THE MULTIVARIATE t-DISTRIBUTION ASSOCIATED WITH A SET OF NORMAL SAMPLE DEVIATES

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Summary

This paper gives a short account of the more important properties of the multivariate t-distribution, which arises in association with a set of normal sample deviates.

I. INTRODUCTION

The multivariate t-distribution described herein was encountered in the course of a recent investigation of the frequency distribution of the spectrographic error which occurs in the D.C. arc excitation of samples of soil and plant ash (Oertel and Cornish 1953). The original observations consisted of triplicate and sextuplicate determinations of copper, manganese, molybdenum, and tin in samples containing varying amounts of each of these elements, and it was found that the variance of measured line intensity was proportional to the square of the mean intensity. A logarithmic transformation stabilized the variance, which, for any individual element, could then be estimated from all the samples involving that element, and by subsequent standardization all the values of the new metric could be placed on a comparable basis. The standardized variates appeared to be normally distributed, and it became desirable, for various reasons, to test this point.

In providing tests of normality, two general lines of approach have been followed :

(1) a normal distribution is fitted to the sample data, and the χ^2 test of goodness of fit is applied ;

(2) certain functions of the sample moments are calculated, and the significance of their departure from expectations based on the assumption of normality is examined,

but neither of these procedures, as ordinarily used, was suitable owing to the particular nature of the data. For this reason, and as the situation is likely to arise frequently in practice, other means were sought to make the necessary tests. A very limited number of exact tests is available (Fisher 1946), and reasonably accurate approximate tests have been devised by Pearson and Welch (1937) using as a base either a standard test of type (2) above or the work of Geary (1935, 1936). To meet the situation, other exact tests for small samples

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are required to supplement those given by Fisher, and consequently this particular t-distribution has been studied because it offered distinct possibilities in this connexion.

II. Distribution Function of Normal Sample Deviates

Suppose $x_1, x_2, \ldots, x_n$ is a random sample of observations from a normal distribution specified by a mean ξ and variance σ^2.

Let

$$\bar{x} = \sum_{i=1}^{n} x_i/n$$

be the sample mean, and

$$y_i = x_i - \bar{x}$$

be the deviates from the sample mean. The distribution function of one deviate is a classical result; Fisher (1920) gave the distribution of two deviates, and Irwin (1929) established the general non-singular distribution of $(n-1)$ deviates, which, without loss of generality, may be assumed to be the first $(n-1)$. In conformity with the notation used herein, the general distribution may be stated in the following manner:

If $x_1, x_2, \ldots, x_n$ are distributed in a multivariate normal distribution with variance-covariance matrix $\sigma^2\mathbf{I}_n$,* then $y_1, y_2, \ldots, y_{n-1}$ and $\bar{x}$ are distributed in a non-singular multivariate normal distribution with variance-covariance matrix

$$\mathbf{V} = \sigma^2 \begin{bmatrix} (n-1)/n & -1/n & \cdot & \cdot & \cdot & -1/n & 0 \\ -1/n & (n-1)/n & \cdot & \cdot & \cdot & -1/n & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1/n & -1/n & \cdot & \cdot & \cdot & (n-1)/n & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1/n \end{bmatrix}$$

of order $n \times n$.

The deviates $y_1, y_2, \ldots, y_{n-1}$ are thus distributed independently of $\bar{x}$. Alternatively, $y_1, y_2, \ldots, y_n$ are distributed in a singular multivariate normal distribution with variance-covariance matrix

$$\sigma^2 \begin{bmatrix} (n-1)/n & -1/n & \cdot & \cdot & \cdot & -1/n \\ -1/n & (n-1)/n & \cdot & \cdot & \cdot & -1/n \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1/n & -1/n & \cdot & \cdot & \cdot & (n-1)/n \end{bmatrix}$$

of order $n \times n$ and rank $(n-1)$.

III. The Multivariate t-Distribution

Let $\mathbf{B}$ denote the leading submatrix of order $(n-1)\times(n-1)$ in the matrix $\mathbf{V}$ of Section II. If s^2 is an estimate of σ^2, based on ν degrees of freedom and

* The symbol $\mathbf{I}$ will designate the unit matrix, and the attached subscript will indicate its order.

distributed independently of $y_1, y_2, \ldots, y_{n-1}$, the distribution function of the y_i and s may be written

$$\frac{|\mathbf{B}|^{-\frac{1}{2}}}{(2\pi)^{(n-1)/2}} \frac{\nu^{\nu/2}}{2^{(\nu-2)/2}\sigma^{\nu}\Gamma\nu/2} \int \cdots \int e^{-\mathbf{y}'\mathbf{B}^{-1}\mathbf{y}/2} s^{\nu-1} e^{-\nu s^2/2\sigma^2} ds\,d\mathbf{y},^{*}$$

the multiple integral being taken over the region defined by the inequalities

$$-\infty \leqslant y_i \leqslant Y_i, \quad i=1,2,\ldots,n-1,$$
$$0 \leqslant s \leqslant S.$$

$\mathbf{B}^{-1}$ is the reciprocal matrix of $\mathbf{B}$, and $\mathbf{y}'$ is the row vector $[y_1\, y_2 \ldots y_{n-1}]$.

Make the non-singular transformation indicated by the matrix equation

$$\mathbf{t} = \mathbf{Q}\mathbf{y},$$

where $\mathbf{Q}$ is a diagonal matrix, whose diagonal elements are each equal to

$$\frac{1}{s}\sqrt{\frac{n}{n-1}}.$$

The jacobian is $|\mathbf{Q}|^{-1}$, and the distribution function becomes

$$\frac{\nu^{\nu/2}}{(2\pi)^{(n-1)/2}2^{(\nu-2)/2}\sigma^{\nu}\Gamma\nu/2}\int \cdots \int |\mathbf{QBQ}'|^{-\frac{1}{2}} e^{-\mathbf{t}'(\mathbf{QBQ}')^{-1}\mathbf{t}/2} e^{-\nu s^2/2\sigma^2} s^{\nu-1} d\mathbf{t}\,ds,$$

in which the domain of integration is defined by

$$-\infty \leqslant t_i \leqslant T_i, \quad i=1,2,\ldots,n-1,$$
$$0 \leqslant s \leqslant S.$$

Since

$$\mathbf{QBQ}' = \frac{\sigma^2}{s^2}\begin{bmatrix} 1 & -1/(n-1) & \cdot & \cdot & \cdot & -1/(n-1) \\ -1/(n-1) & 1 & \cdot & \cdot & \cdot & -1/(n-1) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1/(n-1) & -1/(n-1) & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$
$$= \frac{\sigma^2}{s^2}\mathbf{R} \text{ say,}$$

it follows that

$$|\mathbf{QBQ}'|^{-\frac{1}{2}} = \left(\frac{s}{\sigma}\right)^{n-1} |\mathbf{R}|^{-\frac{1}{2}},$$

and integrating for s from 0 to ∞, the multivariate distribution function of $t_1, t_2, \ldots, t_{n-1}$ is

$$\frac{\Gamma(\nu+n-1)/2\,|\mathbf{R}|^{-\frac{1}{2}}}{(\pi\nu)^{(n-1)/2}\Gamma\nu/2}\int \cdots \int (1+\mathbf{t}'\mathbf{R}^{-1}\mathbf{t}/\nu)^{-(\nu+n-1)/2} d\mathbf{t}, \quad \ldots . \quad (1)$$

the integral being taken over the region defined by the inequalities

$$-\infty \leqslant t_i \leqslant T_i, \quad i=1,2,\ldots,n-1.$$

The limiting form of the distribution function (1), as $\nu \to \infty$, is that of a multivariate normal distribution with variance-covariance matrix $\mathbf{R}$.

* The notation $\Gamma m/2$ denotes $\Gamma(m/2)$ throughout.

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IV. Properties of the Distribution

(a) *Mean Values*

The vector of mean values, $E(\mathbf{t})$, is obviously null, and consequently the variance-covariance matrix is

$$\begin{aligned} E(\mathbf{tt}') &= E\left(\frac{n}{s^2(n-1)}\mathbf{yy}'\right) \\ &= E\left(\frac{1}{s^2}\right)E\left(\frac{n}{n-1}\mathbf{yy}'\right) \\ &= \frac{\nu}{\sigma^2(\nu-2)} \cdot \sigma^2\mathbf{R} \\ &= \frac{\nu}{\nu-2}\mathbf{R}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2) \end{aligned}$$

(b) *Distribution of Linear Functions*

Suppose

$$\mathbf{x} = \mathbf{Ht}$$

are any $p \leqslant (n-1)$ linearly independent linear functions of the t_i. The distribution of these functions can be derived directly but is more conveniently obtained from the corresponding result for normally distributed variates, namely, that if $y_1, y_2, \ldots, y_{n-1}$ are distributed in a multivariate normal distribution with variance-covariance matrix $\mathbf{B}$, then the variates

$$\mathbf{z} = \mathbf{Hy}$$

are distributed in a multivariate normal distribution with variance-covariance matrix $\mathbf{HBH}'$.

The distribution function of the z_i is thus

$$\frac{|\mathbf{HBH}'|^{-\frac{1}{2}}}{(2\pi)^{p/2}}\int\ldots\int e^{-\mathbf{z}'(\mathbf{HBH}')^{-1}\mathbf{z}/2}d\mathbf{z}$$

over the region defined by the inequalities

$$z_i \leqslant Z_i, \qquad i=1,2,\ldots,p,$$

and consequently, the distribution function of

$$\mathbf{x} = \mathbf{Ht} = \mathbf{HQy} = \mathbf{Qz}$$

and s is

$$\frac{\nu^{\frac{1}{2}\nu}}{(2\pi)^{p/2}2^{(\nu-2)/2}\sigma^\nu\Gamma\nu/2}\int\ldots\int|\mathbf{QHBH}'\mathbf{Q}'|^{-\frac{1}{2}}e^{-\mathbf{x}'(\mathbf{QHBH}'\mathbf{Q}')^{-1}\mathbf{x}/2}e^{-\nu s^2/2\sigma^2}s^{\nu-1}ds\,d\mathbf{z}$$

over the region defined by

$$\begin{aligned} x_i &\leqslant X_i, \qquad i=1,2,\ldots,p, \\ s &\leqslant S. \end{aligned}$$

Since

$$\mathbf{QHBH}'\mathbf{Q}' = \frac{\sigma^2}{s^2}\mathbf{HRH}',$$

integration for s from 0 to ∞ leaves the distribution of $x_1, x_2, \ldots, x_p$ in the form

$$\frac{\Gamma(\nu+p)/2 \mid \mathbf{HRH}' \mid^{-\frac{1}{2}}}{(\pi\nu)^{p/2}\Gamma\nu/2}\int \ldots \int \{1+\mathbf{x}'(\mathbf{HRH}')^{-1}\mathbf{x}/\nu\}^{-(\nu+p)/2}d\mathbf{x}. \quad \ldots (3)$$

over the region defined by

$$x_i \leqslant X_i, \qquad i=1,2,\ldots,p,$$

that is, a multivariate t-distribution of order p characterized by the matrix $(\mathbf{HRH}')^{-1}$.

(c) *Marginal Distribution of* $\mathrm{t}_1, \mathrm{t}_2, \ldots, \mathrm{t}_\mathrm{r}$

The marginal distribution of $t_1, t_2, \ldots, t_r$, $r<(n-1)$, follows from the previous result by taking

$$\mathbf{H}=\begin{bmatrix}\mathbf{I}_r & \cdot \\ \cdot & \cdot\end{bmatrix},$$

and is thus a multivariate t-distribution of order r, characterized by the matrix $\mathbf{R}_1^{-1}$, $\mathbf{R}_1$ being the leading submatrix of order $r\times r$ in $\mathbf{R}$. The variance-covariance matrix is $\{\nu/(\nu-2)\}\mathbf{R}_1$, and consequently, the variances and covariances of the marginal distribution are identical with their values in the original distribution. The limiting form of this distribution as $\nu\to\infty$ is the marginal multivariate normal distribution with variance-covariance matrix $\mathbf{R}_1$.

(d) *Conditional Distribution of* $\mathrm{t}_1, \mathrm{t}_2, \ldots, \mathrm{t}_\mathrm{r}$

To find the conditional distribution of $t_1, t_2, \ldots, t_r$ when $t_{r+1}, \ldots, t_{n-1}$ are given specified values, first partition the matrix $\mathbf{R}^{-1}$ so that it takes the form

$$\begin{bmatrix}\mathbf{R}_1 & \mathbf{R}_3 \\ \mathbf{R}_3' & \mathbf{R}_2\end{bmatrix}$$

where the submatrices $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$ respectively are of orders $r\times r$, $\{n-(r+1)\}\times\{n-(r+1)\}$, and $r\times\{n-(r+1)\}$ and the row vector $\mathbf{t}'$ so that

$$\mathbf{t}'=[t_1 \ldots t_r \mid t_{r+1} \ldots t_{n-1}],$$

which may be written

$$[\mathbf{t}_1' \quad \mathbf{t}_2'].$$

Since

$$\begin{bmatrix}\mathbf{I}_r & \cdot \\ -\mathbf{R}_3'\mathbf{R}_1^{-1} & \mathbf{I}_{n-(r+1)}\end{bmatrix}\begin{bmatrix}\mathbf{R}_1 & \mathbf{R}_3 \\ \mathbf{R}_3' & \mathbf{R}_2\end{bmatrix}\begin{bmatrix}\mathbf{I}_r & -\mathbf{R}_1^{-1}\mathbf{R}_3 \\ \cdot & \mathbf{I}_{n-(r+1)}\end{bmatrix}=\begin{bmatrix}\mathbf{R}_1 & \cdot \\ \cdot & \mathbf{R}_2-\mathbf{R}_3'\mathbf{R}_1^{-1}\mathbf{R}_3\end{bmatrix}, \quad \ldots\ldots (4)$$

on taking reciprocals

$$(\mathbf{R}_2-\mathbf{R}_3'\mathbf{R}_1^{-1}\mathbf{R}_3)^{-1}=\mathbf{R}_2,$$

where $\mathbf{R}_2$ is the submatrix in $\mathbf{R}$ corresponding to $\mathbf{R}_2$ in $\mathbf{R}^{-1}$.

The marginal distribution of $t_{r+1}, t_{r+2}, \ldots, t_{n-1}$ may thus be written

$$\frac{\Gamma\{\nu+n-(r+1)\}/2 \mid \mathbf{R}_2-\mathbf{R}_3'\mathbf{R}_1^{-1}\mathbf{R}_3 \mid^{\frac{1}{2}}}{(\pi\nu)^{\{n-(r+1)\}/2}\Gamma\nu/2}\{1+\mathbf{t}_2'(\mathbf{R}_2-\mathbf{R}_3'\mathbf{R}_1^{-1}\mathbf{R}_3)\mathbf{t}_2/\nu\}^{-\{\nu+n-(r+1)\}/2}d\mathbf{t}_2.$$

Express the quadratic form $\mathbf{t}'\mathbf{R}^{-1}\mathbf{t}$ as

$$\mathbf{t}'\mathbf{R}^{-1}\mathbf{t}=\mathbf{t}_1'\mathbf{R}_1\mathbf{t}_1+2\mathbf{t}_2'\mathbf{R}_3'\mathbf{t}_1+\mathbf{t}_2'\mathbf{R}_3'\mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{t}_2+\mathbf{t}_2'(\mathbf{R}_2-\mathbf{R}_3'\mathbf{R}_1^{-1}\mathbf{R}_3)\mathbf{t}_2,$$

and take as the fixed set of values for the variables in $\mathbf{t}_2$, the elements of a vector $\mathbf{a}$. Using the determinantal relation found by taking determinants of both sides of (4), the conditional distribution function of $\mathbf{t}_1$ is

$$\frac{\Gamma(\nu+n-1)/2\,|\,\mathbf{R}_1\,|^{\frac{1}{2}}}{\pi\nu)^{r/2}\Gamma\{\nu+n-(r+1)\}/2}\{1+\mathbf{a}'(\mathbf{R}_2-\mathbf{R}_3'\mathbf{R}_1^{-1}\mathbf{R}_3)\mathbf{a}/\nu\}^{\{\nu+n-(r+1)\}/2}$$
$$\times\int\cdots\int\left\{1+\frac{(\mathbf{t}_1+\mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{a})'\mathbf{R}_1(\mathbf{t}_1+\mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{a})+\mathbf{a}'(\mathbf{R}_2-\mathbf{R}_3'\mathbf{R}_1^{-1}\mathbf{R}_3)\mathbf{a}}{\nu}\right\}^{-(\nu+n-1)/2}d\mathbf{t}_1, \qquad \text{(5)}$$

integration being taken over the domain specified by

$$t_i\leqslant T_i, \qquad i=1,2,\ldots,r.$$

The distribution function (5) is dependent upon the particular set of values chosen for the variables in the vector $\mathbf{t}_2$, and its limiting form, as $\nu\to\infty$, is the conditional multivariate normal distribution of order r, with vector of means $-\mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{a}$, and variance-covariance matrix $\mathbf{R}_1^{-1}$.

The mean value of $\mathbf{t}_1$ is

$$E(\mathbf{t}_1)=\frac{\Gamma(\nu+n-1)/2\,|\,\mathbf{R}_1\,|^{\frac{1}{2}}}{(\pi\nu)^{r/2}\Gamma\{\nu+n-(r+1)\}/2}\{1+\mathbf{a}'(\mathbf{R}_2-\mathbf{R}_3'\mathbf{R}_1^{-1}\mathbf{R}_3)\mathbf{a}/\nu\}^{\{\nu+n-(r+1)\}/2}$$
$$\times\int_{-\infty}^{\infty}\cdots\int\mathbf{t}_1\left\{1+\frac{(\mathbf{t}_1+\mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{a})'\mathbf{R}_1(\mathbf{t}_1+\mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{a})+\mathbf{a}'(\mathbf{R}_2-\mathbf{R}_3'\mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{a})}{\nu}\right\}^{-(\nu+n-1)/2}d\mathbf{t}_1.$$

To evaluate this integral, first make the change of variable

$$\mathbf{t}_1+\mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{a}=\mathbf{z}\,;$$

the jacobian is 1, and, omitting the constant factor, the integral becomes

$$\int_{-\infty}^{\infty}\cdots\int(\mathbf{z}-\mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{a})\left\{1+\frac{\mathbf{z}'\mathbf{R}_1\mathbf{z}+\mathbf{a}'(\mathbf{R}_2-\mathbf{R}_3'\mathbf{R}_1^{-1}\mathbf{R}_3)\mathbf{a}}{\nu}\right\}^{-(\nu+n-1)/2}d\mathbf{z}$$
$$=-\mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{a}\int_{-\infty}^{\infty}\cdots\int\left\{1+\frac{\mathbf{z}'\mathbf{R}_1\mathbf{z}+\mathbf{a}'(\mathbf{R}_2-\mathbf{R}_3'\mathbf{R}_1^{-1}\mathbf{R}_3)\mathbf{a}}{\nu}\right\}^{-(\nu+n-1)/2}d\mathbf{z}.$$

Since $\mathbf{R}_1$ is a real, positive definite symmetric matrix, the quadratic form $\mathbf{z}'\mathbf{R}_1\mathbf{z}$ may be reduced to a sum of squares, after which the integration is easily performed, and finally yields

$$E(\mathbf{t}_1)=-\mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{a}, \qquad \text{(6)}$$

which is a linear function of the elements of the vector $\mathbf{a}$. The regression of $\mathbf{t}_1$ on $\mathbf{t}_2$ thus exists and is linear.

The variance-covariance matrix is then

$$E(\mathbf{t}_1+\mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{a})(\mathbf{t}_1+\mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{a})'=\frac{\Gamma(\nu+n-1)/2\mid \mathbf{R}_1\mid^{\frac{1}{2}}}{(\pi\nu)^{r/2}\Gamma\{\nu+n-(r+1)\}/2}\{1+\mathbf{a}'(\mathbf{R}_2-\mathbf{R}_3'\mathbf{R}_1^{-1}\mathbf{R}_3)\mathbf{a}/\nu\}^{\{\nu+n-(r+1)\}/2}$$

$$\times\int_{-\infty}^{\infty}\cdots\int(\mathbf{t}_1+\mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{a})(\mathbf{t}_1+\mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{a})'$$

$$\left\{1+\frac{(\mathbf{t}_1+\mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{a})'\mathbf{R}_1(\mathbf{t}_1+\mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{a})+\mathbf{a}'(\mathbf{R}_2-\mathbf{R}_3'\mathbf{R}_1^{-1}\mathbf{R}_3)\mathbf{a}}{\nu}\right\}^{-(\nu+n-1)/2}d\mathbf{t}_1,$$

and this integral is also readily evaluated after making the congruent transformation

$$\mathbf{t}_1+\mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{a}=\mathbf{Pz}$$

such that $\mathbf{P}'\mathbf{R}_1\mathbf{P}=\mathbf{I}_r$.

The variance-covariance matrix is

$$\frac{\nu+\mathbf{a}'(\mathbf{R}_2-\mathbf{R}_3'\mathbf{R}_1^{-1}\mathbf{R}_3)\mathbf{a}}{\nu+n-(r+3)}\mathbf{R}_1^{-1}, \quad \ldots\ldots\ldots\ldots\ldots \quad (7)$$

which depends upon the particular values chosen for the variables in the vector $\mathbf{t}_2$.

The limiting form of the matrix (7), as $\nu\to\infty$, is $\mathbf{R}_1^{-1}$, which is the variance-covariance matrix of the limiting conditional multivariate normal distribution, independent, as it should be, of the values of the fixed variates.

(*e*) *Distribution Function of* $\mathbf{t}'\mathbf{R}^{-1}\mathbf{t}$

(i) From the transformation of Section III

$$\mathbf{t}=\mathbf{Qy},$$

and so

$$\mathbf{t}'\mathbf{R}^{-1}\mathbf{t}=\mathbf{y}'\mathbf{Q}'\mathbf{R}^{-1}\mathbf{Q}\mathbf{y}$$

$$=\sum_{i=1}^{n} y_i^2/s^2.$$

Consequently

$$\mathbf{t}'\mathbf{R}^{-1}\mathbf{t}/(n-1)$$

is distributed as e^{2z} with $(n-1)$ and ν degrees of freedom.

(ii) The distribution of $\mathbf{t}'\mathbf{R}^{-1}\mathbf{t}$ may now be examined when the variables are subject to the linearly independent linear homogeneous conditions represented by the matrix equation

$$\mathbf{St}=\mathbf{0},$$

where $\mathbf{S}$ is of order $p\times(n-1)$ and rank $p<(n-1)$.

Construct the matrix $\mathbf{H}$ such that

$$\mathbf{HRH}'=\mathbf{I}_{n-p-1},$$

and

$$\mathbf{HRS}'=\mathbf{0},$$

and make the non-singular transformation

$$\mathbf{x}=\mathbf{Pt}=\begin{bmatrix}\mathbf{H}\\ \mathbf{S}\end{bmatrix}\mathbf{t}.$$

The distribution function of $\mathbf{t'R^{-1}t}$ thus becomes

$$\frac{|\mathbf{PRP'}|^{-\frac{1}{2}}\Gamma(\nu+n-1)/2}{(\pi\nu)^{(n-1)/2}\Gamma\nu/2}\int\ldots\int\{1+\mathbf{x}'(\mathbf{PRP}')^{-1}\mathbf{x}/\nu\}^{-(\nu+n-1)/2}d\mathbf{x},$$

where the domain of integration is defined by

$$\mathbf{x}'(\mathbf{PRP}')^{-1}\mathbf{x}\leqslant Q, \text{ say.}$$

Now impose the conditions

$$\mathbf{St}=\mathbf{0}.$$

This makes

$$x_{n-p}=x_{n-p+1}=\ldots=x_{n-1}=0, \quad \ldots\ldots\ldots\ldots \quad (8)$$

and, using the results of Section IV (d), the conditional distribution of $x_1, x_2, \ldots, x_{n-p-1}$ when (8) holds is

$$\frac{\Gamma(\nu+n-1)/2}{(\pi\nu)^{(n-p-1)/2}\Gamma(\nu+p)/2}\int\ldots\int(1+\mathbf{x}_1'\mathbf{x}_1/\nu)^{-(\nu+n-1)/2}d\mathbf{x}_1,$$

where $\mathbf{x}_1'$ is the vector $[x_1, x_2, \ldots, x_{n-p-1}]$ and the integral is taken over the region defined by

$$\mathbf{x}_1'\mathbf{x}_1\leqslant Q.$$

A spherical polar transformation in $(n-p-1)$ dimensions, followed by a change of variable which makes the square of the radius vector equal to $\frac{\nu}{\nu+p}\chi^2$ reduces this integral to the form

$$\frac{\Gamma(\nu+n-1)/2}{(\nu+p)^{(n-p-1)/2}\Gamma(n-p-1)/2\;\Gamma(\nu+p)/2}\int_0^Q(\chi^2)^{(n-p-3)/2}\{1+\chi^2/(\nu+p)\}^{-(\nu+n-1)/2}d(\chi^2),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

which is equivalent to Fisher's z-distribution with $(n-p-1)$ and $(\nu+p)$ degrees of freedom. The transfer of p degrees of freedom from one set of degrees of freedom to the other is a consequence of the fact that the conditional distribution is dependent upon the values of the fixed variates.

The distinction between the distribution (9) and that of $\mathbf{t'R^{-1}t}=\sum_{i=1}^{n} y_i^2/s^2$ when the y_i are subject to the restrictions $\mathbf{Sy}=\mathbf{0}$, should be noted. The latter distribution is, of course, Fisher's z-distribution with degrees of freedom $(n-p-1)$ and ν. Since s is essentially positive, either set of restrictive conditions then implies the other, and at first sight it might seem that the two distributions are identical.

(f) *Distribution Function of* $\mathbf{t'At}$

The distribution function of the quadratic form $\mathbf{t'At}$, of rank $r\leqslant(n-1)$, is given by

$$\frac{\Gamma(\nu+n-1)/2\,|\mathbf{R}|^{-\frac{1}{2}}}{(\pi\nu)^{(n-1)/2}\Gamma\nu/2}\int\ldots\int(1+\mathbf{t'R^{-1}t}/\nu)^{-(\nu+n-1)/2}d\mathbf{t},$$

where the domain of integration is defined by

$$\mathbf{t}'\mathbf{A}\mathbf{t} \leqslant Q.$$

Make the congruent transformation

$$\mathbf{t} = \mathbf{H}\mathbf{x},$$

the matrix $\mathbf{H}$ being chosen so that

$$\mathbf{H}\mathbf{R}^{-1}\mathbf{H}' = \mathbf{I}_{n-1},$$

and

$$\mathbf{H}\mathbf{A}\mathbf{H}' = \boldsymbol{\Lambda},$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix whose diagonal elements are the roots of the equation

$$| \lambda\mathbf{R}^{-1} - \mathbf{A} | = 0,$$

or, alternatively, the latent roots of the matrix $\mathbf{RA}$.

The jacobian is $| \mathbf{R} |^{\frac{1}{2}}$, so that the distribution function becomes

$$\frac{\Gamma(\nu+n-1)/2}{(\pi\nu)^{(n-1)/2}\Gamma\nu/2}\int \cdot \cdot \cdot \int (1+\mathbf{x}'\mathbf{x}/\nu)^{-(\nu+n-1)/2}d\mathbf{x},$$

the integral being taken over the region defined by

$$\mathbf{x}'\boldsymbol{\Lambda}\mathbf{x} \leqslant Q,$$

or

$$\sum_{i=1}^{r} \lambda_i x_i^2 \leqslant Q,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the non-zero latent roots of $\mathbf{RA}$.

After integrating for $x_{r+1}, x_{r+2}, \ldots, x_{n-1}$, the distribution function becomes

$$\frac{\Gamma(\nu+r)/2}{(\pi\nu)^{r/2}\Gamma\nu/2}\int \cdot \cdot \cdot \int (1+\mathbf{x}_1'\mathbf{x}_1/\nu)^{-(\nu+r)/2}d\mathbf{x}_1, \quad \ldots\ldots \quad (10)$$

where $\mathbf{x}_1'$ is the vector $[x_1\, x_2 \ldots x_r]$, and the domain of integration is defined by

$$\sum_{i=1}^{r} \lambda_i x_i^2 \geqslant Q.$$

Consequently, the necessary and sufficient condition that the distribution function (**10**) is equivalent to the z-distribution with degrees of freedom r and ν, is that the non-zero latent roots of the matrix $\mathbf{RA}$ are all equal to unity.

V. Parametric Values of the Multivariate Distribution

In further discussion of the conditional distribution and of regression and correlation among the t-variates, consideration is given in particular to the case $r=1$, since this value is of the greatest importance in practice.

(a) *Conditional Distribution*

The determinant

$$| \mathbf{R} | = n^{n-2}/(n-1)^{n-1},$$

and, if R_{ij} denotes the co-factor of the (ij)th element in $| \mathbf{R} |$, then

$$R_{ii} = 2n^{n-3}/(n-1)^{n-2},$$
$$R_{ij} = n^{n-3}/(n-1)^{n-2}, \qquad i \neq j,$$

and hence

$$\mathbf{R}^{-1}=\begin{bmatrix} 2(n-1)/n & (n-1)/n & \cdot & \cdot & \cdot & (n-1)/n \\ (n-1)/n & 2(n-1)/n & \cdot & \cdot & \cdot & (n-1)/n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (n-1)/n & (n-1)/n & \cdot & \cdot & \cdot & 2(n-1)/n \end{bmatrix}.$$

Taking $r=1$, the relation

$$\mathbf{R}^{-1}=\begin{bmatrix} \mathbf{R}_1 & \mathbf{R}_3 \\ \mathbf{R}_3' & \mathbf{R}_2 \end{bmatrix}$$

gives

$$\mathbf{R}_1=2(n-1)/n,$$

and

$$\mathbf{R}_3=[(n-1)/n \quad (n-1)/n \ldots (n-1)/n],$$

a row vector of order $(n-2)$.

The conditional mean value of the variate t_1 is then

$$-\mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{a}=-\tfrac{1}{2}\sum_{j=2}^{n-1} a_j.$$

Moreover,

$$\mathbf{R}_2-\mathbf{R}_3'\mathbf{R}_1^{-1}\mathbf{R}_3=\begin{bmatrix} 3(n-1)/2n & (n-1)/2n & \cdot & \cdot & \cdot & (n-1)/2n \\ (n-1)/2n & 3(n-1)/2n & \cdot & \cdot & \cdot & (n-1)/2n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (n-1)/2n & (n-1)/2n & \cdot & \cdot & \cdot & 3(n-1)/2n \end{bmatrix},$$

and hence

$$\nu+\mathbf{a}'(\mathbf{R}_2-\mathbf{R}_3'\mathbf{R}_1^{-1}\mathbf{R}_3)\mathbf{a}=\nu+\sum_j 3(n-1)a_j^2/2n+2\sum_{j<k}(n-1)a_ja_k/2n.$$

The conditional variance of the variate t_1 thus becomes

$$\frac{\nu+3(n-1)\sum_j a_j^2/2n+(n-1)\sum_{j<k} a_ja_k/n}{\nu+n-4}\;\frac{n}{2(n-1)},$$

and the average value of this quantity, for all possible values of $a_2, a_3, \ldots, a_{n-1}$, is

$$\frac{\nu+\dfrac{3(n-1)}{2n}(n-2)\dfrac{\nu}{\nu-2}-\dfrac{n-1}{n}\tfrac{1}{2}(n-3)(n-2)\dfrac{\nu}{(\nu-2)(n-1)}}{\nu+n-4}\;\frac{n}{2(n-1)}=\frac{\nu}{\nu-2}\;\frac{n}{2(n-1)}.$$

(b) *Regression and Correlation*

From (2), the variance-covariance matrix of the determining variates $t_2, t_3, \ldots, t_{n-1}$ is

$$\frac{\nu}{\nu-2}\mathbf{R}_2,$$

of which the determinant is

$$\frac{2n^{n-3}}{(n-1)^{n-2}}\left(\frac{\nu}{\nu-2}\right)^{n-2},$$

and, consequently, the reciprocal matrix

$$\left[\frac{\nu}{\nu-2}\mathbf{R}_2\right]^{-1}=\begin{bmatrix} \frac{3(n-1)}{2n}\ \frac{\nu-2}{\nu} & \frac{n-1}{2n}\ \frac{\nu-2}{\nu} & \cdot\ \cdot\ \cdot & \frac{n-1}{2n}\ \frac{\nu-2}{\nu} \\ \frac{n-1}{2n}\ \frac{\nu-2}{\nu} & \frac{3(n-1)}{2n}\ \frac{\nu-2}{\nu} & \cdot\ \cdot\ \cdot & \frac{n-1}{2n}\ \frac{\nu-2}{\nu} \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \frac{n-1}{2n}\ \frac{\nu-2}{\nu} & \frac{n-1}{2n}\ \frac{\nu-2}{\nu} & \cdot\ \cdot\ \cdot & \frac{3(n-1)}{2n}\ \frac{\nu-2}{\nu} \end{bmatrix}.$$

Also from (**2**), the covariance of t_1 with each of $t_2, t_3, \ldots, t_{n-1}$ is

$$-\nu/(\nu-2)(n-1),$$

so that the vector of regression coefficients in the multiple regression of t_1 on $t_2, t_3, \ldots, t_{n-1}$ is

$$\left[\frac{\nu}{\nu-2}\mathbf{R}_2\right]^{-1}\begin{bmatrix} \frac{-\nu}{(\nu-2)(n-1)} \\ \frac{-\nu}{(\nu-2)(n-1)} \\ \cdot \\ \cdot \\ \cdot \\ \frac{-\nu}{(\nu-2)(n-1)} \end{bmatrix}=\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \cdot \\ \cdot \\ \cdot \\ -\frac{1}{2} \end{bmatrix},$$

in agreement with the coefficients of the linear function for the conditional mean value of t_1.

The residual variance of t_1 with respect to $t_2, t_3, \ldots, t_{n-1}$ is

$$\frac{\left|\frac{\nu}{\nu-2}\mathbf{R}\right|}{\left|\frac{\nu}{\nu-2}\mathbf{R}_2\right|}=\frac{\nu}{\nu-2}\ \frac{n}{2(n-1)},$$

independent of the values of the determining variates, and equal to the average conditional variance given above.

The square of the multiple correlation of t_1 with $t_2, t_3, \ldots, t_{n-1}$ is

$$1-\frac{\left|\frac{\nu}{\nu-2}\mathbf{R}\right|}{\frac{\nu}{\nu-2}\left|\frac{\nu}{\nu-2}\mathbf{R}_2\right|}=(n-2)/2(n-1),$$

and the partial correlation of t_1 with t_j $(j=2,3,\ldots,n-1)$ is

$$\frac{-\left(\frac{\nu}{\nu-2}\right)^{n-2}R_{1j}}{\sqrt{\left(\frac{\nu}{\nu-2}\right)^{n-2}R_{11}\left(\frac{\nu}{\nu-2}\right)^{n-2}R_{jj}}}=-\tfrac{1}{2}.$$

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