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Solitary Magnetosonic Waves in a Relativistic Plasma

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Abstract

We have analysed the formation of solitary magnetosonic waves propagating in a direction perpendicular to the magnetic filed in a relativistic two component plasma. Our approach is that of the effective potential. Variations of the effective potential and the solitary wave in relation to the Mach number and other parameters are discussed.

1. Introduction

The study of nonlinear waves in both magnetised and unmagnetised plasmas is an important topic which has made tremendous progress over the last two decades. An initial attempt to analyse the characteristics of solitary waves in a magnetised plasma was undertaken by Gardner and Morikawa (1968). Later, Kakuntani *et al.* (1968) and Berezin and Karpman (1964) derived the KdV equation for the magnetosonic wave for nonrelativistic plasmas with zero β value. Nonlinear evaluation of the magnetosonic wave plays an important role in plasma turbulence (Lacombe and Mangency 1969), the trapping of ions (Lewbege *et al.* 1983) and the development of shock structure (Jager 1985). Magnetosonic shock waves are believed to be responsible for the heating of the solar corona (Kuperns *et al.* 1981). In some theoretical studies these magnetosonic waves have been described by a KdV-like equation. For example Vito and Pantano (1984) have shown that such a wave in a cold (nonrelativistic) plasma can be described by the KdV equation. Recently it has been recognised that nonlinear fast magnetosonic waves can strongly accelerate trapped ions by a $\boldsymbol{V} \times \boldsymbol{B}$ type acceleration in a direction perpendicular to the magnetic field in a relativistic plasma (Ohsawa 1985). More recently, solitary waves in a relativistic plasma have been discussed by Das and Paul (1985, 1987) and Roy Chowdhury *et al.* (1988).

On the other hand, it is known that a KdV-type equation describes only small amplitude waves due to the approximations involved in the derivative via the reductive perturbative scheme. So here in this paper we study the formation and propagation of nonlinear magnetosonic waves which propagate in a direction perpendicular to the magnetic field, in a relativistic plasma without assuming

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that the wave amplitude is small. Our approach is that of the effective potential which is capable of treating both large and small amplitude waves. It has been demonstrated already by Sagdeev (1966), Schamel (1973) and Sochmel (1976) that such an approach is very effective in the theoretical analysis of large amplitude plasma waves.

2. Formulation

To start we make the usual assumption that the relativistic two fluid plasma under consideration can be described by the two hydrodynamic equations

$$\frac{\partial n_j}{\partial t} + \operatorname{div}(n_j \boldsymbol{v}_j) = 0\,, \tag{1a}$$

$$m_j\left(\frac{\partial}{\partial t} + \boldsymbol{v}_j \cdot \operatorname{grad}\right) r_j \boldsymbol{v}_j = q_j \boldsymbol{E} + \frac{q_j}{c} \boldsymbol{v}_j \times \boldsymbol{B}\,, \tag{1b}$$

where the subscript $j = \mathrm{i}$ for the ion and $j = \mathrm{e}$ for the electron, with Maxwell's equations

$$\operatorname{curl} \boldsymbol{B} = \left(\frac{U_0^*}{c}\right)^2 \frac{\partial E}{\partial t} + \frac{4\pi e}{c}(n_\mathrm{i} \boldsymbol{v}_i - n_\mathrm{e} \boldsymbol{v}_\mathrm{e})\,, \tag{2a}$$

$$\operatorname{curl} \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}\,, \tag{2b}$$

$$\operatorname{div} \boldsymbol{B} = 0\,, \tag{2c}$$

$$\operatorname{div} \boldsymbol{E} = 4\pi e(n_\mathrm{i} - n_\mathrm{e})\,. \tag{2d}$$

Here n_i and n_e denote the ion and electron densities respectively, $v_{x\mathrm{i}}$ and $v_{x\mathrm{e}}$ denote the x-component of the velocities, r_j is the Lorentz factor, $\boldsymbol{B}$ the magnetic field and $\boldsymbol{E}$ the electric field. All quantities have been normalised with respect to the characteristic number density n_0^*, the characteristic speed U_0^*, the characteristic length L_0^* and the characteristic magnetic field B_0^* (Kakutani 1974). Furthermore we assume that the magnetic field is in the z-direction and that the wave is propagating in the x-direction. To simplify the ensuing computation we have furthermore assumed the quasi-neutrality condition $n_\mathrm{i} \approx n_\mathrm{e} = n$, by virtue of equation (2d). Then from (1) we get

$$\frac{\partial n_\mathrm{e}}{\partial t} + \frac{\partial}{\partial x}(n_\mathrm{e} v_{x\mathrm{e}}) = 0\,,$$

$$\frac{\partial n_\mathrm{e}}{\partial t} + \frac{\partial}{\partial x}(n_\mathrm{e} v_{xi}) = 0\,.$$

Subtracting we get

$$\frac{\partial}{\partial x}(n_{\mathrm{e}}\, v_{x\mathrm{e}} - n_{\mathrm{e}}\, v_{x\mathrm{i}}) \;=\; 0\,,$$

whence $v_{x\mathrm{e}} = v_{x\mathrm{i}} + k/n$.

On the other hand, from Maxwell's equations (2a)–(2d) with the assumptions that the wave is propagating in the x-direction and that the magnetic field is in the z-direction (constant in magnitude) we get $\partial E_y/\partial x = 0$; so, $E_y = \text{constant} = E_{y_1}$ (say). Thus, from the rest of the equation we get

$$\frac{\mathrm{d}B}{\mathrm{d}x} = -\frac{4\pi ne}{c}(v_{yi} - v_{y\mathrm{e}}) + \left(\frac{U_0^*}{c}\right)^2 \frac{\partial E_y}{\partial t}\,. \tag{2e}$$

Since we are concerned with hydromagnetic waves, for which $U_0^*/c \ll 1$, we can neglect the displacement current in (2e) (Kakutani 1974), and then we get

$$\frac{\mathrm{d}B}{\mathrm{d}x} = -\frac{4\pi ne}{c}(v_{y\mathrm{i}} - v_{y\mathrm{e}})\,. \tag{3}$$

Setting $v_{x\mathrm{i}} = v_x$ and $v_{x\mathrm{e}} = v_x + k/n$, we can simplify equations (1) and (2) as follows:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv_x) = 0\,, \tag{4}$$

$$m_{\mathrm{i}}\,\frac{\partial}{\partial t}(r_{\mathrm{i}}\, v_x) + m_{\mathrm{i}}\,\frac{\partial}{\partial x}(r_{\mathrm{i}}\, v_x) = eE_x + \frac{e}{c}\, v_{y\mathrm{i}}\, B\,, \tag{5}$$

$$m_{\mathrm{i}}\,\frac{\partial}{\partial t}(r_{\mathrm{i}}\, v_{y\mathrm{i}}) + m_{\mathrm{i}}\, v_x\,\frac{\partial}{\partial x}(r_{\mathrm{i}}\, v_{y\mathrm{i}}) = eE_{y\mathrm{i}} - \frac{e}{c}\, v_z\, B\,, \tag{6}$$

$$\begin{aligned} m_{\mathrm{e}}\,\frac{\partial}{\partial t}(r_{\mathrm{e}}\, v_x) + m_{\mathrm{e}}\, k\,\frac{\partial}{\partial t}(r_{\mathrm{e}}/n) + m_{\mathrm{e}}\, v_x\,\frac{\partial}{\partial x}(r_{\mathrm{e}}\, v_x) + m_{\mathrm{e}}\, v_x\, k \\ \times\,\frac{\partial}{\partial x}(r_{\mathrm{e}}/n) + m_{\mathrm{e}}\,\frac{k}{n}\,\frac{\partial}{\partial x}(r_{\mathrm{e}}\, v_x) \\ + m_{\mathrm{e}}\,\frac{k^2}{n}\,\frac{\partial}{\partial x}(r_{\mathrm{e}}/n) = -eE_x - \frac{e}{c}\, v_{y\mathrm{e}}\, B\,, \end{aligned} \tag{7}$$

$$m_{\mathrm{e}} \frac{\partial}{\partial t}(r_{\mathrm{e}} n_{y\mathrm{e}}) + m_{\mathrm{e}} v_x \frac{\partial}{\partial x}(r_{\mathrm{e}} v_{y\mathrm{e}}) + \frac{m_{\mathrm{e}} k}{n} \frac{\partial}{\partial x}(r_{\mathrm{e}} v_{y\mathrm{e}})$$

$$= -eE_{y\mathrm{i}} + \frac{e}{c} v_x B + \frac{e}{c} \frac{k}{n} B, \quad (8)$$

where

$$r_j = \frac{1}{(1 - v_j^2/c^2)^{\frac{1}{2}}}.$$

We now make a simple change of variable $(x, t) \rightarrow (\xi, \tau)$ defined by

$$\xi = x - Mt, \qquad \tau = t,$$

and go to the moving frame of reference of the wave by setting the 'τ' derivative of all quantities equal to zero.

Whence we get from (4)

$$n = \frac{n_1(v_1 - M)}{v_x - M}, \quad (9)$$

where n_1, v_1 are integration constants.

Now in the present calculations the Alfvén speed is assumed to be smaller than the speed of light. In the nonrelativistic situation (Adlam and Allen 1958) it has been shown that the speed of the fluid electrons and trapped ions exceeds c if the Alfvén speed is fairly large and if the Alfvén Mach number M_{A} is not too close to unity. Since the particle speed cannot exceed c this indicates that we have to use a relativistic theory for a magnetosonic wave having an Alfvén speed v_{A} comparable or greater than $c(m_{\mathrm{e}}/m_{\mathrm{i}})^{\frac{1}{2}}$. In our case we are interested in the situation where v_{A} is of the order of $c(m_{\mathrm{e}}/m_{\mathrm{i}})^{\frac{1}{2}}$ and the electron velocity is close to c, although the ions have a velocity much smaller than c.

Now $v_{\mathrm{i}} = (v_x^2 + v_{y\mathrm{i}}^2)^{\frac{1}{2}}$ and $v_{\mathrm{e}} = (v_{x\mathrm{e}}^2 + v_{y\mathrm{e}}^2)^{\frac{1}{2}}$. For computational simplicity we take $v_{x\mathrm{i}} < v_{x\mathrm{e}}$ and $v_{y\mathrm{i}} \ll v_{y\mathrm{e}}$. So we can take the value of k smaller than c. From equations (6) and (8) and with the help of (9) we can write

$$-(M - v_x)\left[m_{\mathrm{i}}(r_i v_{y\mathrm{i}}) + m_{\mathrm{e}}\left(1 + \frac{k}{n_1(v_1 - M)}\right) r_{\mathrm{e}} v_{y\mathrm{e}}\right] = f(\xi) \frac{ek(v_x - M)}{cn_1(v_1 - M)}$$

or

$$v_{y\mathrm{i}} = -\frac{m_\mathrm{e}}{m_\mathrm{i}\, r_\mathrm{i}}\left(1 + \frac{k}{n_1(v_1 - M)}\right) r_\mathrm{e}\, v_{y\mathrm{e}} + \frac{ek}{cn_1\, m_\mathrm{i}\, r_\mathrm{i}}\, f(\xi)\,. \tag{10}$$

Furthermore, we have to always ensure that the ion speed is much smaller than c; so, $r_\mathrm{i} \approx 1$. Since

$$\begin{aligned} r_\mathrm{i} &= \left(1 - \frac{v_\mathrm{i}^2}{c^2}\right)^{-\frac{1}{2}} \approx 1 + \frac{v_\mathrm{i}^2}{2c^2} \\ &= 1 + \frac{1}{2c^2}(v_x^2 + v_{y\mathrm{i}}^2)\,, \end{aligned} \tag{11}$$

and since $v_x \ll c$, we get

$$r_\mathrm{i} \approx 1 + v_{y\mathrm{i}}^2/2c^2\,. \tag{12}$$

Then, substituting from equation (10) for $v_{y\mathrm{i}}$ we get

$$2(r_1 - 1)r_\mathrm{i}^2 = \left(1 + \frac{k}{n_1(v_1 - M)}\right)^2 \frac{v_{y\mathrm{e}}^2\, r_\mathrm{e}^2}{c^2(m_i/m_\mathrm{e})^2}\,. \tag{13}$$

Again $v_{y\mathrm{e}}$ is of the order of c, and so we get the condition

$$\left(1 + \frac{k}{n_1(v_1 - M)}\right) r_\mathrm{e} \ll \frac{m_\mathrm{i}}{m_\mathrm{e}} \quad \text{for } r_\mathrm{i} \approx 1\,. \tag{14}$$

Now from equations (5) and (7) we get

$$\begin{aligned} &- m_\mathrm{i}\, M\, \frac{\partial}{\partial\xi}(r_\mathrm{i}\, v_x) + m_\mathrm{i}\, v_x\, \frac{\partial}{\partial\xi}(r_\mathrm{i}\, v_x) + (M - v_x) \\ &\times\, m_\mathrm{e}\left(1 + \frac{k}{n_1(v_1 - M)}\right)^2 \frac{\partial}{\partial\xi}(r_\mathrm{e}\, v_x) + (M - v_x) m_\mathrm{e} \\ &\times\, \frac{Mk}{n_1(v_1 - M)}\left(1 + \frac{k}{n_1(v_1 - M)}\right)\frac{\partial r_\mathrm{e}}{\partial\xi} = 2eE_x + \frac{e}{c}(v_{y\mathrm{i}} + v_{y\mathrm{e}})B\,, \end{aligned} \tag{15}$$

$$- (M - v_x)\frac{\partial v_x}{\partial \xi} + (M - v_x)\frac{\partial}{\partial \xi}\left[v_x \frac{m_e}{m_i} r_e \left(1 + \frac{k}{n_1(v_1 - M)}\right)^2\right]$$

$$+ (M - v_x)\frac{Mk}{n_1(v_1 - M)}\frac{\partial}{\partial \xi}\left[\left(1 + \frac{k}{n_1(v_1 - M)}\right)\frac{m_e}{m_i} r_e\right]$$

$$= \frac{2eE_x}{m_i} + \frac{e}{cm_i}(v_{yi} + v_{ye})B\,. \tag{16}$$

Now since $[1 + k/n_1(v_1 - M)]r_e \ll m_i/m_e$ we get

$$(v_x - M)\frac{\partial v_x}{\partial \xi} \approx \frac{2eE_x}{m_i} + \frac{e}{m_i\, c} v_{ye}\, B\,, \tag{17}$$

$$(v_x - M)\frac{\partial v_x}{\partial \xi} = -\frac{B}{4\pi n m_i}\frac{\mathrm{d}B}{\mathrm{d}\xi}\,, \tag{18}$$

whence (17) and (18) lead to

$$E_x = -\frac{v_{ye}\, B}{2c} - \frac{B}{4\pi n m_i}\frac{\mathrm{d}B}{\mathrm{d}\xi}\,. \tag{19}$$

On the other hand, integrating (18) with the condition that $B = b_1$ when $v = v_1$, we get $v_x = \sigma v_1$, with σ given as

$$\sigma = 1 - \frac{B^2 - B_1^2}{8\pi m_1\, n_1(v_1^2 - Mv_1)}\,. \tag{20}$$

Now going back to equation (3) we observe that under the approximation $r_i \approx 1$ we can write

$$\frac{m_e}{m_i}\left(1 + \frac{k}{n_1(v_1 - M)}\right) r_e\, v_{ye} - \frac{ek}{cn_1\, m_i} f(\xi) + v_{ye} = \frac{c}{4\pi n e}\frac{\mathrm{d}B}{\mathrm{d}\xi}\,, \tag{21}$$

from which we deduce

$$v_{ye} \approx \frac{c(\sigma - M/v_1)}{4\pi m_i\, e(1 - M/v_1)}\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} + \frac{ek}{cn_i\, m_i} f(\xi)\,. \tag{22}$$

Now we compute explicitly the relativistic Lorentz factor for the electron,

$$
\begin{aligned}
r_{\mathrm{e}} &= \left(1-\frac{v_{xe}^2+v_{ye}^2}{c^2}\right)^{-\frac{1}{2}} \\
&= \left(1-\frac{(v_x+k/n)^2}{c^2}-\frac{v_{ye}^2}{c^2}\right)^{-\frac{1}{2}} \qquad (23)\\
&= \left[1-\sigma^2\left\{\frac{1}{c^2}\left(v_1+\frac{k(1-M/v_1\sigma)}{n_1(1-M/v_1)}\right)^2+\left(P\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2}+Qf\right)^2\right\}\right]^{-\frac{1}{2}},
\end{aligned}
$$

where

$$
P=\frac{1-M/v_1\,\sigma}{4\pi n_1\,e(1-M/v_1)}, \qquad Q=\frac{ek}{c^2\sigma n_1\,m_{\mathrm{i}}}.
$$

Since v_1 is of the order of the Alfvén speed v_{A}, which in turn is assumed to be much smaller than c, then $(v_1/c)^2$ is much smaller than unity; finally, the approximate expression for r_{e} turns out to be

$$
\begin{aligned}
r_{\mathrm{e}} \approx{}& \left[1-\left(\sigma-\frac{M}{v_1}\right)^2\left\{\left(\frac{k}{n_1(1-M/v_1)}\right)^2+\left[\frac{1}{4\pi n_1\,e}\left(1-\frac{M}{v_1}\right)^{-1}\right.\right.\right. \\
&\times\left.\left.\left.\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2}+\frac{ek}{c^2 n_1\,m_{\mathrm{i}}(\sigma-M/v_1)}\,f\right]^2\right\}\right]^{-\frac{1}{2}}. \qquad (24)
\end{aligned}
$$

So, finally from equation (8) after using these approximations, we get

$$
\begin{aligned}
&\frac{m_{\mathrm{e}}}{e}(4\pi n_1\,e)v_{ye}\,\frac{\partial}{\partial\xi}\left(\frac{v_{ye}}{(1-v_{ye}^2/c^2)^{\frac{1}{2}}}\right) \\
&\quad=\tfrac{1}{2}\left[\frac{2}{(1-M/v_1)+k/n_1\,v_1}\left(1-\frac{kM}{n_1\,v_1(1-M/v_1)}\,B-\frac{cE_{y_1}}{v_1}\right)\right. \\
&\qquad\left.-\frac{2(B^2-B_1^2)}{16\pi m_{\mathrm{i}}\,n_1\,v_1^2(1-M/v_1)^2}\,2B\,\frac{\mathrm{d}B}{\mathrm{d}\xi}\right]. \qquad (25)
\end{aligned}
$$

Using the identity

$$
f(x)\,\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{f(x)}{[1-f^2(x)]^{\frac{1}{2}}}=\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{1}{[1-f^2(x)]^{\frac{1}{2}}},
$$

we get after a series of algebraic manipulations

$$A\left(\frac{d^2 f}{d\xi^2}\right)^2 + Bf\frac{d^2 f}{d\xi^2} + Cf^2$$

$$= -1 + \left\{\frac{\omega_{\mathrm{pe}}}{2\omega_{\mathrm{ce}}^2}\left[(\tilde{B}-1)S\left(T(\tilde{B}+1) - \frac{2cE_{y1}}{v_1 B_1}\right) - \frac{(\tilde{B}^2-1)^2}{4M_{\mathrm{A}}^2(1-M/v_1)^2}\right] + \left[1 - \left(\frac{\sigma_1\,\omega_{\mathrm{ce}}}{\omega_{\mathrm{pe}}}\frac{d\tilde{B}}{d\tilde{x}_1}\right)^2\right]^{-\frac{1}{2}}\right\}^{-2}, \tag{26}$$

where

$$A = \frac{(\sigma - M/v_1)^2}{[4\pi e n_1(1-M/v_1)]^2},$$

$$B = \frac{2e^2k^2}{c^2 n_1^2 m_i^2}, \qquad C = \frac{ek}{cn_1 m_i},$$

$$S = \left(1 - \frac{kM}{n_1 v_1(1-M/v_1)}\right)\Big/\left(1 - \frac{M}{v_1} + \frac{k}{n_1 v_1}\right),$$

$$T = 1 - \frac{kM}{n_1 v_1(1-M/v_1)}. \tag{27}$$

Since the second and third terms on the left-hand side of (26) contain a $1/c^2$ factor, it is not difficult to establish that these terms are small compared with the first. We neglect these terms and arrive at

$$\tfrac{1}{2}\left(\frac{d\tilde{B}}{d\tilde{x}}\right)^2 + \phi(\tilde{B}) = 0, \tag{28}$$

which is similar to the energy equation for particle motion in a potential. In equation (28), $\phi(\tilde{B})$ actually stands for the negative right-hand side of (26). Here $\tilde{B}$ and $\tilde{x}$ are the normalised magnetic field and ξ, respectively, defined as

$$\tilde{B} = B/B_1, \qquad \tilde{x} = \xi(c/\omega_{\mathrm{pe}}), \tag{29}$$

where $\omega_{\mathrm{pe}} = (4\pi n_1 e^2/m_e)^{\frac{1}{2}}$ and $\omega_{\mathrm{ce}} = B_1 c/4\pi n_1 e$.

3. Analysis of the Effective Potential

Equation (28) is known as the effective potential equation, where $\frac{1}{2}(d\tilde{B}/d\tilde{x})^2$ is similar to the kinetic energy of a particle of unit mass and $\phi(\tilde{B})$ is the potential energy in which it is moving. From the form of equation (28) it is clear that it cannot be treated analytically so we take recourse to numerical methods. But before that we proceed to discuss some important and salient features of the effective potential. The first and most important is that in the nonrelativistic limit the function $\phi(\tilde{B})$ tends to

$$\phi_{\rm NR}(\tilde{B}) = -\tfrac{1}{2}\Big[(\tilde{B}-1)\Big(\tilde{B}+1-\frac{2cE_{y1}}{v_1 B_1}\Big)\Big(1+\frac{k}{n_1 v_1}\Big)^{-1} - \frac{v_{\rm A}^2}{4v_1^2}(\tilde{B}^2-1)^2+\Big(\frac{{\rm d}\tilde{B}}{{\rm d}\tilde{x}}\Big)^2\Big], \qquad (30)$$

when $M/v_1 = 0$, which was actually deduced by Adlam and Allen (1958), Davis *et al.* (1958) and Sagdeev (1966) in their original nonrelativistic analysis. Next observe that where $\tilde{B}$ attains its maximum value $\tilde{B}_{\rm m}$, then ${\rm d}\tilde{B}/{\rm d}\tilde{x}$ should vanish, whence $\phi(\tilde{B}_{\rm m}) = 0$. Also, using the expression for ϕ, we get

$$M_{\rm A} = \Big(1-\frac{M}{v_1}+\frac{k}{n_1 v_1}\Big)^{\frac{1}{2}} \Big/ \Big[2\Big(1-\frac{M}{v_1}\Big) \times\Big(1-\frac{kM}{n_1 v_1(1-M/v_1)}\Big)\Big](\tilde{B}_{\rm m}+1)\,. \qquad (31)$$

Using this expression ϕ take the form

$$\phi(\tilde{B}) = \frac{\omega_{\rm pe}^2(1-M/v_1)^2}{2\omega_{\rm ce}^2(\sigma-M/v_1)^2}(-1+Q^{-2})\,, \qquad (32)$$

where

$$\sigma = 1-\frac{B^2-B_1^2}{8\pi m_1 n_1(v_1^2-Mv_1)}$$
$$= 1-2(\tilde{B}^2-1)\Big(1-\frac{M}{v_1}\Big)\Big(1-\frac{kM}{n_1 v_1(1-M/v_1)}\Big)^2 \Big/ \times\Big[\Big(1-\frac{M}{v_1}+\frac{k}{n_1 v_1}\Big)(\tilde{B}_{\rm m}+1)^2\Big]\,. \qquad (33)$$

On the other hand Q can be written as

$$Q = 1+\frac{\omega_{\rm ce}^2}{2\omega_{\rm pe}^2}\Big[\Big(1-\frac{kM}{n_1 v_1(1-M/v_1)}\Big)^2 \Big/ \Big(1-\frac{M}{v_1}+\frac{k}{n_1 v_1}\Big) \times(\tilde{B}-1)^2\Big(1-\frac{(\tilde{B}+1)^2}{(\tilde{B}_{\rm m}+1)^2}\Big)\Big]\,. \qquad (34)$$

From the condition that σ cannot be zero it follows that $1 < \tilde{B}_{\rm m} < 3$. It may be noted that we can obtain the nonrelativistic analogue of equation (28) by setting

$M = 0$, the corresponding expression being the same as that given by Adlam and Allen (1958) and Davis *et al.* (1958).

Now from equation (28) it follows that the solitary wave solution will exist if $\phi(\tilde{B}) < 0$ and from equation (32) we see that $\phi(\tilde{B})$ is negative if $1 < \tilde{B} < \tilde{B}_{\mathrm{m}}$. Furthermore, we have for $\tilde{B} \approx 1$

$$\phi(\tilde{B}) \approx -\tfrac{1}{2}\left[\left(1 - \frac{kM}{n_1 v_1(1 - M/v_1)}\right)^2 \Big/ \left(1 - \frac{M}{v_1} + \frac{k}{n_1 v_1}\right)\right] \times (\tilde{B} - 1)^2\left(1 - \frac{(\tilde{B}+1)^2}{(\tilde{B}_{\mathrm{m}}+1)^2}\right), \tag{35}$$

and for $\tilde{B} \approx \tilde{B}_{\mathrm{m}}$

$$\begin{aligned}\phi(\tilde{B}) \approx\ & \frac{(1 - M/v_1)^2}{(\sigma - M/v_1)^2} \\ & \times \left[\left(1 - \frac{kM}{n_1 v_1(1 - M/v_1)}\right) \Big/ \left(1 - \frac{M}{v_1} + \frac{k}{n_1 v_1}\right)\right] \\ & \times \frac{(\tilde{B}_{\mathrm{m}} - 1)^2(\tilde{B} - \tilde{B}_{\mathrm{m}})}{\tilde{B}_{\mathrm{m}}^2 + 1} \\ =\ & \left[\left(1 - \frac{M}{v_1} - \frac{kM}{n_1 v_1}\right)^2 \Big/ \left(1 - \frac{M}{v_1} + \frac{k}{n_1 v_1}\right)\right] \\ & \times \frac{(\tilde{B}_{\mathrm{m}}+1)(\tilde{B}_{\mathrm{m}} - 1)^2(\tilde{B} - \tilde{B}_{\mathrm{m}})}{[B_{\mathrm{m}}(2A - 1 + M/v_1) - (2A + 1 - M/v)]^2},\end{aligned} \tag{36}$$

where

$$A = \left[\left(1 - \frac{M}{v_1}\right)^2 - \frac{kM}{n_1 v_1}\right]^2 \Big/ \left(1 - \frac{M}{v_1}\right)\left(1 - \frac{M}{v_1} + \frac{k}{n_1 v_1}\right).$$

It is well known that in the nonrelativistic case the width of the magnetosonic soliton is of the order of the electron inertial length divided by the square root of the amplitude, i.e. $\Delta \sim (c/\omega_{\mathrm{p}})/(\delta\tilde{B})^{\frac{1}{2}}$, $\delta\tilde{B} \sim (\tilde{B}_{\mathrm{m}} - B_1)/B_1$. Similar considerations can also be made in our case. On the other hand, for $\tilde{B} - 1 \sim \mathrm{O}(1)$ and $\tilde{B} - \tilde{B}_{\mathrm{m}} \sim \mathrm{O}(1)$, that is for the region where the amplitude is not too small and not too close to the peak value, we get

$$\Delta \sim \frac{c}{\omega_{\mathrm{pe}}}\frac{\omega_{\mathrm{ce}}}{\omega_{\mathrm{pe}}}\,\delta\tilde{B}. \tag{37}$$

In the highly relativistic case we get $\omega_{\mathrm{ce}}^2/2\omega_{\mathrm{pe}}^2 \gg 1$. The behaviour of the relativistic soliton is quite different from the nonrelativistic situation.

In Fig. 1 we have plotted the effective potential ϕ as a function of the magnetic field. An important event to observe is that the form of ϕ is of the single well type,

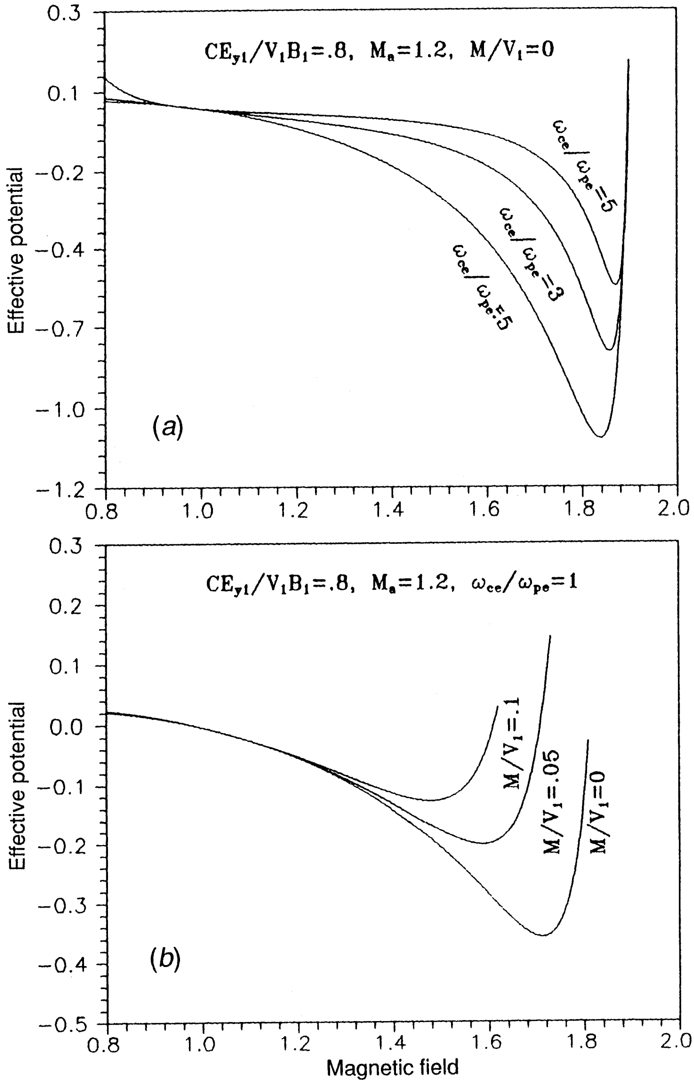

Fig. 1. Variation of the effective potential with magnetic field for different values of (a) $\omega_{\rm ce}/\omega_{\rm pe}$ and (b) M/v_1.

which shows the possibility of trapping the particles giving rise to the formation of a solution. In Fig. 1a we show the variation of the potential with respect to $\omega_{\rm ce}/\omega_{\rm pe}$ and it may be noted that for small values of $\omega_{\rm ce}/\omega_{\rm pe}$ the probability of trapping is greater. On the other hand, in Fig. 1b the variation with respect to M/v_1 is given, where M is the Mach number. In Fig. 2a an interesting situation occurs for the variation with respect to $y = cE_{y_1}/v_1\, B_1$, where values of $y \geq 1{\cdot}2$ have been considered. The well structure of the potential completely disappears. The same is also true for values of $M/v_1 > 1$ (as displayed in Fig. 2b).

We then integrated equation (28) numerically and the solitary wave so obtained is exhibited in Fig. 3 for various values of the parameters. It may be observed

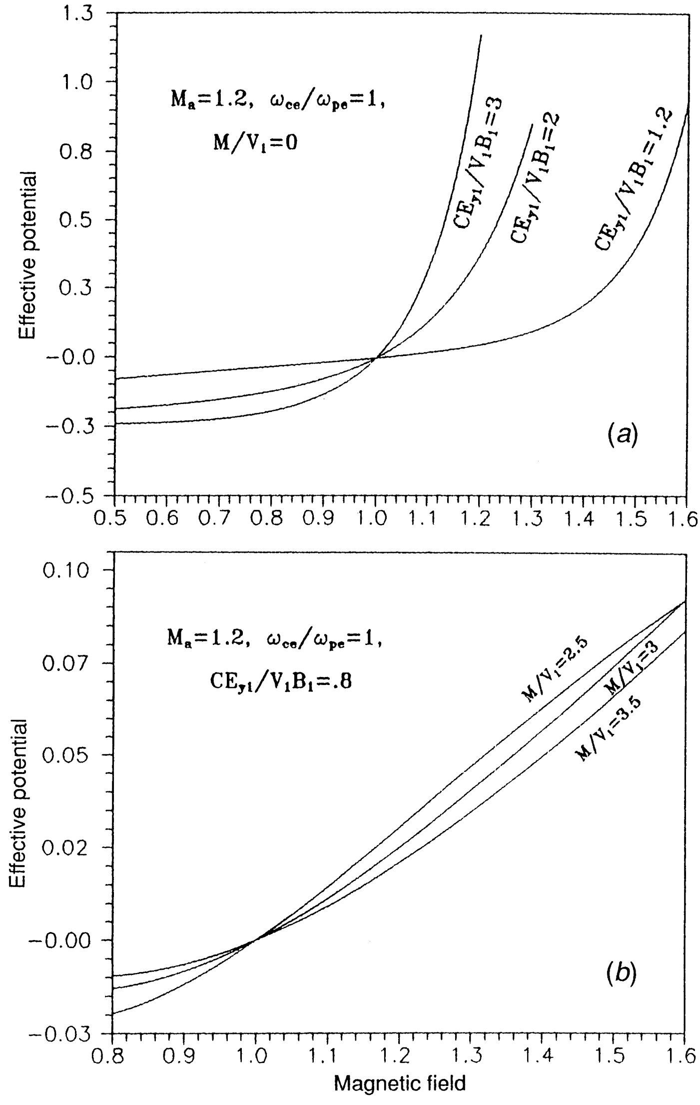

Fig. 2. Variation of the effective potential with (a) $cE_{y_1}/v_1 B_1$ and (b) M/v_1, showing the disappearance of the well structure.

that the peak of the solitary wave increases with an increase in the value of $cE_{y_1}/v_1 B_1$ (Fig. 3b), but on the other hand it becomes flatter for large values of M/v_1 and ω_{ce}/ω_{pe}.

Lastly, in view of equation (20), we have

$$v_{ye} = \frac{C(\sigma - M/v_1)}{4\pi n_1 e(1 - M/v_1)} \frac{dB}{d\xi} \,. \tag{38}$$

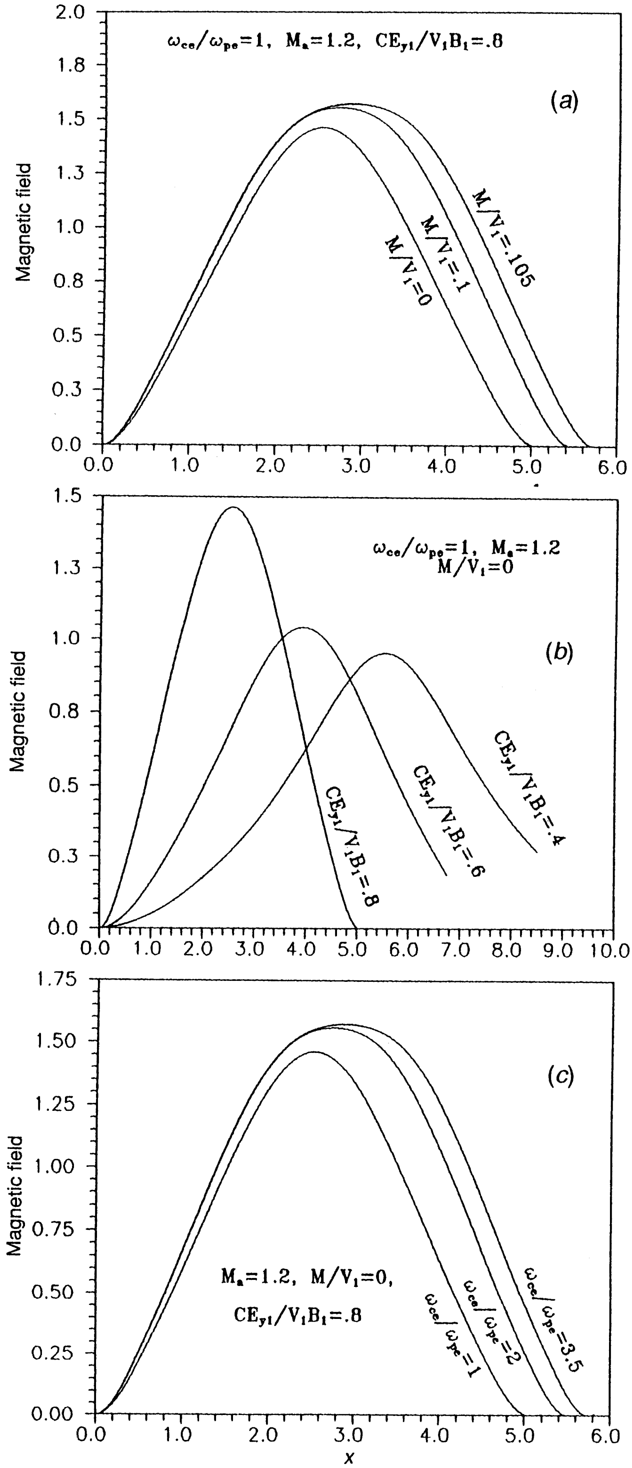

Fig. 3. Change in solitary wave structure with (*a*) M/v_1, (*b*) $cE_{y1}/v_1 B_1$ and (*c*) ω_{ce}/ω_{pe}.

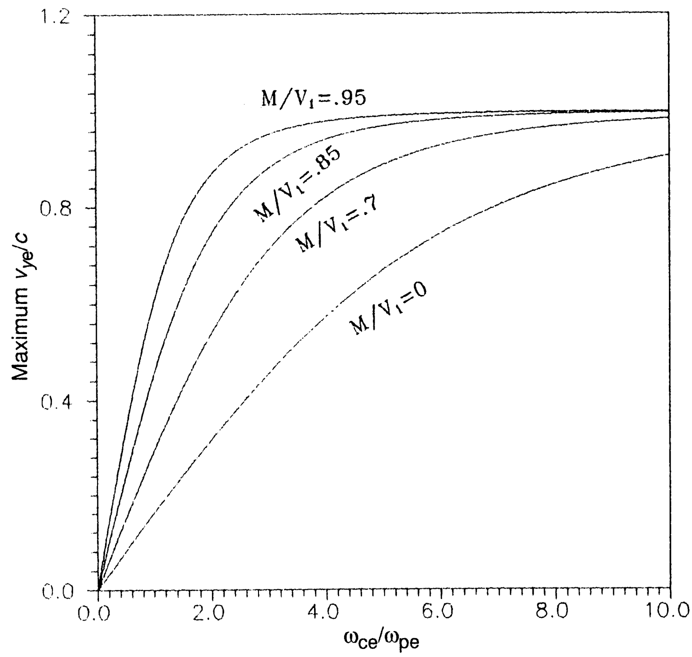

Fig. 4. Maximum electron velocity for different values of M/v_1, with $M_a = 1 \cdot 2$.

So when the magnetic field shows a solitonic structure we can evaluate the corresponding electron velocity. In Fig. 4 we show the behaviour of v_{ye} for various values of M/v_1.

4. Discussion

In our analysis we have studied the formation of magnetosonic solitary waves in a relativistic magnetised plasma, where both the ions and electrons have been considered to be relativistic. Such a situation is usually seen to take place in solar bursts or ionospheric plasmas. The phenomenon can be of great importance in high energy laser–plasma interactions.

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