rho_0}{\partial r}+\gamma H\frac{\partial p_0}{\partial r}\right)\cos\omega\right.$$
$$\left.+\left(\frac{2B_0F}{\mu}\frac{\partial B_0}{\partial z}-\eta_3\frac{\partial\rho_0}{\partial z}+\gamma H\frac{\partial p_0}{\partial z}\right)\sin\omega\right\}$$
$$-\frac{\gamma\eta_2}{U}\left\{p_0+\frac{\lambda-1}{\lambda}U^2\rho_0+\left(\frac{1}{\gamma}+\frac{1}{2\lambda}-\frac{\lambda}{2}\right)\frac{\lambda B_0^2}{\mu}\right\}\frac{\cos\omega}{r},\tag{16a}$$

$$\frac{\delta\omega}{\delta t}=\frac{\eta_5}{\lambda-1}\left\{-\sin2\omega\frac{\partial u}{\partial r}+\cos2\omega\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}\right)+\sin2\omega\frac{\partial v}{\partial z}\right\}$$
$$+U\left\{\left(\eta_6\frac{\partial p_0}{\partial r}+\eta_7\frac{\partial\rho_0}{\partial r}+\eta_8\frac{\partial B_0}{\partial r}\right)\sin\omega-\left(\eta_6\frac{\partial p_0}{\partial z}+\eta_7\frac{\partial\rho_0}{\partial z}+\eta_8\frac{\partial B_0}{\partial z}\right)\cos\omega\right\}$$
$$+\lambda\eta_5\frac{b_0^2\sin\omega}{Ur},\tag{16b}$$

where

$$\eta_1=3E\lambda^2-3E\lambda-1,\qquad \eta_2=\lambda(\lambda-1)/\eta_1\rho_0,$$
$$\eta_3=Fb_0^2+Hc_0^2-\tfrac{1}{3},\qquad \eta_4=\{1+(3-\lambda)\lambda E\}(\lambda-1),$$
$$\eta_5=\frac{\{1-(\lambda-1)\lambda E\}\lambda}{\eta_4},\qquad \eta_6=\frac{2\gamma H-\{1-(\lambda-1)\lambda E\}\lambda/U^2}{\eta_4\rho_0},$$
$$\eta_7=\frac{1-(\lambda-1)\lambda E-2Fb_0^2-2Hc_0^2}{\eta_4\rho_0},\qquad \eta_8=\frac{[4F-\{1-(\lambda-1)\lambda E\}\lambda/U^2]B_0}{\eta_4\rho_0\mu}.$$

These are the equations which have to be solved together with the equations of the shock ray in order to obtain the run of λ and ω in the medium through any assigned initial point. The last terms in these equations, those with a factor $1/r$, are seen to be the effect of the model and geometry in modifying the shock parameters. However, these equations still contain the spatial derivatives of the velocity components which are as yet unknown. It is possible to obtain further equations by using the compatibility relations along the bicharacteristics of the system of equations (1) through a particular point in order to be able to eliminate these derivatives. But, looking at the distribution of U in the right-hand sides of equations (16) we find that the terms containing these derivatives are small compared with the other terms. In this paper we neglect these small terms and solve the equations

$$\frac{\mathrm{d}r}{\mathrm{d}t} = U\cos\omega\,, \tag{17}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = U\sin\omega\,, \tag{18}$$

$$\begin{aligned}\frac{\mathrm{d}\lambda}{\mathrm{d}t} = -3\eta_2\, U\bigg\{&\left(\frac{2B_0F}{\mu}\frac{\partial B_0}{\partial r} - \eta_3\frac{\partial \rho_0}{\partial r} + \gamma H\frac{\partial p_0}{\partial r}\right)\cos\omega \\ &+\left(\frac{2B_0F}{\mu}\frac{\partial B_0}{\partial z} - \eta_3\frac{\partial \rho_0}{\partial z} + \gamma H\frac{\partial p_0}{\partial z}\right)\sin\omega\bigg\} \\ &-\frac{\gamma\eta_2}{U}\left\{p_0 + \frac{\lambda-1}{\lambda}U^2\rho_0 + \left(\frac{1}{\gamma}+\frac{1}{2\lambda}-\frac{\lambda}{2}\right)\frac{\lambda B_0^2}{\mu}\right\}\frac{\cos\omega}{r}\,,\end{aligned} \tag{19}$$

and

$$\begin{aligned}\frac{\mathrm{d}\omega}{\mathrm{d}t} = U\bigg\{&\left(\eta_6\frac{\partial p_0}{\partial r} + \eta_7\frac{\partial \rho_0}{\partial r} + \eta_8\frac{\partial B_0}{\partial r}\right)\sin\omega \\ &-\left(\eta_6\frac{\partial p_0}{\partial z} + \eta_7\frac{\partial \rho_0}{\partial z} + \eta_8\frac{\partial B_0}{\partial z}\right)\cos\omega\bigg\} + \frac{\lambda b_0^2\,\eta_5}{U}\frac{\sin\omega}{r}\,,\end{aligned} \tag{20}$$

simultaneously, with

$$U^2 = \frac{(2c_0^2+\gamma b_0^2)+(2-\gamma)\lambda b_0^2}{(\gamma+1)-(\gamma-1)\lambda}\lambda\,,$$

to determine the position of the shock element at any time t and the corresponding values of the shock parameters λ and ω. ω gives the orientation of the shock element which is continuously changing, since the shock is moving out in an inhomogeneous medium and in a nonuniform magnetic field. Further, the knowledge of the values of λ and ω enables us to find the flow immediately behind the shock element with the help of the shock relations (9).

VI. Numerical Integration

A solution of the equations (17)–(20) in closed form is not possible and, in general, numerical integration has to be employed. To this end, we consider the polytrope $n = 3$ and assume its central density and radius to be identical with the solar values, so that

$$\rho_c = 7{\cdot}5858\times 10^4 \qquad \text{and} \qquad \bar{R} = 6{\cdot}9598\times 10^8\,,$$

where $\bar{R}$ is the radius; MKS units have been used throughout the calculations. γ has been taken to be equal to $\frac{5}{3}$. The shock is assumed to develop and start spherically outwards at the surface $R = 0{\cdot}3\,\bar{R}$ which approximately forms the inner core.

As indicated in Section I, we follow Ledoux and Walraven (1958) to estimate a reasonable value for the initial shock Mach number, $M_{\mathrm{S,i}}$. The energy liberated inside the surface $R = 0 \cdot 3\,\bar{R}$ is used up in pushing the shock forward (Ôno and Sakashita 1961), which gives the relation

$$PVS_0 = \epsilon Q, \tag{21}$$

where S_0 is the area of the initial surface, Q is the total energy inside it, and ϵ is a constant. The energy

$$Q = \iiint \left(\frac{p_0}{\gamma - 1} + \frac{B_0^2}{2\mu} + \frac{m_0\,\rho_0\,G}{R}\right) R^2 \sin\theta\, \mathrm{d}R\, \mathrm{d}\theta\, \mathrm{d}\phi$$

is the sum of the thermal, magnetic, and gravitational potential energy inside the surface. Shock relations and equilibrium values are used and equation (21) is solved for $M_{\mathrm{s,i}}$ in the equatorial plane for different values of β^2 and ϵ. The curves in Figure 3 reveal the nature of variation of $M_{\mathrm{s,i}}$ with ϵ. We take $M_{\mathrm{s,i}} = 5$ to be a reasonable value, and the rate of energy liberation required for the generation of a shock of this Mach number is read from Figure 3 to be $6 \cdot 3\%$ of Q for $\beta^2 = 0 \cdot 01$ and $5 \cdot 8\%$ of Q for $\beta^2 = 0 \cdot 05$.

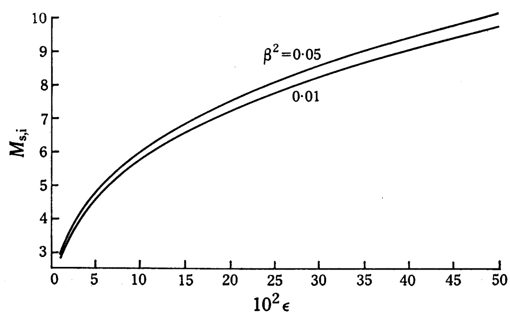

Fig. 3.—Profiles illustrating the variation of $M_{\mathrm{s,i}}$ with ϵ for two values of β^2.

A study has been made of the propagation of shock in magnetic fields of various intensities, which have been chosen to correspond to $\beta^2 = 0 \cdot 01$, $0 \cdot 03$, $0 \cdot 05$, and $0 \cdot 1$ (see Section III). For these characteristic values the maximum value of the field inside the polytrope is of the order of 10^8 G, which agrees well with its estimated value in the stellar interiors. The curves in Figure 1 give some idea of the nature of the magnetic field for $\beta^2 = 0 \cdot 01$; the general pattern remains the same for different values of β^2.

Owing to the symmetry of the magnetic field, the problem is also symmetrical about the r axis. The invariance of the basic equations (1) under the transformation ($z \to -z$, $v \to -v$) confirms this. Therefore, the numerical integration has been carried out only in the upper half of the azimuthal plane (r–z quadrant) in which the initial shock position now becomes a quadrant of the circle $R = 0 \cdot 3\,\bar{R}$.

With the above-mentioned initial and other relevant data, equations (17)–(20) have been integrated in conjunction with the equations (6)–(8) using a fourth-order Runge–Kutta method, on the Monash University CDC 3200 computer, with time steps $\Delta t = 0 \cdot 1$. The integration has been carried out for 10 different points on the initial shock position, unfolding the shock rays nearly as far out as the surface of the polytrope. After every 100 steps of integration in each case the values of the variables have been recorded and the values of the physical quantities just behind the shock computed with the help of equations (9). The position of the shock front at any subsequent time t is obtained as the locus of the positions at time t on the different rays. The front is distorted as the shock moves out and, as in an earlier paper (Sinha 1968*b*), we have expressed the different radial distances obtained for the front at any particular time t as the series

$$R_0(t) + R_2(t)\,\mathrm{T}_2(\cos\theta) + R_4(t)\,\mathrm{T}_4(\cos\theta) + \ldots,$$

where $\mathrm{T}_n(Z) = \cos(n\cos^{-1}Z)$ are the Chebyshev polynomials. The first term $R_0(t)$ gives the mean radius of the front at time t, whereas each of the remaining terms introduces a nonspherical distortion. Similarly, the density ratio λ, pressure ratio Π, and shock Mach number M_s at different times t have also been expressed as the respective series

$$\lambda_0(t) + \lambda_2(t)\,\mathrm{T}_2(\cos\theta) + \lambda_4(t)\,\mathrm{T}_4(\cos\theta) + \ldots,$$

$$\Pi_0(t) + \Pi_2(t)\,\mathrm{T}_2(\cos\theta) + \Pi_4(t)\,\mathrm{T}_4(\cos\theta) + \ldots,$$

and

$$M_0(t) + M_2(t)\,\mathrm{T}_2(\cos\theta) + M_4(t)\,\mathrm{T}_4(\cos\theta) + \ldots.$$

The first term in each case gives the mean value of the function over the front at time t whereas the remaining terms introduce the nonspherically symmetric variations in these quantities.

VII. Results and Discussion

Shock waves with different initial parameters have been studied. For convenience of reference, the various cases corresponding to different initial parameters considered are labelled according to the following scheme:

Case	1	2	3	4	5
$M_{s,i}$	5	5	5	5	7
β^2	0·01	0·03	0·05	0·10	0·01

The labelling of curves in the following graphs also follows this scheme. The results obtained for the geometry of the shock front and the flow behind it as it propagates out in the polytrope are presented and discussed below.

(i) *Behaviour of* λ

Figure 4 depicts the variations of λ_0 and λ_2 with $\bar{R}_0$, where $\bar{R}_0 = R_0/\bar{R}$. The parameter λ_0, and hence λ, increases continuously with $\bar{R}_0$. It is, however, bounded in each case and never exceeds the value four. This result is quite in agreement with the well-known limit for λ given by (10). λ_0 depends significantly on $M_{s,i}$ and increases with increasing values of $M_{s,i}$. Thus the greater the $M_{s,i}$, or the initial shock velocity, the greater is the compression of the matter in the medium. The strength of the magnetic field has no significant influence on λ_0. The parameter λ_2 is small in comparison with λ_0 and it depends much more appreciably on the field strength than on $M_{s,i}$. This is so because λ_2, and similar terms in other Chebyshev expansions described above, is the result of the magnetic field under consideration. λ_2 is found to decrease considerably as the field strength decreases. This result points to the pure gas-dynamical limit ($\beta^2 \to 0$), in which the ratio of the values of any physical variable across the front is uniform over the front.

The ratio of the magnetic field across the front is also equal to λ and so a similar inference can be drawn for the variations of the magnetic field.

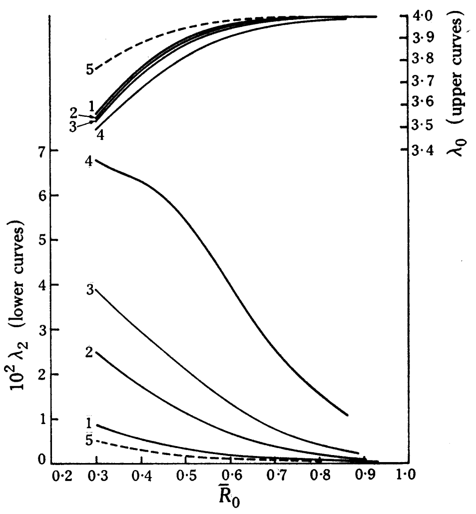

Fig. 4.—Profiles illustrating the density ratio λ as a function of $\bar{R}_0$ for different initial characteristics.

(ii) *Behaviour of Π*

The variations of $\log \Pi_0$ and $\log \Pi_2$ with $\bar{R}_0$ are plotted in Figures 5(*a*) and 5(*b*) respectively. Π_2 is very small in comparison with Π_0 except in the layers beyond $0 \cdot 85\, \bar{R}$. It must be borne in mind that in this region effects due to radiation, which we have not included in this paper, begin to be significant and so our results in the region close to the surface may not be relevant physically. Π_0, and hence Π, continuously increases with $\bar{R}_0$ and does so more rapidly as $M_{s,i}$ increases. However, increasing the magnetic field inhibits this amplification. This is in agreement with equations (9) and also with the "magneto-restraining effect" described by Ôno, Sakashita, and Yamazaki (1960). Π_2 decreases with decreasing field strength.

(iii) *Behaviour of M_s*

As observed in Figures 6(*a*) and 6(*b*) respectively, both M_0 and M_2 continuously increase, though the latter is always very small compared to the former. The magnetic field has a tendency to inhibit the growth of M_s and in the figures this effect becomes very clear in regions far from the centre where the field strength is very small. This agrees with equation (9g). M_2, which is the main nonspherical part of M_s, decreases as the field strength decreases. The overall variation of M_s is more sensitive to a change of $M_{s,i}$ than that of field strength. M_s increases with increasing $M_{s,i}$. The results near the surface may again be deceptive, more so in this case due to the rapidly vanishing density of matter in which the shock is moving.

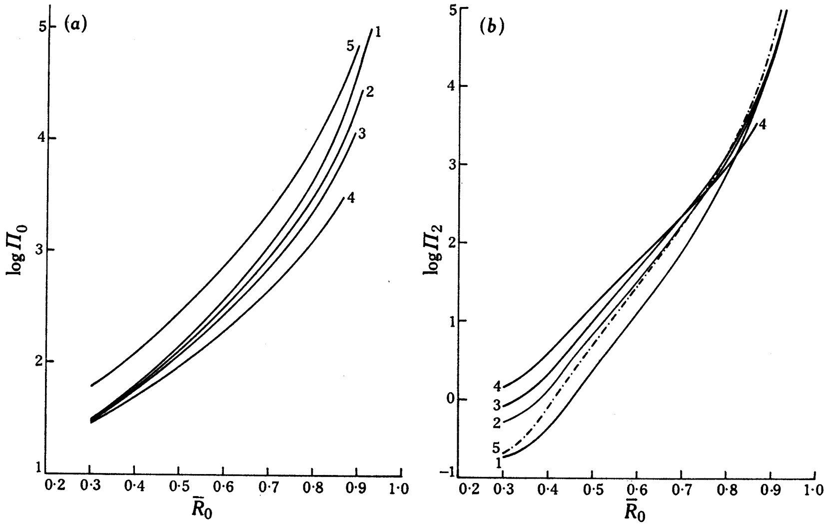

Fig. 5.—Variation of (*a*) $\log \Pi_0$ and (*b*) $\log \Pi_2$ as functions of $\bar{R}_0$ for different initial characteristics.

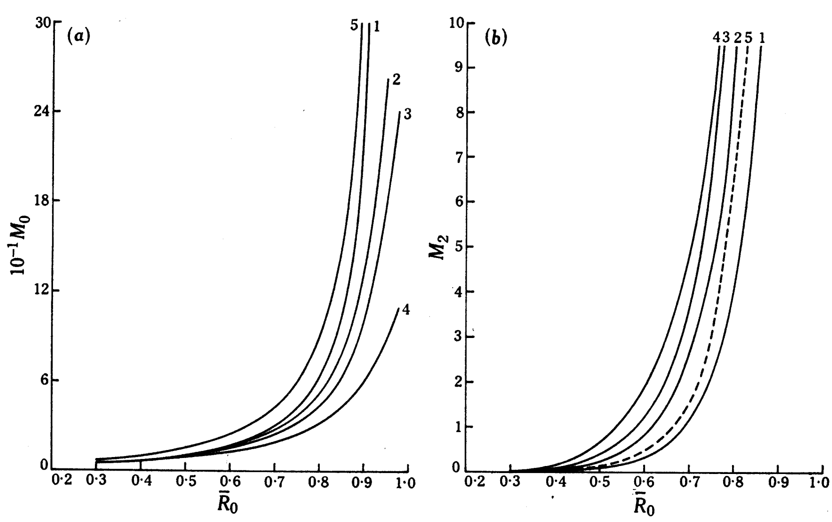

Fig. 6.—Profiles illustrating the variation of (*a*) M_0 and (*b*) M_2 with $\bar{R}_0$ for different initial characteristics.

The shock velocity itself is also continuously increasing, nevertheless, non-uniformly at different points on the front. However, U increases slowly compared with M_s. For example, starting at $0 \cdot 3\,\bar{R}_0$, its value increases (in 10^6 units) (1) from

$1\cdot95$ to $7\cdot47$ at $0\cdot76\,\bar{R}_0$ in case 1, (2) from $1\cdot96$ to $6\cdot91$ at $0\cdot79\,\bar{R}_0$ in case 3, and (3) from $2\cdot73$ to $10\cdot02$ at $0\cdot75\,\bar{R}_0$ in case 5.

VIII. Geometry of the Front

The coefficients R_{2j} ($j = 1, 2, 3, \ldots$) in the Chebyshev expansion of R diminish rapidly so that

$$R = R_0 + R_2\,\mathrm{T}_2(\cos\theta)$$

may be taken as a first approximation. This shows that the front is elliptical and, since $\mathrm{T}_2(\cos\theta)$ is $+1$ along the axis and -1 at the equator, its oblateness δ is given by

$$\delta = 2R_2/R_0\,.$$

Figure 7 depicts the variation of δ with $\bar{R}_0$. The oblateness preserves its nature throughout the propagation and tends to vanish as β^2 becomes vanishingly small, which is the pure gas-dynamical limit. δ is not altered appreciably by a change in

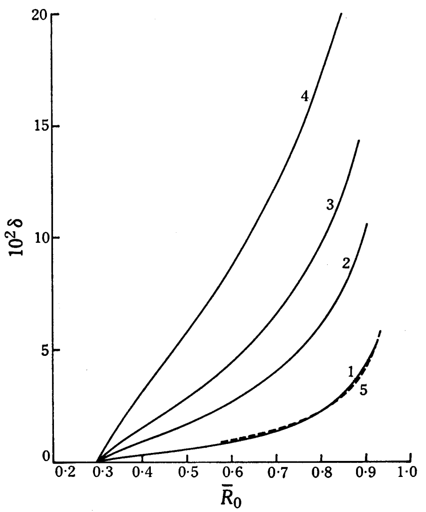

Fig. 7.—Variation of the oblateness δ of the shock front as a function of $\bar{R}_0$ for different initial characteristics.

$M_{\mathrm{s,i}}$, but the field has a remarkable influence upon its variation. The more intense is the field, the greater is the distortion that the front undergoes. This reveals an interesting feature of the shock that, in the event of its reaching the surface, it reaches first at the poles and then successively at places on the surface with diminishing latitudes down to the equator. This order of reaching the surface is opposite to the one obtained for a shock propagating in the presence of a poloidal magnetic field (Sinha 1968*b*), where it is found that the shock reaches first at the equator and then successively through the intermediate points to the poles.

IX. Particle Velocity

The particle velocity behind the shock continuously rises as the shock propagates outwards. This is a consequence of equations (9) and the fact that U as well as λ continuously increase. The escape velocity $V_{\rm esc}$ at $\xi = \xi_1$ is given by

$$\begin{aligned} V_{\rm esc}^2 &= N\left(\frac{1}{\xi_1}\int_0^{\xi_1} \rho_0\,\xi^2\,\mathrm{d}\xi + \int_{\xi_1}^{\xi_0} \rho_0\,\xi\,\mathrm{d}\xi\right) \\ &= \rho_{\rm c} N\left\{\left(\theta_0+\beta^2(\Psi_0+\tfrac{2}{3}\xi^2\theta_0^3)\right)_{\xi_1} - \xi_0\left(\frac{\mathrm{d}\theta_0}{\mathrm{d}\,\xi}\right)_{\xi_0} - \beta^2\left(\xi\frac{\mathrm{d}\Psi_0}{\mathrm{d}\xi}+\Psi_0\right)_{\xi_0}\right\}, \end{aligned}$$

on using equations (6)–(8), together with $N = 4\pi G\alpha^2$. In the present case it is found that, with $M_{\rm s,i} = 5$ and 7, $V_{\rm esc}$ is always attained well inside the polytrope. However, as $M_{\rm s,i}$ decreases $V_{\rm esc}$ is attained farther and farther from the centre. Nevertheless, it follows that a shock of moderate strength will always lead to an ejection of mass from the polytrope.

The toroidal magnetic field tends to impede the rise of particle velocity by restraining the growth of U, and hence plays a negative role in the ejection of mass from the polytrope due to a shock. This is in contrast to the effect of a poloidal magnetic field (Sinha 1968*b*) which tends to increase particle velocity. This may be of importance in the study of novae explosions. The magnetic field may bring forth other effects also. Since the particle velocity is no longer uniform over the front, complex ejection patterns may be formed in the event of a mass separation. However, if $V_{\rm esc}$ is not attained anywhere inside the polytrope, the nonuniform rise in temperature brought about by the passage of the shock in the layer behind it may lead to the generation of circulations in the regions close to the surface.

The results of the solution of equations (16), supplemented by equations (17) and (18) are dealt with in the following paper (Part II, pp. 605–12).

X. References

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