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Statistical Symmetries in Physics

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Abstract

Every law of physics is invariant under some group of transformations and is therefore the expression of some type of symmetry. Symmetries are classified as geometrical, dynamical or statistical. At the most fundamental level, statistical symmetries are expressed in the field theories of the elementary particles. This paper traces some of the developments from the discovery of Bose statistics, one of the two fundamental symmetries of physics. A series of generalizations of Bose statistics is described. A supersymmetric generalization accommodates fermions as well as bosons, and further generalizations, including parastatistics, modular statistics and graded statistics, accommodate particles with properties such as 'colour'. A factorization of elements of $ggl(n_b, n_f)$ can be used to define truncated boson operators. A general construction is given for q-deformed boson operators, and explicit constructions of the same type are given for various 'deformed' algebras; these include a rather simple Q-deformed variety as well as the well known q-deformed variety. A summary is given of some of the applications and potential applications.

1. Introduction

The purpose of science is the discovery of order in the apparent chaos of natural phenomena. The most fundamental of scientific achievements are formulated in the laws of physics, and it has gradually become apparent in the last hundred years that, without exception, these laws are best understood as revealing underlying symmetries in nature. There is a group of transformations and an invariance associated with every symmetry, and it is the discovery of the invariance which may be regarded as the ultimate aim of both experimental and theoretical physics.

A symmetry may be classified as geometrical, dynamical or statistical. The laws of relativity express the geometrical symmetries of objective space and time. The laws of mechanics, and especially of quantum mechanics, express the dynamical symmetries relating the experiences of different observers. At the most basic level, the laws of field theory express the statistical symmetries of the elementary particles. The centenary of S. N. Bose (1894–1994) is a suitable occasion to recognize the absolute importance of his achievement (Bose 1924) in the discovery of Bose statistics, the first known type of quantum statistics, and one of the two fundamental symmetries of nature. The primary object of this paper is to trace some of the subsequent developments in quantum statistics from this discovery to the present day.

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Bose's original application of quantum statistics was to the quantum theory of black-body radiation, but there have been profound consequences throughout experimental and theoretical physics, leading on the one hand to the understanding of phenomena such as superfluidity, superconductivity and the stimulated emission of radiation, as in lasers, and on the other hand to the solution of problems in quantum mechanics, quantized field theory, quantum statistical mechanics and the Bose–Einstein condensation. In this paper we shall discuss algebraic methods, including some which are new, for the treatment of such problems, in the context of supersymmetry.

In fact, the simplest formulation of any statistical symmetry is in algebraic terms. With a photon, or any other boson, we may associate a creation operator b^*, and a corresponding annihilation operator b; these can be represented by semi-infinite matrices satisfying

$$bb^* - b^*b = 1\,, \tag{1}$$

where 1 is the unit matrix. Though this matrix formulation seems far removed from Bose's discovery, it has the same meaning: the matrix $N_b = b^*b$, representing the number of bosons with the same spin and momentum, can have any non-negative integral eigenvalue, yet they do not satisfy classical statistics, inasmuch as the interchange of two bosons of the same type leaves the state unchanged.

The invention of Fermi–Dirac statistics, which provides a natural formulation of Pauli's exclusion principle, followed in 1926. For an electron, or any other fermion, the matrices a^* and a, corresponding to b^* and b, satisfy the very similar relations

$$aa^* + a^*a = 1\,, \tag{2}$$

but, because of the difference in sign, the matrix $N_f = a^*a$, representing the number of fermions of a particular type, has only the eigenvalues 0 and 1 in an irreducible two-dimensional representation.

It is surely no coincidence that matrices of both types also have dynamical applications. Maxwell's equations imply that the photon is also a linear oscillator, and in the dynamical context a position and momentum can be associated with an oscillator, given by $x = i(b - b^*)/2^{\frac{1}{2}}$ and $p = (b + b^*)/2^{\frac{1}{2}}$ in suitable units, respectively; these matrices satisfy the commutation relation $[x,p] \equiv xp - px = i$, first stated by Born and Jordan. Similarly, the there-or-not-there property of an electron is matched by the dynamical up-or-down property of its spin, and in the dynamical context a and a^* are simply operators which shift the spin from the value $\frac{1}{2}$ to $-\frac{1}{2}$ and back again. The energy E of an oscillator, in terms of $\hbar\omega$, where ω is the angular frequency, is given by

$$e = E/(\hbar\omega) = \tfrac{1}{2}(b^*b + bb^*) = N_b \,+\, \tfrac{1}{2}\,; \tag{3}$$

N_b still has non-negative integral eigenvalues, and the additional term $\frac{1}{2}$ represents the 'zero-point' energy of an harmonic oscillator that is peculiar to quantum mechanics. In another dynamical context, the up-or-down component of the spin S is given by

$$S = \tfrac{1}{2}(a^*a - aa^*) = N_f - \tfrac{1}{2}\,, \tag{4}$$

where N_f still has the eigenvalues 0 and 1.

As a matter of interest, it follows from (1) and (3) that

$$be - eb = b, \qquad eb^* - b^*e = b^*\,, \tag{5}$$

and these relations, together with (3), define the Lie superalgebra $osp(2,1)$ in Cartan form. In the dynamical context, it was noticed by Wigner (1950) and Yang (1951) that this superalgebra has representations more general than implied by (1)—though of course they did not express it in that way! It was easy to conclude (Green 1953) that there must be generalizations of Bose statistics for which (1) was also not satisfied. But it follows in the same way from (2) and (4) that

$$aS - Sa = a, \qquad Sa^* - a^*S = a^*\,, \tag{6}$$

and these relations, together with (4), define the Lie algebra $so(3)$ in Cartan form, and it is well known that in the dynamical context there is a whole series of representations more general than implied by (2).

The expression of dynamical quantities in terms of boson and fermion matrices, or equivalently in terms of representations of the Lie algebras $osp(2,1)$ [or $so(2,1)$] and $so(3)$, can be used to obtain exact solutions to many elementary problems of quantum mechanics. This algebraic method, developed in principle by Born, Jordan, Heisenberg, Pauli and Dirac, is so much simpler than the method based on Schrödinger's equation that its neglect over a long period is surprising. It is also easily extended to a wide variety of problems that do not have exact solutions by means of a technique of 'codiagonal perturbations' (Green and Triffet 1969). However, in recent years, through a series of generalizations of the simple boson and fermion algebras, the range of exactly soluble problems has also been widely extended. This has been brought about especially by the 'q-deformation' of the symmetries associated with the classical algebras and superalgebras (Sklyanin 1982; Jimbo 1990). But this is not the only type of deformation compatible with symmetry and supersymmetry, and in the course of this paper we shall uncover general relations between the classical and deformed superalgebras, including deformed superalgebras with more than one parameter that are rather simple and have not been studied previously. In the final Section of this paper we shall illustrate some of the applications with the help of these 'Q-deformed' algebras.

Certain of the extensions to be considered are adapted to the requirements of quantized field theory. From the ordinary boson and fermion creation and annihilation operators, it is in fact easy to construct the bare bones of field theories capable of representing particles of various types; but it is necessary to provide for the possibility of bosons or fermions in different states, distinguished by spin, momentum, and possibly other properties such as charge and colour. The bosons and fermions can both be accommodated in a supersymmetric algebra. If $v_j = 1$ in bosonic states, or -1 in fermionic states, and $b^j = v_j b_j^*$ creates while b_j annihilates a particle in the jth state ($j = 1, ..., n$), then

$$[b_k, b^j] = b_k b^j - v_{-k,j} b^j b_k = v_j \delta_k^j\,, \tag{7}$$

where the brackets denote the generalized commutator, and

$$v_{-k,j} = 1 - \tfrac{1}{2}(1 - v_{-k})(1 - v_j)\,.$$

The notation implies that in general the subscripts of commutation factors like v_{jk} are to be treated as grading vectors depending on the types of particles, and

$$v_{jk}v_{kj} = 1, \qquad v_{j+k,l} = v_{jl}v_{kl}\,. \tag{8}$$

For ordinary bosons and fermions, the subscripts are $\mathbf{Z}_2$-graded, but there are other possibilities which will be mentioned below.

We remark here that although we are treating the number of types of particles (n) as finite, there are no insuperable problems when $n \to \infty$. It is well known that there is a matrix representation for these operators in a direct product of representations corresponding to the individual particles; this is particularly simple for bosons.

If b^j_k and the generalized anticommutator $\{b^j, b_k\}$ are defined by

$$b^j_k = \{b^j, b_k\} - \tfrac{1}{2}\delta^j_k = b^j b_k\,, \tag{9}$$

it is easy to verify with the help of (7) that

$$[b^j_k, b^l] = b^j_k b^l - v_{j-k,l} b^l b_{jk} = v_l \delta^l_k b^j, \qquad [b_j, b^k_l] = b_j b^k_l - v_{-j,k-l} b^j b_k = v_k \delta^k_j b_l\,,$$

$$[b^j_k, b^l_m] = b^j_k b^l_m - v_{j-k,l-m} b^l_m b^j_k = v_l \delta^l_k b^j_m - v_{j-k,l-m} v_j \delta^j_m b^l_k\,. \tag{10}$$

Now, although (7) and (9) imply (10), equation (10), even in conjuction with (9), does not imply (7). In fact the last equality of (10) implies that the b^j_k are elements of and define the superalgebra $ggl(n_b, n_f)$, where n_b and n_f are the numbers of bosonic and fermionic states respectively. Also, (9) and (10) together imply that the b^j, b_k, and b^j_k, together with the generalized anticommutators $\{b^j, b^l\}$ and $\{b_k, b_m\}$ are elements of the superalgebra $osp(2n_b, 2n_f + 1)$. If only bosonic states are present ($n_f = 0$), the superalgebra reduces to $osp(2n, 1)$; if only fermionic states are present ($n_b = 0$), it reduces to $so(2n+1)$. If b^j and b_k belong to the simplest of the represesentations of the superalgebra $osp(2n_b, 2n_f + 1)$, (7) is in fact satisfied and a synthesis of Bose and Fermi statistics results. But there is a whole series of other representations, which correspond to the generalization of quantum statistics known as parastatistics.

The symmetries of this generalization of Bose and Fermi statistics are implied by the structure of $osp(2n_b, 2n_f + 1)$; the represention theory has been investigated by Drühl *et al.* (1970), Bracken and Green (1972), Gray and Hurst (1975), and Ohnuki and Kamefuchi (1982), among others. Parastatistics is old enough (Green 1953) for its applications to have been thoroughly investigated (Messiah and Greenberg 1969; Green 1972). It is noteworthy that Greenberg and Macrae (1983) have shown that it is quite possible to formulate a viable gauge theory of quarks and gluons in terms of parastatistics. But, except in the instances of parastatistics of orders 1 and 2, field theories based on parastatistics present certain difficulties (Gray 1973), and it seemed to the author worth enquiring whether there might not be another and perhaps simpler kind of generalized quantum statistics, better suited to quantum chromodynamics.

The result (Green 1975, 1976) was called modular statistics, because observable particles (e.g. baryons or mesons) were identified as modules consisting of a number of intrinsically unobservable particles (e.g. quarks or gluons). This idea has some affinity with modern string theories. The cyclic symmetry invoked is one which is often overlooked because it is so simple, but has applications in many areas of physics (Green 1994).

Modular statistics of of order 1 are indistinguishable from ordinary quantum statistics, and modular statistics of order 2 are the same as parastatistics of order 2, but for $m > 2$ there are differences. If b^j and b_k are creation and annihilation operators for modular particles of order m, the defining relations are

$$b_j(b_i...b_l)b_k - v_{-j,-k}b_k(b_i...b_l)b_j = 0\,,$$

$$(b_i...b_l)b_jb^k - v_{k,-j}b^kb_j(b_i...b_l) = v_k\delta_j^k(b_i...b_l)\,, \tag{11}$$

and their conjugates, if the product $(b_i...b_l)$ has $m-1$ factors. They generalize in a natural way the relations of modular statistics for any number of bosons and fermions with m colour states. There are 'colourless' modules, defined as products of the b^j and b_k, like b^jb_k and $(b_i...b_l)b_k$, that satisfy ordinary commutation and anticommutation relations, and it is these that are supposed to create or annihilate observable particles, such as baryons and mesons. If this concept of colour is correct, quarks and gluons can only occur in modules, and will never be seen in isolation.

Colour statistics is a generalization of ordinary quantum statistics that is closely related to modular statistics (Kleeman 1983, 1985), and has defining relations identical with (7)–(9) above, except that the commutation coefficients v_{jk} in (8) are not restricted to the values -1 and $+1$, but are complex in general, and typically mth roots of unity, if m different colours are required. Again it is only 'colourless' products of the colour operators which commute or anti-commute, and only these can create or destroy the observable particles. With the more general values of the v_{jk}, the commutation relations in (7), or (9) and (10), define a colour superalgebra that is a particular instance of the algebras introduced by Rittenberg and Wyler (1978). Just as bosonic and fermionic operators can be used to construct the elements of ordinary superalgebras, coloured boson and fermion operators can be used to constuct the elements of colour superalgebras, at least in certain representations.

There are other modifications of quantum statistics. One kind of modified boson was introduced by Lohe and Hurst (1971) to construct general representations of ordinary Lie algebras. Another current modification of quantum statistics is found in 'fractional' statistics, with a kind of modified boson called anyons (see Aneziris *et al.* 1991; Sen and Chitra 1992), whose use, however, is limited by topological considerations to two-dimensional problems. In the following Sections, we shall study yet another kind of generalized quantum statistics which, like parastistics, makes use of the relations (10), but has a different construction for the particle creation and annihilation operators.

2. Factorizations of $ggl(n_b,n_f)$

Modular or coloured bosons and fermions can be constructed from ordinary bosons and fermions and inherit their symmetry; even the parabosons and parafermions inherit the symmetries of the superalgebra $osp(2n_b, 2n_f + 1)$. We shall now discuss the possibilities inherent in a somewhat different generalization of boson and fermion creation and annihilation operators, defined simply as elements of a particular representation of the superalgebra $ggl(n_b + 1, n_f)$. This algebra can also be regarded as a supersymmetric generalization of the 'truncated' boson algebra studied by Buchdahl (1967) and Kleeman (1981).

The representations of $ggl(n_b, n_f)$ in terms of ordinary bosonic and fermionic operators, with elements of the form

$$b^j_k = b^j b_k + \lambda\delta^j_k\,, \tag{12}$$

where λ is an invariant, are well known. A factorization of the generators of this particular type is only available in states of maximal symmetry and, if $n_b > 0$, in infinite-dimensional representations; however, it may be generalized in the following way.

We consider completely reducible representations in which $v_j b^k_j$ is the hermitean conjugate of $v_k b^j_k$. In an irreducible finite-dimensional representation, the highest weight vector ψ is defined by the conditions $b^j_k\psi = 0$ for $1 \leq j < k \leq n = n_b + n_f$, and the set of highest weights $(l_1, l_2, ..., l_n)$ as the eigenvalues of the $v_j b^j_j$ on this vector:

$$v_j b^j_j \psi = l_j \psi\,. \tag{13}$$

The eigenvalue of the positive definite matrix $v_j v_k b^j_k b^k_j = b^k_j b^j_k + v_j v_k b^j_j - b^k_k$ cannot be negative, so $v_k(l_j - l_k) \geq 0$ when $j < k$. All other vectors of the representation space can be obtained from ψ by multiplying it by products of the b^l_m with $n \geq l > m \geq 1$.

The highest weights serve to label irreducible representations (assuming they are completely reducible), and can be regarded as eigenvalues of a set of invariants $(\lambda_1, \lambda_2, ..., \lambda_n)$ of the universal enveloping superalgebra.

Now, $ggl(n_b, n_f)$ is a subalgebra of $ggl(n_b+1, n_f)$, and the latter is spanned by the larger set of elements b^j_k with values of j and k up to $n+1$. In general, the irreducible representation of $ggl(n_b + 1, n_f)$ with highest weights $(l_1,\ l_2, ...,\ l_{n+1})$ contains many irreducible representations of $ggl(n_b,\ n_f)$, but if $v_{n+1} = v_n$ and $l_{n+1} = l_n$ it contains only the representation of $ggl(n_b, n_f)$ with highest weights $(l_1,\ l_2, ...,\ l_n)$; let us choose this representation, and suppose that the highest weight vector of $ggl(n_b + 1, n_f)$ is ψ, so that j is allowed to take values up to $n+1$ in (12). Then $b^j_{n+1}\psi = 0$ for all $j < n+1$, and by multiplying this equation by products of the b^l_m having $n \geq l > m \geq 1$, we see that it is also true that $b^j_{n+1}\phi = 0$ if ϕ is any vector of the space carrying the irreducible representation of $ggl(n_b, n_f)$. It then follows from the commutation relations of $ggl(n_b + 1, n_f)$ in (10) that

$$b^j_{n+1} b^{n+1}_k \phi = v_n(b^j_k - v_{j-n,n-k} v_j \delta^j_k l_n)\phi\,, \tag{14}$$

and that, within any tensor representation of $ggl(n_b, n_f)$, the factorization

$$b^j_k = c^j c_k + l_n v_{j-n,n-k} v_j \delta^j_k \quad (c^j = v_n b^j_{n+1}, \quad c_k = b^{n+1}_k) \tag{15}$$

is possible, where l_n is an invariant of $ggl(n_b, n_f)$. We note that when $v_n = 1$, the choice $l_n = 0$ is possible, and still permits tensor representations corresponding to any type of symmetry.

This is an analogue of the result (11) that is valid within a much less restricted class of representations. It can also be regarded as an application of the factorization method for finding the least eigenvalue l_n of b^j_j when $v_n = 1$. As c^j is the hermitean conjugate of $v_j c_j$, the eigenvalues of the

$$N_j = v_j c^j c_j \tag{16}$$

are non-negative integers. We note that $c^j c_k$ is an element of $ggl(n_b, n_f)$ in an irreducible representation with highest weights $(l_1 - l_n, l_2 - l_n, ..., 0)$, though its factors are well-defined only within the specified representation, with $l_n = l_{n+1}$, of $ggl(n_b + 1,\ n_f)$. We shall interpret the c^j and c_k as generalized creation and annihilation operators; they are clearly different from the corresponding operators of parastatistics, and cannot be used with the same freedom, since $c_k c^j$ vanishes within the irreducible representation of $ggl(n_b,\ n_f)$ considered. They have finite-dimensional representations, and in purely bosonic representations have the same algebraic properties as the 'truncated' bosons of Buchdahl (1967) and Kleeman (1981), with the commutation relations

$$[c^j c_k, c^l] = c^j c_k c^l - v_{j-k,l} c^l c^j c_k = v_l \delta^l_k c^j \,,$$

$$[c_j,\ c^k c_l] = c_j c^k c_l - v_{-j,k-l} c^k c_l c_j = v_k \delta^k_j c_l \,. \tag{17}$$

The factorization will be used in the following to construct representations of various 'deformed' generalizations of $ggl(n_b, n_f)$.

3. Q-Particle Algebras

The possibility of further significant generalizations of quantum statistics appeared with the development of 'q-deformed' algebras or 'quantum groups' by Sklyanin (1982), Jimbo (1990) and others. This has been followed by several different generalizations of (1) and (2). The first, developed by Greenberg (1990, 1991) and Mohapatra (1990), was based on the generalized boson commutation relation

$$b_q b^*_q - q b^*_q b_q = 1 \qquad (-1 \le q \le 1)\,, \tag{18}$$

which reduces to (1) when $q = 1$ and (2) when $q = -1$.

Another generalization was found by Biedenharn (1989) as a by-product of a boson construction of the q-deformed algebra $su_q(2)$. His commutation relations can be written in the form

$$b_q b^*_q - q b^*_q b_q = q^{-\lambda N}, \qquad N b_q - b_q N = b_q, \qquad b^*_q N - N b^*_q = b^*_q\,, \tag{19}$$

with $\lambda = 1$. Obviously (18) corresponds to $\lambda = 0$; this value, and $\lambda = 2$, were considered by Mohapatra (1990) in a discussion of possible physical applications of this type of generalized boson, which we shall call a q-boson. The quantum statistical mechanics of these particles has been formulated in different ways by Martin-Delgardo (1991) and Lee (1992). The interesting limit $q \to 0$ was studied by Govorkov (1983), and in finite-dimensional representations the resulting algebra is closely related to the 'truncated' boson algebra of Buchdahl and Kleeman, which was discussed in the last Section.

For arbitrary values of q, the generalized boson creation and annihilation operators of these types can be expressed in terms of the ordinary boson creation and annihilation operators satisfying $bb^* - b^*b = 1$; it is in fact easy to verify that the relations (19) are satisfied by the substitutions

$$b_q = bf(N), \qquad b_q^* = f^*(N)b^*, \qquad N = b^*b\,,$$

$$f(N)f^*(N) = (q^{-\lambda N} - q^{-N})/[N(q^{1-\lambda} - 1)]\,, \tag{20}$$

and the ordinary boson algebra is recovered in the limit $q \to 1$.

The algebra for a single type of q-boson is easily extended to allow for any number of states, with different spins, momenta etc., if the creation and annihilation operators in different states are allowed to commute or anticommute, and matrix representations can be found in the usual way in terms of direct products of matrices representing the individual q-bosons. Again, the representations are particularly simple when creation and annihilation operators for different types commute.

However, in the following we shall consider objects of a different kind, which will be called Q-particles. For the sake of simplicity, we consider only those of bosonic type ($n_f = 0$). A single particle of this kind has creation and annihilation operators e^* and e satisfying a relation of the type

$$ee^* - e^*e = \Delta(N)\,, \tag{21}$$

where N has non-negative eigenvalues, just as in (19) and (20). The function $\Delta(N)$ is not unique, but has two special forms that will be obtained unambiguously below, by assuming that e and e^* are elements of generalizations of $gl(n) = ggl(n,0)$, called $gl_Q(n)$ and $U_q[gl(n)]$, respectively.

We first state the generalized commutation relations satisfied by the elements e^j_k (j, $k = 1, \ldots n$) of $gl_Q(n)$. These are

$$[e^j_k, e^l_m] \equiv e^j_k e^l_m - v^{jl}_{km} e^l_m e^j_k = \delta^l_k e^j_m - v^{jl}_{km} \delta^j_m e^l_k\,,$$

$$v^{jl}_{km} = w^l_k / w^j_m, \qquad w^j_k = 1 + (Q_j - 1)\delta^j_k\,, \tag{22}$$

and it should be noted that, apart from the e^j_k, only constants are involved in this definition. When $Q_j = 1$, $w^j_k = 1$, and these relations define the Lie algebra $gl(n)$, the irreducible representations of which are well known and were included in the discussion of the previous Section. But it is not immediately clear that they allow matrix representations for any values of the Q_j. They are self-consistent for interchange of the pairs of affixes (j, k) and (l, m), and also under hermitean conjugation, but this is of course not sufficient. The question will be settled by obtaining explicit expressions for the e^j_k in terms of the c^j and c_k introduced in the previous Section, but first we obtain a few general results.

From $e^j_j e^j_k = e^j_k(Q_j e^j_j + 1)$, with $k \neq j$, it follows that

$$e^j_j = (Q_j^{N_j} - 1)/(Q_j - 1)\,, \tag{23}$$

if we require that, when the $Q_j \to 1$, the representation should reduce the representation of $gl(n)$, with $l_n = 0$, in (15). As in (16), N_j has non-negative integral eigenvalues and $N_j e^j_k = e^j_k(N_j + 1)$; similarly, $e^j_k N_k = (N_k + 1)e^j_k$ when $k \neq j$. The simplest invariants of the Q-algebra are functions of $\Sigma^n_{j=1} N_j$, but there are polynomials in the e^j_j of this form only if the $\log(Q_j)$ are commensurable.

From (22) it follows that

$$Q_j e^j_k e^k_j - Q_k e^k_j e^j_k = Q_j(Q_j^{N_j} - 1)/(Q_j - 1) - Q_k(Q_k^{N_k} - 1)/(Q_k - 1)\,. \tag{24}$$

To obtain representations in terms of $gl(n)$, we may now set

$$e^j_k = g_j(N_j)c^j c_k g^*_k(N_k), \qquad N_j = c^j c_j\,. \tag{25}$$

For $j \neq k$, (24) then yields a difference equation

$$Q_j G_j(N_j)G_k(N_k + 1) - Q_k G_k(N_k)G_j(N_j + 1)$$
$$= Q_j(Q_j^{N_j} - 1)/(Q_j - 1) - Q_k(Q_k^{N_k} - 1)/(Q_k - 1)$$

with the solution

$$G_j(N_j) \equiv N_j g_j(N_j)g^*_j(N_j) = (Q_j^{N_j} - 1)/(Q_j - 1)\,, \tag{26}$$

which determines $g_j(N_j)$, apart from an arbitrary phase factor. For the eigenvalue 0 of N_j, $g_j(N_j)$ may be given its limiting value 1.

We have now succeeded in satisfying all the relations (22) defining $gl_Q(n)$ in terms of the generalized creation and annihilation operators

$$e^j = g_j(N_j)c^j, \qquad e_k = c_k g^*_k(N_k)\,. \tag{27}$$

It is also possible to write

$$e^j_k = \tfrac{1}{2}(\{e^j, e_k\} - \Delta_j(N_j)\delta^j_k), \qquad e^{jk} = \tfrac{1}{2}\{e^j, e^k\}, \qquad e_{jk} = \tfrac{1}{2}\{e_j, e_k\}\,, \tag{28}$$

where $\Delta_j(N_j) = [e_j, e^j] = Q_j^{N_j}$, thus obtaining a representation of all the generators of $osp_Q(2n, 1)$ in terms of Q-particle creation and annihilation operators. We have also obtained the form of the function $\Delta(N)$ in (21), corresponding to this type of deformation.

We now assume that the Q_j have the same value (Q) for all j. The resulting Q-deformed algebra is a supersymmetric generalizaton of one obtained by the method of infinitesimal transformations elsewhere (Green 1994), and $gl_Q(n)$ still differs essentially from the q-deformed algebra denoted in the literature by $U_q[gl(n)]$. However, there is a relation between them. If we write

$$\epsilon_j = \varphi(N_j)e^j_{j+1}\varphi^*(N_{j+1}), \qquad \epsilon^j = \varphi(N_{j+1})e^{j+1}_j\varphi^*(N_j)\,,$$

$$\varphi(N)\varphi^*(N) = q^{1-N}, \qquad q^2 = Q\,, \tag{29}$$

and make use of (24), we find

$$[\epsilon_j, \epsilon^j] \equiv \epsilon_j\epsilon^j - \epsilon^j\epsilon_j = (q^{h_j} - q^{-h_j})/(q - q^{-1}), \qquad h_j = N_j - N_{j+1}\,, \tag{30}$$

and that if

$$\epsilon'_j = [\epsilon_j, \epsilon_{j+1}] \equiv \epsilon_j\epsilon_{j+1} - q\epsilon_{j+1}\epsilon_j\,, \tag{31}$$

then

$$[\epsilon_j, \epsilon'_j] \equiv \epsilon_j\epsilon'_j - q^{-1}v\epsilon'_j\epsilon_j = 0\,. \tag{32}$$

So, these and the other commutation relations of the q-deformed algebra $U_q[gl(n)]$ (Jimbo 1990) are satisfied identically by the substitutions of (29). It is also possible to obtain a supersymmetric generalization corresponding to the q-deformed algebra $U_q[ggl(n_b, n_f)]$ (Bracken *et al.* 1990). In the above we have considered 'deformations' determined by only a single parameter q, but there may well be q-deformations, like those obtained for $gl_Q(n)$ above, involving n or more parameters q_j; the existence of such generalizations of $U_q[ggl(n)]$ is in fact suggested by the work of Manin (1989), but a more general statement and verification of the commutation relations is not at all simple and will not be attempted here. In illustrating the applications, we shall therefore restrict attention to the Q-deformed variety of Lie algebras.

4. Applications of Generalized Statistics

We conclude with a brief summary of the more important uses and potential uses of generalized statistics, in most instances using the Q-deformed algebras for the purpose of illustration.

(1) The deformed algebras were introduced originally with the motivation of finding new solutions of the Yang–Baxter equation (Jimbo 1990), which has applications in quantum mechanics, quantized field theory and the statistical mechanics of crystal lattices. These have been discussed in a separate paper (Green 1994) and will not be elaborated here.

(2) The applications to quantized field theories and thence to high-energy particle physics of various types of generalized statistics have already been mentioned. The deformed creation and annihilation operators could have some use in the formulation of nonlinear field theories, but the development of such field theories is outside the scope of this paper.

(3) The simplest applications are to quantum mechanics. Here, for the purpose of illustration, we shall consider a system of n generalized oscillators. With e^j and e_j $(j = 1, ...n)$ defined as in (27), the energy of the j-th oscillator will be assumed to be a Q-analogue of the energy E_j of the simple oscillator in (3):

$$E_j/(\hbar\omega_j) = \tfrac{1}{2}(e^j e_j + e_j e^j) = G_j(N_j) + \tfrac{1}{2}\Delta_j(N_j)\,,$$

$$\Delta_j(N_j) = G_j(N_j + 1) - G_j(N_j)\,, \tag{33}$$

with $G_j(N_j)$ given by (26), and in units of $\hbar\omega$. This is clearly a nonlinear function of the number operator N_j in general and the energy levels are therefore not equally spaced; however, it is still true that e^j raises the eigenvalue of N_j by one unit, and e_j lowers it by one unit; therefore, like boson creation and annhilation operators, they are ladder operators which can be used to move from one energy level to another.

If mutual interactions are neglected, the energy of a system of oscillators of this type is

$$E(N) = \sum_j E_j(N_j) = \sum_j \tfrac{1}{2}\{e^j, e_j\}\hbar\omega_j \tag{34}$$

and is a transcendental function of the N_j in general. However, in constructing interactions for the Q-oscillators, we might wish to make use of the elements of $gl_Q(n_b)$ given in (25). Typically, these raise the eigenvalue of one of the N_j by one unit and lower the eigenvalue of another by one unit, and leave $\Sigma_j N_j$ unchanged. Thus, it may be appropriate to construct a Q-deformed energy that is conserved, and this is always possible. For the sake of simplicity, assume that $Q_j = Q_k = Q$, so that $G_j(N) = G_k(N) = G(N)$ and $\Delta_j(N) = \Delta_k(N) = \Delta(N)$. Then, with the help of identities such as $\Delta(N_j{+}N_k) = \Delta(N_j)\Delta(N_k)$ and $G(N_j + N_k) = \Delta(N_j)G(N_k) + G(N_j)$, it is easy to construct invariants of $ggl(n)$ in terms of which the combined energy of any set of interacting Q-oscillators might be expressed.

(4) The statistical mechanics of a set of M Q-deformed oscillators within a region of unit volume, neglecting energy of interaction between different oscillators, can be derived from the canonical partition function

$$Z_M = \exp(-\beta F_M) = \prod_j \sum_{N_j} \exp[-\beta E_j(N_j)]\,, \tag{36}$$

where F_M is the free energy for the set of $M = \Sigma_j N_j$ particles, and the summation is over all non-negative values of N_j. As in (34), the oscillators are distinguished only by their frequencies ω_j, and, as usual, $\beta = 1/(k_B T)$, where k_B is Boltzmann's constant and T is the absolute temperature.

The above canonical partition function can also be used for an independent-particle model of a set of particles in a region of unit volume, with energy levels determined by the number of particles in each level but approximating to those of the oscillators. In this application, N_j represents the number of particles occupying the jth energy level, and we can set $E_j(N_j) = E(N_j) = [G(N_j) + \frac{1}{2}\Delta(N_j)]\epsilon_j$. The corresponding grand partition function is

$$Z = e^{\beta p} = \exp\left(\sum_N \exp\{\beta[\mu N - E(N)]\}\right), \tag{37}$$

where p is the pressure and μ is the chemical potential per particle. The number density ρ and the internal energy U per unit volume are given by

$$\rho = \frac{1}{\beta Z}\frac{\partial Z}{\partial \mu}, \qquad U = \frac{1}{\beta Z}\frac{\partial Z}{\partial \beta}. \tag{38}$$

The internal energy can also be computed for the density $M = \Sigma_j N_j$, using the canonical instead of the grand partition function, and this was done to compute the curves in Fig 1, which correspond to five different

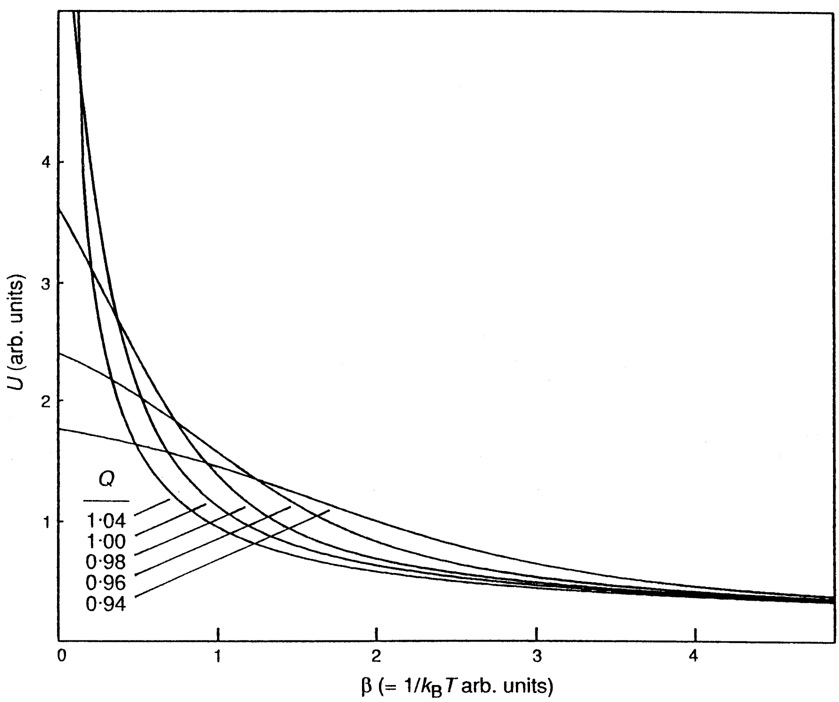

Fig. 1. Internal energy U, for various values of Q, as a function of inverse temperature for a set of oscillators with a Q-deformed energy spectrum.

values (0·94, 0·96, 0·98, 1·00 and 1·04) of Q. The curve with $Q = 1$ of course corresponds to the equally spaced energy levels of a Bose gas. The curve for $Q = 1\cdot 04$ reflects an increased internal energy at higher temperatures (small β), but not at lower temperatures because of a much reduced probability of the occupation of the higher levels, whereas the reverse is true for $Q < 1$.

(5) The theory of the Bose–Einstein condensation, which is linked to quantum statistical mechanics, depends rather sensitively on the value of Q. The critical temperature, where a condensation into the state of lowest energy ($j = 0$) occurs, is reached when the chemical potential reaches a value near to $\frac{1}{2}\epsilon_0$, which would cause the summation over N_0 in (37) to diverge if extended to infinity. For $Q = 1$ this occurs when N_0 has an expectation value given by $\beta N_0(N_0)(\mu - \frac{1}{2}\epsilon_0) \approx 1$. However, for more general values of Q the corresponding condition is

$$\beta N_0\mu - [G(N_0) + \tfrac{1}{2}\Delta(N_0)]\epsilon_0 \approx 1\,. \tag{39}$$

The theory of superfluidity in liquid helium, due to London (1938) and Tisza (1938), attributes the transition between the normal and superfluid states to the Bose–Einstein condensation. The predicted temperature of the phase transition, assuming $Q = 1$, is about 10% too high, but this is easily remedied by the use of a value of $Q \approx 1 + 1/(5N_0)$ a little greater than unity.

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