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Nuclear Equation of State as Determined by the Quantum Virial Expansion

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Abstract

The quantum virial expansion is used to investigate the properties of neutron matter in the temperature range $20 < kT < 60$ MeV. The pressure as a function of temperature is determined for various densities of neutron matter. Also the stiffness, the energy and the coefficient of volume expansion all as a function of temperature and density are studied.

1. Introduction

In this communication we discuss some of the thermal properties of nuclear matter for the intermediate temperature range, when the value of kT (k being the Boltzmann constant and T the temperature) lies between 20 and 60 MeV. We start from a quantum mechanical partition function and use the quantum cluster expansion (Kilpatrick *et al.* 1954; Hirschfelder *et al.* 1954; Huang 1987). We calculate the temperature dependence of the second virial coefficient and study the equation of state for neutron matter using the nucleon–nucleon phase shifts.

The great advantage of this method is that it does not depend on a detailed knowledge of the interaction which is not so well known, but instead relies on a knowledge of the experimentally well-established energy dependence of the phase shifts. The nuclear equation of state has been investigated on numerous occasions (Küpper *et al.* 1974; Bonche *et al.* 1985; Levit and Bonche 1985; ter-Haar and Malfliet 1987; Jaqaman *et al.* 1983, 1984; Bandyopadhyay *et al.* 1989; Sperber 1989; Suraud 1987; Brack *et al.* 1985) in the low energy regime or for $kT < 20$ MeV. The other extreme of the temperature range where kT is above 150 MeV has also been studied to some extent to determine if a quark–gluon plasma can be formed (Satz 1985; Karsch 1988). There is virtually no information on the thermostatic properties of nuclei or nuclear matter in the temperature regime which we are reporting. Various methods have been used to determine the equation of state and other thermostatic properties of finite nuclei as well as the properties of nuclear matter. In most cases, one first considers infinite nuclear matter and later makes appropriate corrections for finite nuclei. There are difficulties in a rigorous derivation of an equation of state for charged matter due to the long range of the electric Coulomb force. Hence, the corrections due to the Coulomb energy can only be introduced in a

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semi-phenomenological way. We therefore focus on uncharged infinite nuclear matter. In particular, we carry out our calculations for neutron matter, as the results may have some astrophysical significance. The virial expansion was used previously (Jaqaman *et al.* 1983) to study the nuclear equation of state. In the present paper we consider the most important contributions to the virial expansion and limit ourselves to the first two terms in the nuclear equation of state. First, we determine the second virial coefficient $B_2(T)$. We then use this information to study the dependence of the pressure on temperature for various nuclear densities. We study the energy per nucleon as a function of temperature, as well as the stiffness and the volume expansion coefficient.

2. Theory

The virial-cluster expansion is a very useful tool to obtain an equation of state for a classical gas. This expansion allows one to write the pressure as a power series in the density. The expansion coefficients may be obtained from the partition function and are temperature dependent, such that

$$P = kT\rho\{1 + B_2(T)\rho + B_3(T)\,\rho^2 + \ldots\}. \tag{1}$$

Here P is the pressure and ρ the density. The virial expansion yields the equation of state for a real gas, and the terms beyond the first term are due to the short range two-body interaction and represent the deviation from an ideal classical gas. For example, the second classical virial coefficient can be written as

$$B_2(T) = -2\pi\int_0^\infty [\exp\{-U(r)/kT\} - 1]r^2\,\mathrm{d}r, \tag{2}$$

where $U(r)$ is a central two-body potential.

Since the classical partition function has a quantum mechanical analog, one can use a similar formalism to determine the quantum virial coefficients. However, there is a notable difference between the classical and quantum mechanical case. In the classical case, when there is no interaction, all the virial coefficients vanish and one obtains an equation of state for a classical ideal gas. In the quantum mechanical case, even in the absence of interactions, the virial coefficients do not vanish. This is due to specific symmetry requirements imposed on the wavefunction of the system. For a system of fermions which we consider, the effect of the Pauli exclusion principle is equivalent to the existence of a repulsive force, hence the virial coefficients without an interaction are not equal to zero. They are positive, leading to an increase of pressure for an ideal Fermi gas as compared with an ideal classical gas. There are then two terms for any of the virial coefficients for a Fermi gas, one due to the Pauli exclusion principle $B_i^{\mathrm{f}}(T)$ and the other $B_i^{\mathrm{int}}(T)$ to interactions between particles. In particular, the first term for the second virial coefficient is given by

$$B_2^{\mathrm{f}}(T) = 2^{-5/2}\lambda_{\mathrm{T}}^3/g. \tag{3}$$

Here $\lambda_T = h/(2\pi mkT)^{1/2}$ is the thermal wavelength, g a spin degeneracy factor and m the nucleon mass.

As mentioned before, in the case of a classical approach the higher terms in the virial expansion starting with B_2 reflect the deviation from an ideal gas and are attributed to the interactions between the constituents of the gas. The quantum mechanical virial coefficients also depend on the interaction or potential between the particles. Unfortunately, the nuclear interaction is not very well known. For this reason we decided to express the second virial coefficient due to the interaction between particles in terms of experimentally measured and relatively well-known nucleon–nucleon phase shifts. The idea behind this approach is that asymptotically the interaction between nucleons modifies the partition function by changing the density of states in the phase space. The contribution to the second virial coefficient from nucleon–nucleon scattering can be then written (Hirschfelder *et al.* 1954) in the form

$$B_2^{\text{int}}(T) = -2^{3/2}\lambda_T^3 \sum_j (2j+1)\frac{1}{\pi}\int_0^\infty \frac{d\eta_j}{d\chi} \exp\left(-\frac{h^2\chi^2}{2\mu kT}\right) d\chi. \tag{4}$$

Here χ is the wave number, η_j the phase shift and μ the reduced mass. There are no equivalent simple forms in terms of phase shifts for higher virial coefficients. The higher order terms due to interaction in the virial expansion correspond to three- and more-body clusters and their contribution to the equation of state is expected to be considerably smaller than B_2^{int}. The higher terms coming from the Pauli exclusion principle can be easily included. Their values decrease with temperature; for example, the contribution due to B_3^{f} at $kT = 20$ MeV for the range of densities considered is of the order of a few per cent. Luckily enough at higher temperatures, the contributions from higher order clusters and many-body forces are less significant and can be neglected. It is important to recall that the measured phase shifts are for p–n scattering, while we are interested in the equation of state of neutron matter. However, since the wavefunction for an n–n system must be antisymmetric, in the summation over the angular momentum j we consider only contributions from the phase shifts of antisymmetric states properly weighted.

It should be noticed that we use free space phase shifts. It is expected that in nuclear matter the phase shifts will be modified due to the density dependence of the nucleon–nucleon interaction. This effect, as well as the neglect of higher coefficients, is believed to be of second order.

3. Results

We now discuss some of the results which we have obtained. First, we consider the temperature dependence of the second virial coefficient. In Fig. 1 we show $B_2^{\text{int}}(T)$, as calculated from the phase shifts according to equation (4) (dashed curve), and $B_2(T)$ which combines the contribution due to the Pauli principle with the one due to the interaction (solid curve). It is important to notice that, as expected, the Fermi contribution to the second virial coefficient is always positive but decreases with temperature. The interaction term is negative but for high temperatures its magnitude slowly decreases, so one can see the effect of the hard core in the nucleon–nucleon interaction. As can

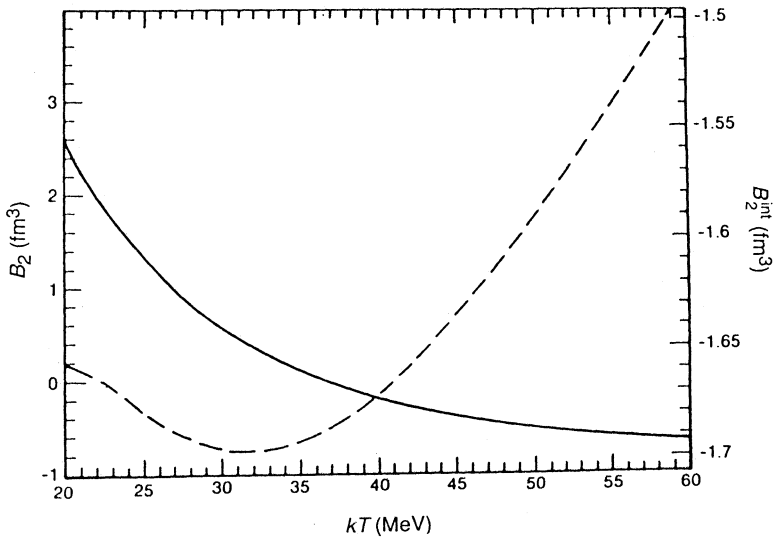

Fig. 1. Values of the second virial coefficients $B_2(T)$. The dashed curve shows the values calculated from the interaction only (right scale) and the solid curve shows the total second virial coefficient.

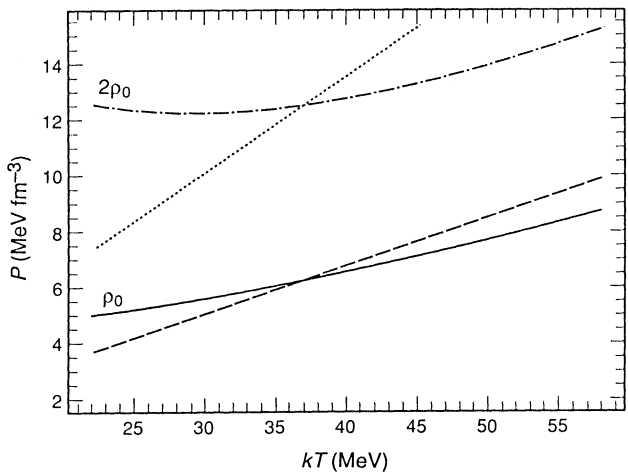

Fig. 2. Dependence of the pressure on temperature. The solid curve is for normal density, $\rho = \rho_0$, and the dot–dash curve for $\rho = 2\rho_0$. The dashed curve corresponds to an ideal gas of normal density, while the dotted curve shows the pressure of an ideal gas of density $\rho = 2\rho_0$.

be seen in Fig. 1 the second virial coefficient is positive for temperatures up to about 35 MeV and then drops below zero. For very high temperatures we found that B_2 asymptotically approaches zero suggesting that, as expected, the gas of interacting fermions approaches an ideal classical gas.

In Fig. 2 we display the pressure dependence on temperature for $\rho = \rho_0$ (solid curve) and $\rho = 2\rho_0$ (dot–dash cruve). Here ρ_0 is the density of normal nuclear matter. The dashed curve corresponds to an ideal gas of density ρ_0, while the dotted curve shows the pressure of an ideal gas of density $\rho = 2\rho_0$. For both densities we see that the behaviour of our equation of state is dominated at lower temperatures by the Pauli principle and at higher temperatures by the attractive nuclear interaction.

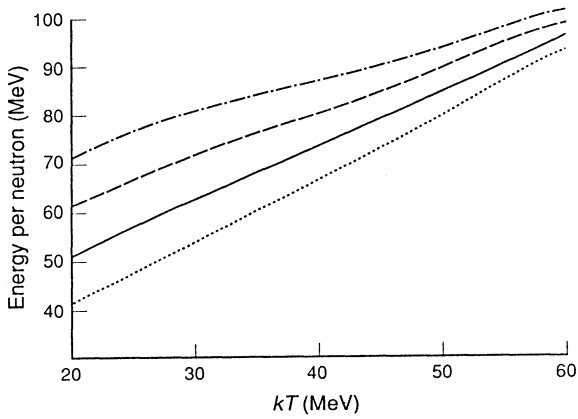

Fig. 3. Energy per neutron as a function of temperature for four different densities. The dotted curve is for $\rho = \rho_0/2$, the solid curve for normal density, the dashed curve for $\rho = 3\rho_0/2$ and the dot–dash curve for $\rho = 2\rho_0$.

In Fig. 3 we show the energy per neutron as a function of temperature for different nuclear densities: $\rho = \rho_0/2$ (dotted curve), $\rho = \rho_0$ (solid curve), $\rho = 3\rho_0/2$ (dashed curve) and $\rho = 2\rho_0$ (dot–dash curve). It is important to note that the energy of an interacting Fermi gas is always above the value for the classical non-interacting gas and it converges to the value of $E = \frac{3}{2}kT$ for very high temperatures.

In Fig. 4 we show the volume expansion coefficient as a function of temperature for the same densities as previously. It is rewarding to note that in the high temperature limit this expansion coefficient is independent of density and converges to the classical limit $1/T$. The coefficient of volume expansion α is defined as

$$\alpha = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P . \tag{5}$$

Finally, in Fig. 5 we show the stiffness as a function of temperature for the two densities $\rho = 2\rho_0$ (dot–dash curve) and $\rho = \rho_0$ (solid curve). This coefficient is defined as (Preston and Bhaduri 1974)

$$K = 9\left(\frac{\partial P}{\partial \rho}\right)_T . \tag{6}$$

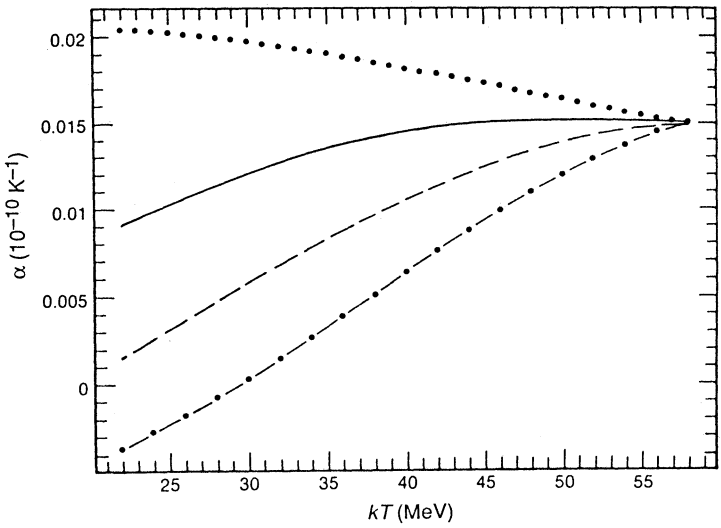

Fig. 4. Volume expansion coefficient as a function of temperature for four densities. The dotted curve is for $\rho = \rho_0/2$, the solid curve for normal density, the dashed curve for $\rho = 3\rho_0/2$ and the dot–dash curve for $\rho = 2\rho_0$.

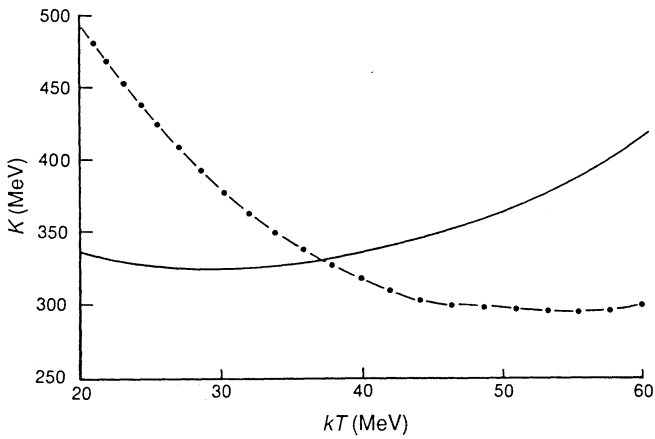

Fig. 5. Stiffness as a function of temperature for two different densities. The solid curve is for normal density and the dot–dash curve for twice the normal density.

So far this coefficient has been studied mainly for nuclear matter at equilibrium. Here we display its values for hot neutron matter at two different densities.

In our study we chose the relatively unexplored region of temperatures $20 < kT < 60$ MeV. In this temperature range the virial expansion for a Fermi gas converges fast even for high densities. We decided to stop our investigation at $kT = 60$ MeV because above this temperature pionic degrees of freedom may play an important role. Clearly, it does not make sense to extend this approach to high temperatures where quark–gluon degrees of freedom dominate the

scene. We feel that in the temperature regime which we investigated, the model used here is realistic. The corrections due to our choice of free space phase shifts and due to the neglect of higher terms in the virial expansion are not significant. The present model can be used as a starting point in bridging the gap in our knowledge of the equation of state between the low and high temperature regimes.

Extrapolating the information from lower temperature regimes (Jaqaman *et al.* 1983) yields results consistent with our conclusions. It is important to keep in mind that the lower energy regime has been investigated in considerable detail, but there are no universally accepted values for the thermostatic parameters. The extrapolation of our results to the lower temperature range is in accord with quoted values.

4. Conclusions

We would like to make some concluding remarks about the virial expansion and its role in investigating the nuclear equation of state. Most studies of the equation of state take advantage of the energy density formalism based on the Thomas–Fermi and extended Thomas–Fermi models. These approaches in turn hinge on mean field theory and in most cases one uses a zero range Skyrme interaction. Within the framework of this approximation very sophisticated and elaborate calculations have been carried out. On the other hand, the work of Jaqaman *et al.* (1983) took advantage of the virial expansion. It is important to note that the approach of those authors and our approach differ. Their work is, in principle, based on mean field theory using a Skyrme force. They calculated directly only the Fermi virial coefficients due to Pauli blocking and reproduced the interaction by replacing the mass by the effective mass. They included more terms in their virial expansion, which is required in the lower temperature regime where the virial expansion does not converge as fast. This is also possible in their approach since the Fermi virial coefficients can be calculated to any order. On the other hand, our expansion hinges on investigating the departure from ideal gas behaviour by studying directly the role of the two-body interaction when replacing the interaction dependence of the second virial coefficient by its dependence on the nucleon nucleon phase shifts. We apply our method to a higher temperature regime where the virial expansion converges much faster.

In summary, we have demonstrated that many thermal properties of very hot neutron matter can be determined using a quantum virial expansion. We calculated the second virial coefficient $B_2(T)$ and included both contributions due to the Pauli principle and to two-body interactions using the nucleon–nucleon phase shifts. The model also allows the determination of other quantities such as specific heats which are not given here. We have thus embarked on the exploration of the equation of state in a previously uncharted temperature regime.

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