ON SOLVING THE EQUATION OF RADIATIVE TRANSFER FOR CONSERVATIVE NON-UNIFORM MEDIA

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Summary

The equation of radiative transfer in a form suitable for non-uniform semi-infinite media in radiative equilibrium is solved for the attenuation coefficient $\varkappa$ compatible with models for total intensity J as functions of two variables, subject to the boundary condition of zero inward flux at the surface.

The method is tested in a case where a solution has been previously obtained and then applied to an axially symmetric medium.

I. INTRODUCTION

Recent studies of solar granulation and sunspot phenomena have emphasized the importance of solutions of the equation of radiative transfer for non-uniform media. Giovanelli (1959) produced the approximate form of the transfer equation, suitable for non-uniform media in radiative equilibrium,

$$\nabla^2 J = \frac{1}{\varkappa} \nabla J \cdot \nabla \varkappa, \tag{1}$$

where J, the total intensity of radiation, and $\varkappa$, the attenuation coefficient of the medium, are both functions of position.

This equation is usually solved subject to the boundary condition that the inward flux is zero at the surface. In Cartesian or cylindrical polar coordinates, if the z-axis is normal to the surface $z=0$, and positive outwards, this takes the form

$$\varkappa J + \frac{2}{3} \frac{\mathrm{d}J}{\mathrm{d}z} = 0, \tag{2}$$

when $z=0$.

In the customary method of solution of (1) and (2), a particular model for the attenuation coefficient $\varkappa$ is assumed and (1) is solved for J subject to (2). Giovanelli (1959) considered the model semi-infinite medium

$$\begin{aligned} \varkappa &= \varkappa_0(1+\alpha \cos lx) \quad &(z<0), \\ &= 0 &(z>0), \end{aligned}$$

where $\varkappa_0$, α, and l are constants. Assuming α small, he obtained the approximate solution

$$J(x,z) = Az + B + C\mathrm{e}^{lz} \cos lx,$$

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where A, B, and C are constants determined by the boundary condition, and C/A is small. Higher harmonic terms are assumed negligible. Wilson (1960), using the model

$$\begin{aligned} \varkappa &= \varkappa_0 e^{-\nu z}(1+\alpha \cos lx) \quad (z<0), \\ &= 0 \quad (z>0), \end{aligned} \tag{3}$$

where ν is constant, obtained a similar solution

$$J(x,z) = A e^{-\nu z} + B + C e^{pz} \cos lx, \tag{4}$$

where

$$p^2 + \nu p - l^2 = 0, \quad p>0, \tag{5}$$

and C/A is small.

As a first step towards the study of radiative transfer in non-uniform regions of the solar atmosphere, this method of solution of (1) has the disadvantage that it is necessary to propose a model for the attenuation coefficient, and assess its likelihood by comparing the solution of (1) for J with the information which can be obtained from observations of the directional intensity at the surface. In all but the simplest cases, it is by no means obvious how best to choose the models for $\varkappa$.

In the present paper a new and, for many purposes, more direct approach to equation (1) is suggested and is tested by application to the sinusoidal case (equations (3), (4), and (5)) which has already been studied. It is then applied to an exponential medium exhibiting axial symmetry.

II. The Solution of the Transfer Equation

Although the total intensity J is not directly observable, its variation near the surface $z=0$ in a direction parallel with the surface must be closely related to the variation in directional intensity at $z=0$. It is therefore logical to assume a model for J throughout the medium, based on observations, and use equations (1) and (2) to find the attenuation coefficient $\varkappa$ required to produce such radiation.

Assuming J and $\varkappa$ to be point functions of two variables (e.g. x and z, or, using cylindrical polar coordinates for an axially symmetric model, r and z), equation (1) may be written

$$\frac{\partial}{\partial x}\left(\frac{1}{\varkappa}\,\frac{\partial J}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\varkappa}\,\frac{\partial J}{\partial z}\right) = 0,$$

which is a necessary and sufficient condition for the existence of a function $\varphi(x,z)$ such that

$$\frac{\partial \varphi}{\partial z} = \frac{1}{\varkappa}\,\frac{\partial J}{\partial x} + \psi(z), \tag{6}$$

$$\frac{\partial \varphi}{\partial x} = -\frac{1}{\varkappa}\,\frac{\partial J}{\partial z} + \eta(x), \tag{7}$$

where $\psi(z)$ and $\eta(x)$ are arbitrary functions of one variable. Thus if $\varphi(x,z)$ can be found, $\varkappa(x,z)$ can be determined from (6) and (7), $\psi(z)$ and $\eta(x)$ being chosen so that $\varkappa$ is unique.

Eliminating $\varkappa$ from (7) yields

$$\frac{\partial\Phi}{\partial z}\bigg/\frac{\partial\Phi}{\partial x}=-\frac{\partial J}{\partial x}\bigg/\frac{\partial J}{\partial z}, \tag{8}$$

where $\partial\Phi/\partial z=\partial\varphi/\partial z-\psi(z)$ and $\partial\Phi/\partial x=\partial\varphi/\partial x-\eta(x)$.
Thus

$$\Phi(x,z)=\varphi(x,z)+X(x)+Z(z), \tag{9}$$

where X and Z are again arbitrary functions of one variable.

Equation (8) yields

$$\frac{\partial\Phi}{\partial x}\,\frac{\partial J}{\partial x}+\frac{\partial\Phi}{\partial z}\,\frac{\partial J}{\partial z}=0,$$

or

$$\nabla\Phi\cdot\nabla J=0, \tag{10}$$

whence, if $J(x,z)$=constant defines a family of curves in the x,z plane, $\Phi(x,z)$=constant defines an orthogonal family. Given $J(x,z)$, therefore, we may always find Φ and hence $\varphi(x,z)$ to within arbitrary functions of one variable, and an accurate solution of (1) is thus available. It is emphasized, however, that any solution of (1) is subject to the boundary condition (2), and that if the accurate solution fails on the boundary, it is necessary to consider approximate solutions which satisfy (1) and (2) to the same order of accuracy.

III. Application to the Sinusoidal Case

It is instructive to apply this method to the model for J given by equation (5), i.e.

$$J(x,z)=A\mathrm{e}^{-\nu z}+B+C\mathrm{e}^{pz}\cos lx.$$

The family of curves $J(x,z)$=constant has gradient given by

$$\frac{\mathrm{d}x}{\mathrm{d}z}=-\frac{A\nu\mathrm{e}^{-\nu z}-Cp\mathrm{e}^{pz}\cos lx}{Cl\mathrm{e}^{pz}\sin lx}.$$

Hence the differential equation for the orthogonal family is

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{A\nu-Cp\mathrm{e}^{(p+\nu)z}\cos lx}{Cl\mathrm{e}^{(p+\nu)z}\sin lx}.$$

Substituting $\zeta=\mathrm{e}^{(p+\nu)z}$ yields

$$\frac{\mathrm{d}\zeta}{\mathrm{d}x}+l\lambda\zeta\cot lx=\frac{\mu}{\sin lx}, \tag{11}$$

where $\lambda=p(p+\nu)/l^2$, and $\mu=A\nu(p+\nu)/Cl$.
This equation has the solution

$$\zeta(\sin lx)^{\lambda}=\int\mu(\sin lx)^{\lambda-1}\mathrm{d}x+\Phi,$$

and the family of curves Φ=constant is given by

$$\Phi(x,z)=e^{(p+\nu)z}(\sin lx)^{\lambda}-\int\mu(\sin lx)^{\lambda-1}dx, \tag{12}$$

whence $\varphi(x,z)$ is determined to within arbitrary functions of one variable. Substitution in (7) yields

$$\varkappa=\frac{Cl}{(p+\nu)}\,\frac{e^{-\nu z}(\sin lx)^{1-\lambda}}{[1+(Cl/A\nu)\eta(x)(\sin lx)^{1-\lambda}/\{1-(Cp/A\nu)e^{(p+\nu)z}\cos lx\}]}, \tag{13}$$

while from (6)

$$\varkappa=\frac{Cl}{(p+\nu)}\,\frac{e^{-\nu z}(\sin lx)^{1-\lambda}}{\{1+\psi(z)e^{-(p+\nu)z}/(p+\nu)(\sin lx)^{\lambda}\}}. \tag{14}$$

A unique function for $\varkappa$ is obtained if the arbitrary functions $\eta(x)$ and $\psi(z)$ are taken to be zero. Thus

$$\varkappa=\varkappa_0 e^{-\nu z}(\sin lx)^{1-\lambda}, \tag{15}$$

where $\varkappa_0=Cl/(p+\nu)$.

If $p^2+\nu p-l^2=0$, as in (5), $1-\lambda=0$ and (15) becomes

$$\varkappa=\varkappa_0 e^{-\nu z}. \tag{16}$$

Applying the boundary condition (2) to the solution (16) at $z=0$ and $\cos lx=\pm 1$ yields

$$\left.\begin{aligned}\left(\varkappa_0-\frac{2}{3}\nu\right)A+\varkappa_0 B+\left(\varkappa_0+\frac{2}{3}p\right)C=0,\\ \left(\varkappa_0-\frac{2}{3}\nu\right)A+\varkappa_0 B-\left(\varkappa_0+\frac{2}{3}p\right)C=0,\end{aligned}\right\} \tag{17}$$

which may be satisfied only if C is zero. If the ratio $C/(A+B)$ is small, (17) is satisfied only if first-order terms are ignored. Thus (16), although an accurate solution to (1), is not a satisfactory solution to the problem and it is necessary to consider approximate solutions.

As a first approximation the ratio C/A is assumed small and in (13) non-zero values of $\eta(x)$ are investigated. Provided $\eta(x)$ is of order not greater than unity, (13) yields an approximation correct to first order terms

$$\varkappa=\varkappa_0 e^{-\nu z}\left[1-\frac{Cl}{A\nu}\eta(x)+O\left(\frac{C}{A}\right)^2\right]$$

or

$$\varkappa=\varkappa_0 e^{-\nu z}\{1+\alpha\eta(x)\}, \tag{18}$$

where α is of order C/A and $\eta(x)$ is arbitrary but of order not greater than unity. The boundary conditions suggest the choice

$$\eta(x)=\cos lx,$$

and in this case (2) is satisfied correct to the first order provided

$$\alpha=\frac{C}{A+B}\left(1+\frac{2p}{3\varkappa_0}\right), \tag{19}$$

and B is related to A by

$$B=A\left(\frac{2\nu}{3\varkappa_0}-1\right).$$

As (18) also satisfies equation (1) to the same order, it is with (4) a satisfactory first-order solution to the problem. Equation (14) has not been used to obtain this approximate solution, since partial differentiation has eliminated the leading constant A.

This method verifies the solutions (3) and (4) previously obtained, but provides a more detailed account of the approximations used.

IV. Application to an Axially Symmetric Medium

As a first step towards a solution of the problem of radiative transfer in the neighbourhood of a sunspot, an axially symmetric model of $J(r,z)$ is chosen which has a minimum value at $r=0$ and increases to a finite limit as r approaches infinity, the dependence of $J(r,z)$ on r decreasing with depth. A function having these properties is

$$J=A\mathrm{e}^{-\nu z}+B+C\mathrm{e}^{pz-kr^2}, \tag{20}$$

where A must be positive, C negative, and p and k must be positive. Equations (6) et seq. apply provided cylindrical polar coordinates are used, x being replaced by r, and $\varkappa(r,z)$ being assumed independent of θ.

Proceeding as before, the function φ is found to be given by

$$\varphi=r^{\lambda}\mathrm{e}^{(p+\nu)z}+R(r)+Z(z),$$

where $\lambda=p(p+\nu)/2k$, and this yields the accurate solution of (1) for $\varkappa$,

$$\varkappa=\varkappa_0 r^{1-\lambda}\mathrm{e}^{-\nu z-kr^2}, \tag{21}$$

where $\varkappa_0=2Ck/(p+\nu)$.

In the special case $p^2+p\nu-2k=0$, $1-\lambda=0$, and the solution becomes

$$\varkappa=\varkappa_0\mathrm{e}^{-\nu z-kr^2}. \tag{22}$$

Applying the boundary condition (2) at $z=0$ yields

$$\varkappa_0\mathrm{e}^{-kr^2}(A+B+C\mathrm{e}^{-kr^2})+\frac{2}{3}(-A\nu+Cp\mathrm{e}^{-kr^2})=0. \tag{23}$$

For a first-order solution, the boundary condition must be satisfied at two values of the horizontal coordinate. At $r=0$ (23) yields

$$A(1-2\nu/3\varkappa_0)+B+C(1+2p/3\varkappa_0)=0, \tag{24}$$

and at any finite value $r=R$,

$$A(1-2\nu\mathrm{e}^{kR^2}/3\varkappa_0)+B+C(\mathrm{e}^{-kR^2}+2p/3\varkappa_0)=0. \tag{25}$$

Thus, provided the ratios C/A and B/A satisfy (24) and (25), (20) and (22) form a first-order solution of (1) and (2) in the axially symmetric case. Since the features of the total intensity J are determined by the attenuation coefficient and the boundary condition, it is to be expected that the solution imposes the restrictions (24) and (25) on the values of A, B, and C. The absolute magnitude of A is determined by the net flux at $z=-\infty$, and $\varkappa_0$ is given by (21).

A difficulty arises, however, in considering the limit as r approaches infinity. Equation (2) becomes

$$J=A\mathrm{e}^{-\nu z}+B,$$

which entails the solution

$$\varkappa=\varkappa_0\mathrm{e}^{-\nu z},$$

whereas in (22) the limit of $\varkappa_0\mathrm{e}^{-\nu z-kr^2}$ is zero. However, equation (25) requires $A=0$ when R approaches infinity, and the case is trivial. Thus the first-order solution given by (22) is available only over a restricted range of the horizontal coordinate.

An approximate solution may be obtained over the range $0\leqslant r<\infty$ by considering the discontinuous function

$$\left.\begin{aligned}\varkappa&=\varkappa_0\mathrm{e}^{-\nu z-kr^2}, & 0\leqslant r\leqslant R,\\ \varkappa&=\varkappa'\mathrm{e}^{-\nu z}, & r>R,\end{aligned}\right\} \tag{26}$$

On the boundary $z=0$, equations (24) and (25) hold for the range $0\leqslant r\leqslant R$, while at $r=\infty$ equation (2) yields

$$\varkappa'(A+B)-\tfrac{2}{3}A\nu=0,$$

therefore

$$\varkappa'=\tfrac{2}{3}\nu/(1+B/A).$$

The solution is continuous at $r=R$ provided R is chosen to satisfy

$$\varkappa'=\varkappa_0\mathrm{e}^{-kR^2}.$$

The partial derivative $\partial\varkappa/\partial r$ is discontinuous at this point, but the discontinuity is small if R is large.

As an alternative approximate solution which is analytic at all points in the range $0\leqslant r<\infty$, equations (26) suggest

$$\varkappa=\varkappa_0\mathrm{e}^{-\nu z}(\beta+\mathrm{e}^{-kr^2}), \tag{27}$$

where β is a constant. On the boundary $z=0$ this solution yields consistent equations at $r=0$, $r=R$, and $r=\infty$. Equation (1) is satisfied correct to first-order terms in C/A when r is small and when r is large. For values of r of order unity, however, it is satisfied only provided terms of order C/A may be ignored.

V. Conclusion

Giovanelli's form of the transfer equation for non-uniform media has been shown to be readily soluble for the attenuation coefficient $\varkappa$, given the total intensity J as a function of two coordinates.

The method has been applied to the case

$$J = A\mathrm{e}^{-\nu z} + B + C\mathrm{e}^{pz} \cos lx,$$

with boundary condition at $z=0$

$$\varkappa J + \tfrac{2}{3}\mathrm{d}J/\mathrm{d}z = 0,$$

and the approximate solution, correct to first-order terms, is found to be in agreement with a previous solution (Wilson 1960).

The axially symmetric case for which

$$J = A\mathrm{e}^{-\nu z} + B + C\mathrm{e}^{pz - kr^2}$$

with a similar boundary condition at $z=0$ yields the solution

$$\varkappa = \varkappa_0 \mathrm{e}^{-\nu z - kr^2},$$

which satisfies the boundary condition to the first order over a finite range of r, provided two relations hold between A, B, and C. This indicates the approximate solution in the range $0 \leqslant r < \infty$ of the form

$$\begin{aligned} \varkappa &= \varkappa_0 \mathrm{e}^{-\nu z - kr^2}, & 0 \leqslant r \leqslant R, \\ \varkappa &= \varkappa' \mathrm{e}^{-\nu z}, & r > R, \end{aligned}$$

where, if R satisfies

$$\varkappa_0 \mathrm{e}^{-kR^2} = \tfrac{2}{3}\nu/(1 + B/A),$$

$\varkappa$ is continuous for $0 \leqslant r < \infty$, but $\partial\varkappa/\partial r$ is discontinuous at $r=R$.

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VII. Rfeerences

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