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Section B: Short Communication

Solving the Three-Dimensional Findpath Problem via Liapunov's Method

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Abstract

This paper presents a method for solving the findpath problem. Commonly known as the second or direct method of Liapunov, the method is used to solve this geometric problem of finding collision-free trajectories of moving solid objects amongst other fixed and moving solid objects. A Liapunov function is proposed for a n-point dynamical system in three-space. Computer simulations are carried out to show the effectiveness of the proposed Liapunov function-based feedback controllers.

Keywords: Liapunov Function, Collision-Avoidance, Controllers, Trajectories.

I. GENERAL INTRODUCTION

In this paper, the problem of generating collision-free trajectories of point objects moving to their targets in a cluttered environment is considered. This problem is known as the *findpath problem*, also referred to as the *robot path planning problem*, since robotics is the most prevalent practical application of findpath problem. The use of robots is significantly increasing in the manufacturing systems, not only for productivity enhancement but also for greater efficiency and versatility. In robotic research, if a workspace is cluttered with fixed and moving obstacles, a collision-free, optimal (smoothest, fastest and safest) trajectory is desired as a solution of findpath problem, that can lead the mobile robot to its designated target (Vanualailai and Ha, 1998). Motion planning in dynamic environments is difficult but is more interesting than problems from static situations. We have to govern the position and velocity components of a object simultaneously for every time $t \geq 0$ from start to the end of an assigned task.

II. INTRODUCTION

Two major lines of research have been done over the years to solve the findpath problem (Günther and Azarm, 1993; Sheu and Xue, 1993):

1. *Graph Search Techniques* include algorithms that employ some kind of technique for graph searching. The method decomposes free space, into simple regions called cells. Then, a collision-free path is generated by constructing and searching a non-direct graph that connect the origin and destination via the vertices of solid obstacles, or via patches of cells. Various methods have been used over the years to construct these connectivity graphs. Although the graph search techniques are elegant theoretically, they tend to be computationally intensive in practice.

2. *Potential Field Techniques* include algorithms that employ some kind of physical analogy. Commonly known as the potential field method, it uses artificial potential fields with repulsive and attractive poles applied to the obstacles and targets. It then utilises the resulting fields to influence the path of the point masses. This technique is easier to implement.

III. THE SECOND METHOD OF LIAPUNOV

Liapunov's second method falls under the latter category. The first ever attempt to use the Liapunov method in the study of findpath problem was by Stonier in 1990. He constructed Liapunov-like functions to provide nonlinear analytic forms of control laws for the movement of point masses. In his research, he made a "right of way" assumption which controlled the movement of the point masses to avoid collision. However, his research findings had two major drawbacks. Firstly, it was difficult to justify the use of position vector components as constants in the Liapunov-like function. Secondly, his assumption could not be justified in systems of more than two moving objects. Vanualailai (1994) solved the two problems, by removing the "right-of-way" assumption and using functions that showed that the moving obstacles could be avoided at ease from variable distances. Also a single Liapunov-like function was constructed for the entire system instead of separate ones for different moving objects. Vanualailai and Ha (1998) added another vital component to the scalar function that guaranteed the function to be zero at the equilibrium points, hence the birth of a complete Liapunov function.

The aim of this paper is to address the findpath problem but in three-space and propose a solution to the system of n-point objects using an ameliorated version of the Liapunov function used by Vanualailai and Ha (1998). Feedback controllers will also be constructed for controlling the trajectory of the i-th object.

IV. STABILITY

Let us first look at the concept of stability via the Liapunov function. Consider the autonomous (time invariant) nonlinear system

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0\ , \quad t_0 \geq 0\ , \tag{1}$$

where $\boldsymbol{f} : \Omega \subset \boldsymbol{R}^n \to \boldsymbol{R}^n$ is assumed to be smooth enough to guarantee existence, uniqueness and continuous dependence of solutions $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ of (1) in Ω, an open set in $\boldsymbol{R}^n$. For the purpose of considering stability concepts in the sense of Liapunov, we assume that $\boldsymbol{f}(\boldsymbol{e}) \equiv \boldsymbol{0}$ so that $\boldsymbol{x}(t) \equiv \boldsymbol{e}$ is the equilibrium state of the system (1) passing through $(t_0, \boldsymbol{x}_0)$ in Ω for all $t > t_0$.

Definition 1 *The equilibrium state $\boldsymbol{x}(t) \equiv \boldsymbol{e}$ of (1) is said to be stable at time t_0 if, for each $\epsilon > 0$, there exists a $\delta(t_0, \epsilon) > 0$ such that*

$$||\boldsymbol{x}(t_0)|| < \delta(t_0, \epsilon) \Rightarrow ||\boldsymbol{x}(t)|| < \epsilon, \ \ \forall \ \ t > t_0.$$

The direct method of Liapunov states that this equilibrium state $\boldsymbol{x}(t) \equiv \boldsymbol{e}$ is stable if there exists a scalar function $L(\boldsymbol{x})$ in the neighborhood of $\boldsymbol{e}$ such that

(i) $L(\boldsymbol{e}) = 0$,

(ii) $L(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \boldsymbol{e}$,

(iii) $\left.\frac{dL(\boldsymbol{x})}{dt}\right|_{(1)} \leq 0$ for all $\boldsymbol{x} \in \Omega$.

When $L(\boldsymbol{x})$ successfully meets the above conditions, it is called the Liapunov function for system (1).

V. DYNAMICS OF MULTIPLE POINT OBJECTS

A Liapunov function is proposed that will work for n number of moving objects in three-space with any recognised number of moving and fixed obstacles. Consider the following system of ordinary differential equations :

$$\left.\begin{array}{ll} \dot{x}_i = r_i\ , & \dot{r}_i = u_i\ , \\ \dot{y}_i = s_i\ , & \dot{s}_i = v_i\ , \\ \dot{z}_i = t_i\ , & \dot{t}_i = w_i\ . \end{array}\right\} \text{ for } i = 1, 2, ..., n. \tag{2}$$

In the xyz-space, we refer to the point (x_i, y_i, z_i) as the position of the i-th object. Therefore, system (2) is a description of the instantaneous velocity $(\dot{x}_i, \dot{y}_i, \dot{z}_i) = (r_i, s_i, t_i)$ and instantaneous acceleration $(\dot{r}_i, \dot{s}_i, \dot{t}_i) = (u_i, v_i, w_i)$ of the i-th point object.

We make the assumption that we can transfer the i-th object from one point to another in the xyz-space by manipulating its instantaneous acceleration (u_i, v_i, w_i). By the Liapunov technique, (u_i, v_i, w_i) for $i = 1, 2, ...n$, are considered as feedback controllers, to be obtained from the proposed Liapunov function associated with system (2).

We shall use the vector notation

$$\boldsymbol{X_i} = (x_i, r_i, y_i, s_i, z_i, t_i) \in \boldsymbol{R}^6\ ,$$

in the xyz-space to refer to the position and velocity components of the i-th object,$i = 1, 2, ..., n$. For generality, we can further state that $\boldsymbol{x} = (\boldsymbol{X}_1, \boldsymbol{X}_2, ..., \boldsymbol{X}_n) \in \boldsymbol{R}^{6n}$.

A. Targets

The targets for system (2) are spherical regions enclosing a set of fixed points. The i-th target set, with center (p_{i1}, p_{i2}, p_{i3}), and radius rp_i, is defined as, for $i = 1, \ldots, n$,

$$\begin{aligned} T_i \ = \ & \left\{(x, y, z) \in \boldsymbol{R}^3 : (x - p_{i1})^2 + (y - p_{i2})^2 \right. \\ & \left. +(z - p_{i3})^2 \leq rp_i^2 \right\}\ , \end{aligned}$$

which becomes the fixed target set of the i-th object.

In connotation, the target will then become the *Fixed AntiTarget* T_j of all the other moving point objects ($j \neq i$). Mathematically, we can generalise this as :

$$FAT_j^i = T_j\ , \qquad i, j = 1, 2, ...n, \ \ i \neq j\ ,$$

where the superscript indicates the point object considered, and the subscript indicates which fixed obstacle it is. Hence $FAT_j^i = T_j$ is the j-th fixed antitarget of object A_i.

B. Point Objects

Now, we define the j-th moving point object A_j with center (x_j, y_j, z_j) and radius (rap_j), $j = 1, \ldots, n$,

$$\begin{aligned} A_j \ = \ & \left\{(x, y, z) \in \boldsymbol{R}^3 \ : [x - x_j(t)]^2 + [y - y_j(t)]^2 \right. \\ & \left. +[z - z_j(t)]^2 \leq [rap_j]^2 \right\}\ . \end{aligned}$$

This j-th moving point object also becomes a *Moving AntiTarget* for all the other moving point objects. Mathematically, we can generalise this as:

$$MAT_j^i = A_j\ , \qquad i, j = 1, 2, ...n, \ \ i \neq j\ ,$$

where the superscript again indicates the point object being considered. However, this time the subscript denotes the particular moving antitarget. Therefore, $MAT_j^i = A_j$ is the j-th moving antitarget of object A_i.

C. Obstacle Avoidance and Target Attraction

In three-dimensional space, the Liapunov function looks like a parabolic "mirror" pointing upward or of a "cup" on a table. In our scheme, we consider the cup-shaped surface as a point object and the bottom of the cup as the center of our target. The Liapunov function decreases along the trajectory of the point masses from the top of the cup to the bottom which is essentially one of the equilibrium points.

Now, in order to obtain the feedback controllers, we need to construct a Liapunov function for system (2) to make attraction to target and avoidance of all fixed and moving obstacles possible to the i-th object. Accordingly, we define the following functions for the i-th object:

1. Attraction to Target

For the attraction to the target T_i, we consider the function

$$\begin{aligned} V_i(\boldsymbol{x}) &= \frac{1}{2}\left[(x_i - p_{i1})^2 + (y_i - p_{i2})^2 + (z_i - p_{i3})^2 \right. \\ &\quad \left. + r_i^2 + s_i^2 + t_i^2\right] , \end{aligned}$$

which is a measure of the distance from object A_i to the target T_i. The function also measures the speed of the i-th object. The inclusion of the velocity components will help in the formulation of control law that provides for a *damping* capability that determines the rate of convergence of object A_i to its target T_i.

In the Liapunov function, $V_i(\boldsymbol{x})$ would act as a attractor by having the i-th object move to its designated target T_i. Note that $V_i(p_{i1}, 0, p_{i2}, 0, p_{i3}, 0) = 0$ and $V_i(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq (p_{i1}, 0, p_{i2}, 0, p_{i3}, 0)$, $i = 1, 2, ...n$.

2. Avoidance of Fixed Obstacles

For the avoidance of the fixed obstacle $FAT_j^i = T_j$, we consider the function

$$\begin{aligned} W_{ij}(\boldsymbol{x}) &= \frac{1}{2}\left[\,(x_i - p_{j1})^2 + (y_i - p_{j2})^2 \right. \\ &\quad \left. + (z_i - p_{j3})^2 - r{p_j}^2\,\right] , \end{aligned}$$

which is a measure of the distance between object A_i and the fixed obstacle T_j. The function is positive over the domain $\{\boldsymbol{x} \in \boldsymbol{R}^{6n} : (x_i - p_{j1})^2 + (y_i - p_{j2})^2 + (z_i - p_{j3})^2 > r{p_j}^2\}$.

In three-dimensional space, the surface, for some i, j with $i \neq j$,

$$s_i = \frac{c}{(x_i - p_{j1})^2 + (y_i - p_{j2})^2 + (z_i - p_{j3})^2 - r{p_j}^2} ,$$

where $c > 0$, is a right circular cylinder with radius rp_i. If this cylinder is a part of the Liapunov potential energy cup, then the i-th object, will naturally slow down as it reaches the saddle-like base of this structure and then avoid the cylinderical structure as it sinks to the bottom of the cup. Since the point object must inevitably be attracted to the target set which is the inherent advantage of the Liapunov function, and $(x_i - p_{j1})^2 + (y_i - p_{j2})^2 + (z_i - p_{j3})^2 \to r{p_j}^2$ implies an increase in energy, would initiate movement of the approaching object away from its fixed obstacle. Hence, object A_i will effectively avoid its fixed obstacle T_j. Also, we cannot have the situation where $(x_1 - p_{21})^2 + (y_1 - p_{22})^2 + (z_1 - p_{23})^2 = r{p_2}^2$. Consequently for the desired and effective avoidance of the fixed obstacles, we have function $W_{ij}(\boldsymbol{x})$ in the denominator of the Liapunov function.

3. Avoidance of the Moving Obstacle

For the avoidance of the moving obstacle $MAT_j^i = A_j$, we consider the function

$$\begin{aligned} V_{ij}(\boldsymbol{x}) &= \frac{1}{2}\left[\,(x_i - x_j)^2 + (y_i - y_j)^2 \right. \\ &\quad \left. + (z_i - z_j)^2 - rap_j^2\,\right] , \end{aligned}$$

which is a measure of the distance from object A_i to the secure avoidance region about object A_j. Function V_{ij} will also appear in the denominator of the Liapunov function for avoidance of the moving obstacles. Note that the function is positive over the domain $\{\boldsymbol{x} \in \boldsymbol{R}^{6n} : (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 > ra{p_j}^2\}$.

4. Obstacle Avoidance and Target Attraction

The Liapunov function of system (2) must be exactly zero at the target center (p_{i1}, p_{i2}, p_{i3}). As a consequence, we introduce a function

$$\begin{aligned} G_i(\boldsymbol{x}) &= \frac{1}{2}\left[\,(x_i - p_{i1})^2 + (y_i - p_{i2})^2 \right. \\ &\quad \left. + (z_i - p_{i3})^2\,\right] \geq 0 \ , \ \forall \ \boldsymbol{x} \in \boldsymbol{R}^{6n} , \end{aligned}$$

with $G(p_{i1}, 0, p_{i2}, 0, p_{i3}, 0) = 0$, for $i = 1, 2, ...n$. This function is a measure of distance between centers of object A_i and its target T_i.

D. Liapunov Function

Introducing $\alpha_{ij} > 0$ and $\beta_{ij} > 0$, which act as the control parameters for the point objects and using functions $V_i(\boldsymbol{x})$, $W_{ij}(\boldsymbol{x})$, $V_{ij}(\boldsymbol{x})$ and $G_i(\boldsymbol{x})$ from above, we define a Liapunov function $L(\boldsymbol{x})$ for target attraction and avoidance of fixed and moving obstacles for the i-th object.

The tentative Liapunov function is, for $i \neq j$,

$$L(\boldsymbol{x}) = \sum_{i=1}^{n} V_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\alpha_{ij}}{W_{ij}} + \frac{\beta_{ij}}{V_{ij}} \right) G_i \, , \qquad (3)$$

which is continuous and positive over the domain, for $i, j = 1, \ldots, n$,

$$D(L) = \{\boldsymbol{x} \in \boldsymbol{R}^{6n} : V_{ij}(\boldsymbol{x}) > 0, W_{ij}(\boldsymbol{x}) > 0\}.$$

We need to obtain the controllers, (u_i, v_i, w_i) so that object A_i can be controlled on its path to the target T_i without any collision with the moving and/or fixed anti-targets. Hence on differentiating the Liapunov function $L(\boldsymbol{x})$ we have,

$$\begin{aligned}\left.\frac{dL(\boldsymbol{x})}{dt}\right|_{(2)} &= \sum_{i=1}^{n} \dot{V}_i \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}\left(\frac{\dot{G}_i}{W_{ij}} - \frac{\dot{W}_{ij} G_i}{W_{ij}^2}\right) \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_{ij}\left(\frac{\dot{G}_i}{V_{ij}} - \frac{\dot{V}_{ij} G_i}{V_{ij}^2}\right).\end{aligned}$$

Differentiating the various components of $L(\boldsymbol{x})$ and treating the velocity components separately, we obtain :

$$\begin{aligned}&\left.\frac{dL(\boldsymbol{x})}{dt}\right|_{(2)} \\ &= \sum_{i=1}^{n}\left\{u_i + (x_i - p_{i1}) \times \left[1 + \sum_{j=1}^{n}\left(\frac{\alpha_{ij}}{W_{ij}} + \frac{\beta_{ij}}{V_{ij}}\right)\right]\right. \\ &\quad \left. - \sum_{j=1}^{n} \frac{\beta_{ij} G_i}{V_{ij}^2}(x_i - x_j) - \sum_{j=1}^{n} \frac{\alpha_{ij} G_i}{W_{ij}^2}(x_i - p_{j1})\right\} r_i \\ &\quad - \sum_{j=1}^{n} \frac{\beta_{ij} G_i}{V_{ij}^2}(x_i - x_j)\, r_j \\ &\quad + \sum_{i=1}^{n}\left\{v_i + (y_i - p_{i2}) \times \left[1 + \sum_{j=1}^{n}\left(\frac{\alpha_{ij}}{W_{ij}} + \frac{\beta_{ij}}{V_{ij}}\right)\right]\right. \\ &\quad \left. - \sum_{j=1}^{n} \frac{\beta_{ij} G_i}{V_{ij}^2}(y_i - y_j) - \sum_{j=1}^{n} \frac{\alpha_{ij} G_i}{W_{ij}^2}(y_i - p_{j2})\right\} s_i \\ &\quad - \sum_{j=1}^{n} \frac{\beta_{ij} G_i}{V_{ij}^2}(y_i - y_j)\, s_j \\ &\quad + \sum_{i=1}^{n}\left\{w_i + (z_i - p_{i3}) \times \left[1 + \sum_{j=1}^{n}\left(\frac{\alpha_{ij}}{W_{ij}} + \frac{\beta_{ij}}{V_{ij}}\right)\right]\right. \\ &\quad \left. - \sum_{j=1}^{n} \frac{\beta_{ij} G_i}{V_{ij}^2}(z_i - z_j) - \sum_{j=1}^{n} \frac{\alpha_{ij} G_i}{W_{ij}^2}(z_i - p_{j3})\right\} t_i \\ &\quad - \sum_{j=1}^{n} \frac{\beta_{ij} G_i}{V_{ij}^2}(z_i - z_j)\, t_j\,.\end{aligned}$$

To extract the controllers, we introduce positive constants(convergence parameters) $\rho_i > 0$, $\gamma_i > 0$ and $\upsilon_i > 0$ to get $\dot{L}_{(2)} = -(\rho_i\, {r_i}^2 + \gamma_i\, {s_i}^2 + \upsilon_i\, {t_i}^2)$, provided the controllers for $i = 1, 2, \ldots n$ are as follows :

$$\begin{aligned}u_i(\boldsymbol{x}) &= -(x_i - p_{i1}) \times \left[1 + \sum_{j=1}^{n}\left(\frac{\alpha_{ij}}{W_{ij}} + \frac{\beta_{ij}}{V_{ij}}\right)\right] \\ &+ \sum_{j=1}^{n} \frac{\beta_{ij} G_i}{V_{ij}^2}(x_i - x_j) \\ &+ \sum_{j=1}^{n} \frac{\alpha_{ij} G_i}{W_{ij}^2}(x_i - p_{j1}) \\ &+ \sum_{j=1}^{n} \frac{\beta_{ji} G_j}{V_{ji}^2}(x_j - x_i)\ - \rho_i\, r_i\,,\end{aligned} \tag{4}$$

$$\begin{aligned}v_i(\boldsymbol{x}) &= -(y_i - p_{i2}) \times \left[1 + \sum_{j=1}^{n}\left(\frac{\alpha_{ij}}{W_{ij}} + \frac{\beta_{ij}}{V_{ij}}\right)\right] \\ &+ \sum_{j=1}^{n} \frac{\beta_{ij} G_i}{V_{ij}^2}(y_i - y_j) \\ &+ \sum_{j=1}^{n} \frac{\alpha_{ij} G_i}{W_{ij}^2}(y_i - p_{j2}) \\ &+ \sum_{j=1}^{n} \frac{\beta_{ji} G_j}{V_{ji}^2}(y_j - y_i)\ - \gamma_i\, s_i\,,\end{aligned} \tag{5}$$

and

$$\begin{aligned}w_i(\boldsymbol{x}) &= -(z_i - p_{i3}) \times \left[1 + \sum_{j=1}^{n}\left(\frac{\alpha_{ij}}{W_{ij}} + \frac{\beta_{ij}}{V_{ij}}\right)\right] \\ &+ \sum_{j=1}^{n} \frac{\beta_{ij} G_i}{V_{ij}^2}(z_i - z_j) \\ &+ \sum_{j=1}^{n} \frac{\alpha_{ij} G_i}{W_{ij}^2}(z_i - p_{j3}) \\ &+ \sum_{j=1}^{n} \frac{\beta_{ji} G_j}{V_{ji}^2}(z_j - z_i)\ - \upsilon_i\, t_i\,,\end{aligned} \tag{6}$$

which gives us the control law as vector, $\boldsymbol{l_i} = (u_i, v_i, w_i)$ for the movement of object A_i.

Finally, if we let $\boldsymbol{X_{e_i}} = (p_{i1}, 0, p_{i2}, 0, p_{i3}, 0) \in \boldsymbol{R}^6$, for $i = 1, 2, \ldots n$, then we have $\boldsymbol{x_e} = (\boldsymbol{X_{e_1}}, \boldsymbol{X_{e_2}}, \ldots \boldsymbol{X_{e_n}}) \in \boldsymbol{R}^{6n}$ as the equilibrium state of system (2).

Properties of the Liapunov Function

- **(i)** $L(\boldsymbol{x})$ is continuous and has first partial derivatives in the region $D(L)$ in the neighborhood of the stable equilibrium state, $\boldsymbol{x_e}$,
- **(ii)** $L(\boldsymbol{x_e}) = 0$,
- **(iii)** $L(\boldsymbol{x}) > 0$,$\forall\, \boldsymbol{x} \in D(L) \setminus \boldsymbol{x_e}$,

TABLE I: **Simulation parameters: Example 1**

Time Interval, RK4 Step Size	$[0, 40]$, 0.01
Targets Centers	$(p_{11}, p_{12}, p_{13}) = (50,\ 50,\ 200)$ $(p_{21}, p_{22}, p_{23}) = (150,\ 130,\ 200)$ $(p_{31}, p_{32}, p_{33}) = (250,\ 50,\ 200)$
Targets Radii	$rp_1 = rp_2 = rp_3 = 5$
Initial positions of Objects	$(x_1, y_1, z_1) = (200,\ 90,\ 200)$ $(x_2, y_2, z_2) = (150,\ 50,\ 200)$ $(x_3, y_3, z_3) = (100,\ 90,\ 200)$
Initial velocities of Objects	$(r_1, s_1, t_1) = (5,\ 5,\ 2)$ $(r_2, s_2, t_2) = (4,\ 4,\ 2)$ $(r_3, s_3, t_3) = (5,\ 5,\ 2)$
Objects Radii	$rap_1 = rap_2 = rap_3 = 4$
Control Parameters	$\alpha_{12} = \alpha_{13} = \alpha_{21} = \alpha_{23} = \alpha_{31} = \alpha_{32} = 6.0$ $\beta_{12} = \beta_{13} = \beta_{21} = \beta_{23} = \beta_{31} = \beta_{32} = 6.0$
Convergence Parameters	$\rho_1 = \gamma_1 = \upsilon_1 = 5.0$ $\rho_2 = \gamma_2 = \upsilon_2 = 6.0$ $\rho_3 = \gamma_3 = \upsilon_3 = 5.0$

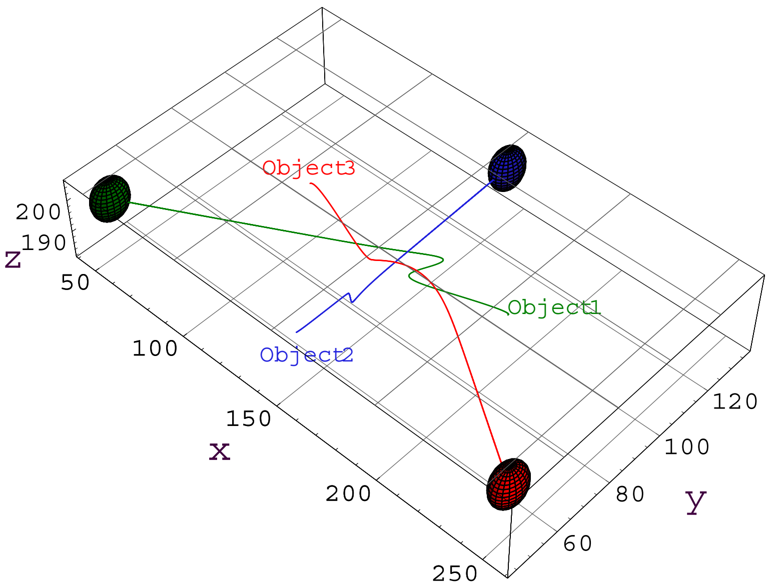

FIG. 1: Simulation results: trajectories of the three moving objects, where each converges to its target in Example 1.

(iv) $\dot{L}_{(2)}(\boldsymbol{x}) \leq 0\ , \forall\ \boldsymbol{x} \in D(L)$.

Hence, the scalar function created is actually a Liapunov function for system (2) and it guarantees stability of system (2) with the following theorem:

Theorem 1 *The equilibrium state* $(\boldsymbol{x_e})$ *of system* (2) *is stable provided* u_i, v_i *and* w_i *for* $i = 1, 2, ...n$, *are defined as equations* (4), (5), *and* (6), *respectively.*

VI. SIMULATIONS

The examples below illustrate the collision-avoidance capabilities of point objects and the optimum trajectories obtained through appropriate manipulation of control and convergence parameters. For the numerical integration of system (2), a fourth-order Range-Kutta method is utilised.

Example 1 We have a triangular configuration for this simulation, where targets are situated at the vertices and the objects start initially at the midpoints. Table I provides details of the initial state of the system. The convergence parameters of Object 2 have been increased which makes it possible for all the moving objects to meet at one place and hence a three-way avoidance could be observed. Figure 1 shows the collision-free trajectories of the moving objects.

Remark 1 In Example 1, we have a trianglular situation in three-space, where the paths of all moving ob-

TABLE II: **Simulation parameters: Example 2**

Time Interval, RK4 Step Size	$[0, 40]$, 0.01
Targets Centers	$(p_{11}, p_{12}, p_{13}) = (100,\ 100,\ 150)$ $(p_{21}, p_{22}, p_{23}) = (190,\ 100,\ 150)$ $(p_{31}, p_{32}, p_{33}) = (280,\ 100,\ 150)$ $(p_{41}, p_{42}, p_{43}) = (150,\ 110,\ 160)$ $(p_{51}, p_{52}, p_{53}) = (220,\ 90,\ 160)$
Targets Radii	$rp_1 = rp_2 = rp_3 = rp_4 = rp_5 = 5$
Initial positions of Objects	$(x_1, y_1, z_1) = (235,\ 100,\ 150)$ $(x_2, y_2, z_2) = (55,\ 100,\ 150)$ $(x_3, y_3, z_3) = (145,\ 100,\ 150)$
Initial velocities of Objects	$(r_1, s_1, t_1) = (4,\ 1,\ 6)$ $(r_2, s_2, t_2) = (4,\ 1,\ 6)$ $(r_3, s_3, t_3) = (4,\ 1,\ 6)$
Objects Radii	$rap_1 = rap_2 = rap_3 = 4$
Control Parameters	$\alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{15} = \alpha_{21} = \alpha_{23} = 6.0$ $\alpha_{24} = \alpha_{25} = \alpha_{31} = \alpha_{32} = \alpha_{34} = \alpha_{35} = 6.0$ $\beta_{12} = \beta_{13} = \beta_{21} = \beta_{23} = \beta_{31} = \beta_{32} = 6.0$
Convergence Parameters	$\rho_1 = \gamma_1 = \upsilon_1 = 7.0$ $\rho_2 = \gamma_2 = \upsilon_2 = 5.0$ $\rho_3 = \gamma_3 = \upsilon_3 = 7.0$

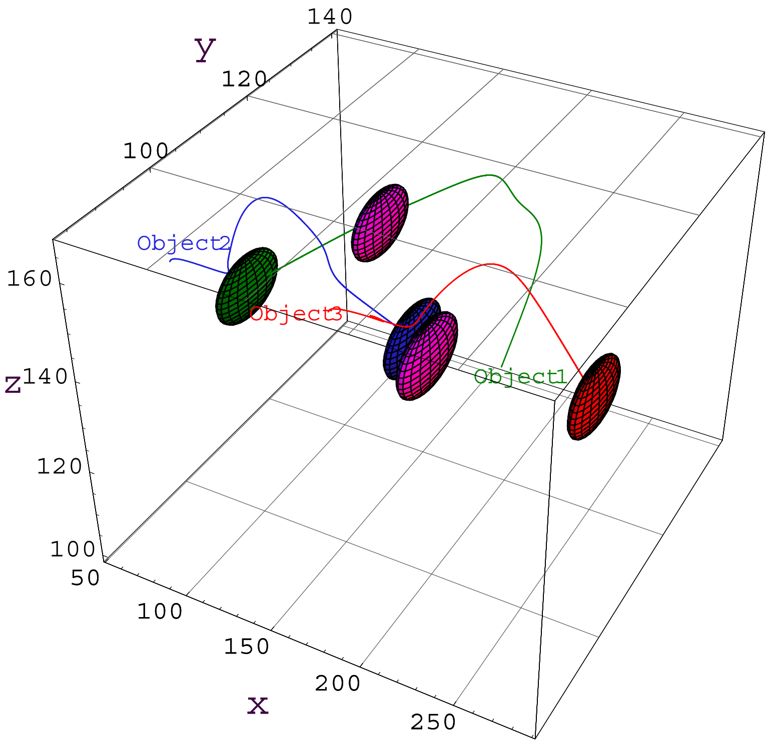

FIG. 2: Simulation results: trajectories of the three moving objects, where each converges to its target in Example 2.

jects meet at a certain place in space. The avoidance and then convergence to the designated targets are seen. The paths obtained are optimal for the initial states provided. Initially, the sizes of the controllers are large which is because of a large degree of control required to start the movement of objects on a certain path. After that, the controllers are reduced to the vicinity of the time axis, which indicates the asymptotic behaviour of the controllers.

Example 2 We have injected two extra fixed antitargets into the workspace. Antitarget T_4 is situated infront of the initial position of object A_2, while antitarget T_5 has been situated to block the trajectories of object A_1 and object A_2. Table II provides details of the initial state

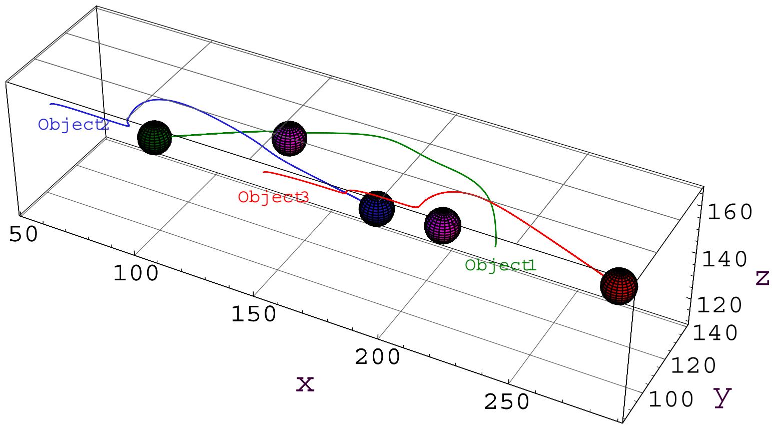

FIG. 3: Simulation results: trajectories of the three moving objects, where each converges to its target in Example 2 from a different viewpoint.

of the system. Figure 2 and Figure 3 show the collision free path paths of the moving objects from two different viewpoints.

Remark 2 In Example 2, we have the targets and the moving objects spaced out in a line with the extra fixed antitargets blocking their original trajectories. The newer paths again lead directly to the target centers. However, we see that object A_1 takes a longer time to reach the target center because it has to avoid antitarget T_4 on its way. Again we see the asymptotic behavior of the controllers.

VII. PARAMETERS

We will briefly consider the effects of the two main parameters - the control and convergence parameters, on the trajectories of the moving objects to their designated target centers. Through proper manipulation of these two parameters, better solutions to the findpath problem can be found. How safe, smooth and quick the trajectory would be, will directly depend on the feedback controllers, which have these parameters instilled. The parameters were first put into desirable effects by Vanualailai and Ha (1998).

A. Control Parameter

Because of the makeup of our Liapunov function, we are at liberty to increase or decrease the control parameters α_{ij} and β_{ij}, as much as we pleased to obtain a desired trajectory. These control parameters allow us to control the trajectory of a moving point object to its target center, hence improving the quality of the solution to the findpath problem. Since α_{ij} appeared in the Liapunov function associated with W_{ij}, it takes care of the avoidance of fixed obstacles. While β_{ij} is associated with V_{ij}, and thus takes care of the avoidance of moving obstacles.

B. Convergence Parameters

Decreasing the convergence parameters ρ_i, γ_i and υ_i will reduce the time taken by the objects to reach the center of the target, however, it will not guarantee increasingly better trajectories. Also, increasing the values of convergence parameters will result in safer and smoother trajectories, however, it will make the system very slow and hence expensive to maintain. Consequently proper manipulation of the control parameters of the particular moving object may improve the trajectories.

VIII. SUMMARY AND CONCLUSION

Optimal navigation in an environment cluttered with fixed and moving obstacles is the ultimate search of researchers. Although various formulations for this problem have surfaced, the search for the indispensable one still prevails. Liapunov's Direct Method has been used extensively to solve the findpath problem. This research carries on from where Vanualailai and Ha (1998) left, and we have tried to get closer to the ideal solution with two major improvements.

Firstly, the Liapunov method has been successfully applied to three-dimensional systems. This is more realistic and less restrictive when compared to the two-dimensional research, although the functions turn out to be quite complex. Secondly, researchers in this field of study have constructed Liapunov functions that only embrace systems with a maximum of two point objects. We, in this research, have generalised the Liapunov function to work for a general n-point system. Computer simulations have provided necessary evidence for this. A generalised Liapunov function is indeed helpful in further research work where a large number of objects, targets and obstacles are considered.

The applicability of this method has been verified by copious extraordinary simulations that have shown that indeed the Liapunov method is indispensable for solving findpath problems. The theoretical concepts developed for this system can also be demonstrated by considering the problem of coordination between n-robotic arms, n-link robot arm and n mobile robots in a workspace. Also, getting feasible solutions of the much fancied *Highly Intelligent Vehicle Systems* is another possibility. These will be considered by the team for future research papers.

However, our Liapunov function can only guarantee stability of the system. This means that for some initial conditions the trajectories may cease motion close to but before reaching the center of their designated targets. We have used trial-and-error method to obtain certain values of control and convergence parameters, which engender asymptotic stability of the system. However, ideally we should be able to construct a Liapunov function that guarantees asymptotic stability of a system irrespective of the initial conditions. This is a problem which will be taken up in future.

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