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Feynman Rules for Magnetised Spin-0 Bosons

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Abstract

Solutions of the Klein–Gordon equation are given in the Landau and cylindrical gauges and these are used to calculate explicit forms of the vertex function. From an $\hat{S}$-matrix expansion analogous to the spin-$\frac{1}{2}$ case we obtain the Feynman rules for magnetised spin-0 particles: the major differences between the spin-$\frac{1}{2}$ and spin-0 cases are the explicit forms of the respective vertex functions and the addition of a two-photon vertex in the spin-0 case. This additional vertex makes possible a whole new class of Feynman diagrams.

1. Introduction

The aim of this paper is to derive the Feynman rules pertaining to spin-0 bosons in an ambient magnetic field. There are two motivations for the calculations. Feynman rules were written down for unmagnetised spin-0 and spin-1 bosons and spin-$\frac{1}{2}$ fermions (e.g. Scadron 1979, p. 199) but only the spin-$\frac{1}{2}$ case has been extended to include an ambient magnetic field (Melrose and Parle 1983*b*, hereinafter MPIII). The present paper bridges the gap between the spin-$\frac{1}{2}$ and spin-0 cases. It is also of some formal interest to compare the form of the rules for the magnetised spin-0 and spin-$\frac{1}{2}$ cases and to consider the effects of spin and statistics (although this is not the purpose here). The second motivation comes from theoretical applications of magnetised spin-0 bosons. For instance, superconductivity is modelled in terms of paired fermions forming pseudo-bosons (e.g. Frölich 1950), which exhibit the Meissner effect in a magnetic field (Schafroth 1955), and the laboratory production of pions in strong fields may be possible (Rafelski *et al.* 1978). There is also application to astrophysics—it is thought, for example, that a pion plasma may form a large part of the core of a neutron star (Shapiro and Teukolsky 1983, p. 251), which may also be magnetised, as in the case of pulsars. The Feynman rules developed here allow, in principle, the calculation of any process relevant to these applications, to any order in the boson's charge. We present a fully self-contained derivation starting with the exact solution of the Klein–Gordon equation in a magnetic field, including the derivation of the Hamiltonian density for photon/boson interactions and of the vertex function and finishing with the Feynman rules obtained from an $\hat{S}$-matrix expansion.

The solution to Dirac's equation in an ambient magnetic field is well known (e.g. Itzykson and Zuber 1980—hereinafter IZ, p. 67; Melrose and Parle

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1983*a*—hereinafter MPI), however the corresponding Klein–Gordon equation had not been considered until recently (Witte, Kowalenko and Hines 1988—hereinafter WKH), who treated it thoroughly and in a number of ways. Here, for completeness and as an introduction to our notation, we present a succinct solution of the equation following that of the Dirac equation (MPI) (the same differential equation describes both bosons and fermions, the only difference being in the form of the energy eigenvalues). We choose to normalise our wavefunctions according to the quantum field theory prescription, equivalent to allowing a single particle in a normalisation volume, V. This normalisation is different from that of WKH and thus so is our expression for operator expectation values.

A calculation of the interaction Hamiltonian density for an unmagnetised boson/photon system, within the framework of quantum field theory, appears in IZ (p. 282). Here we perform the equivalent calculation including an ambient magnetic field. The result is used in deriving appropriate Feynman rules for the magnetised spin-0 plasma but in doing so, Fourier transforms in the usual sense cannot be performed; instead we define a modified Feynman propagator and a vertex function analogous to the magnetised spin-$\frac{1}{2}$ case (MPI). The vertex function is an important quantity in the theory, since it appears as the contribution to the Feynman amplitude from every vertex in a Feynman diagram. We derive explicit forms for it in both the Landau and cylindrical gauges and compare them with the spin-$\frac{1}{2}$ case, then we consider its gauge dependence, which is contained in a factor premultiplying a four-vector, $[\Gamma^{\epsilon\epsilon'}_{qq'}(\mathbf{k})]^\mu$, the definition of which is different to that of its analogue in the spin-$\frac{1}{2}$ case. In deriving the Feynman rules, the $\hat{S}$-matrix expansion is analogous to that of the spin-$\frac{1}{2}$ case (MPIII) except for an additional vertex and the explicit form of the vertex function.

The organisation of the paper is as follows. In Section 2 we solve the Klein–Gordon equation in the Landau and cylindrical gauges and in Section 3 we derive the interaction Hamiltonian. In Section 4 we define the vertex function and give explicit forms for it in the Landau and cylindrical gauges, from which we can obtain its gauge dependence. Section 5 deals with the particle and photon propagators and in Section 6 we obtain the $\hat{S}$-matrix expansion. Finally, Section 7 details the Feynman rules.

In this paper we use the metric tensor $g_{\mu\nu} = \mathrm{diag}\,(+---)$ and natural units, $\hbar = c = 1$.

2. Solution of the Klein–Gordon Equation

The Klein–Gordon equation for the complex scalar wavefunction, Ψ, is

$$(\partial^\mu \partial_\mu + m^2)\Psi = 0\,, \tag{1}$$

where m is the particle mass. To include an ambient magnetic field described by the four-potential, $A^\mu(x)$, we make the minimal coupling replacement

$$\partial^\mu \to D^\mu = \partial^\mu + iqA^\mu(x)\,, \tag{2}$$

where q is the charge on the particle, and obtain

$$(\partial^\mu \partial_\mu + 2iqA_\mu \partial^\mu + iq\partial^\mu A_\mu - q^2 A^2 + m^2)\Psi = 0\,. \tag{3}$$

An important point concerning the use of equation (2) is that in general (as is the case for spin-1 particles) one must make the replacement in the Lagrangian for the particle and not in the wave-equation itself (Berestetskii *et al.* 1971—hereinafter BLP, p. 100).

(a) Landau Gauge

We choose first to solve equation (3) in the Landau gauge

$$A^{\mu}(t,\mathbf{x}) = (0,0,Bx,0)\,, \tag{4}$$

which describes a magnetic field, B, in the positive z direction. Since only the variable x is involved (if the Hamiltonian were written down, it would depend only on x), we anticipate plane wave solutions in the other variables and try a solution of the form

$$\Psi_q^{\epsilon}(t,\mathbf{x}) = f(x)\exp\{-i\epsilon(\mathcal{E}t - p_y y - p_z z)\}\,, \tag{5}$$

where the quantum numbers $\mathcal{E}$, p_y and p_z are the energy and y and z components of momentum respectively. We have chosen to take $\epsilon = +1$ for particles and $\epsilon = -1$ for antiparticles so that the quantum numbers represent the physical energy and momentum of either a particle or antiparticle. On substituting this wavefunction into the Klein–Gordon equation, changing variable to

$$\xi = (|\,q\,|\,B)^{\frac{1}{2}}\left(x - \frac{\epsilon p_y}{qB}\right), \tag{6}$$

and writing

$$\mathcal{E}^2 = (2n+1)\,|\,q\,|\,B + m^2 + p_z^2\,, \tag{7}$$

we obtain

$$\left(\frac{d^2}{d\xi^2} + 2n + 1 - \xi^2\right) f(\xi) = 0\,. \tag{8}$$

This is the equation for a simple harmonic oscillator with normalisable solutions given by the Hermite polynomials, H_n, and it is the same equation as occurs in the fermion case. The change of variable from x to ξ, cf. (6), involves a shift of origin to the centre of gyration, as shown below in Section 2*d* when we consider the position expectation value of the wavefunctions. The energy eigenvalues (7) introduce the positive integral quantum number n which fixes the value of momentum perpendicular to the magnetic field. Here, in the boson case, there is a non-zero ground state energy ($n = 0$), so bosons are, classically speaking, always spiralling along the magnetic field lines ($\langle p_\perp \rangle \neq 0$), and all levels are non-degenerate. By contrast, in the fermion case, the ground state has zero perpendicular momentum so, classically speaking, fermions can move along field lines without gyrating motion, and all but the ground state are doubly degenerate. The difference in energy level degeneracy between the boson case and the fermion case is simply because of the difference in the number of spin states. This difference and its effect on the behaviour of boson and fermion gases should be most obvious in super strong magnetic fields, such as exist in pulsars (up to 10^8 T, e.g. Smith 1977, p.45). In such fields spin-$\frac{1}{2}$ fermions quickly fall into the ground state due to synchrotron losses, whereas the spin-0 bosons retain their gyrating motion (with $p_\perp \sim B^{\frac{1}{2}}$). However, we do not discuss any implications of this here.

The form of the energy eigenvalues suggests that we define a critical magnetic field $B_c = m^2/|q|$ and if the magnetic field exceeds this value then all particles have relativistic energies and relativistic quantum effects cannot be ignored. If we take pions ($\pi^\pm$) as our test bosons then $B_c = 3\cdot 3\times 10^{14}$ T, which is far in excess of proposed pulsar fields (one of the sources of the highest possible fields) so in most cases we need only consider the classical or non-relativistic limits of our quantities.

The solution to (8) is

$$f_n(\xi) = \frac{CH_n(\xi)\exp(-\frac{1}{2}\xi^2)}{(\pi^{\frac{1}{2}}2^n n!)^{\frac{1}{2}}}, \tag{9}$$

where C is a normalisation constant to be discussed later.

(b) Cylindrical Gauge

It is constructive to have solutions to the wave equation in more than one gauge so that the gauge independence of the theory can be considered when we look at the vertex function in Section 4. One other choice of gauge made in the treatment of the Dirac equation (MPI), and one which has an explicit symmetry with respect to x and y coordinates is the cylindrical gauge

$$A^\mu(x,y) = \tfrac{1}{2}(0,-By,Bx,0). \tag{10}$$

Here we anticipate plane-wave like solutions for the t and z coordinates and try

$$\Psi_q^\epsilon(t,\mathbf{x}) = g(r,\phi)\exp\{-i\epsilon(\mathcal{E}t - p_z z)\}, \tag{11}$$

where $x = r\cos\phi$ and $y = r\sin\phi$ define the polar coordinates r and ϕ with $\mathcal{E}$ and p_y as before. In addition the periodicity in the polar angle ϕ requires that the angular dependence of $g(r,\phi)$ be limited to a phase factor so that

$$g(r,\phi) = g(r)e^{il\phi}, \tag{12}$$

where ℓ is an additional quantum number which must be integral for $g(r,\phi)$ to be single valued. Substituting equations (10), (11) and (12) into equation (3) and writing the result in cylindrical polar coordinates, (r,ϕ,z), yields

$$\left\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{\ell^2}{r^2} + |q|B\left(2n+1+\frac{|q|\ell}{q}\right) - \frac{q^2B^2r^2}{4}\right\}g(r) = 0, \tag{13}$$

where we have made use of the energy eigenvalues given earlier. This equation is the same as the resulting equation in the fermion case, apart from changes in notation, and the normalisable solutions involve the generalised Laguerre polynomials. We define the functions (Svetozarova and Tsytovich 1962; MPI)

$$J_\nu^n(x) = \left\{ \frac{n!}{(n+\nu)!} \right\}^{\frac{1}{2}} \exp\left(-\tfrac{1}{2}x\right) x^{\frac{1}{2}\nu} L_n^\nu(x), \tag{14}$$

with n and ν integral and where $L_n^\nu(x)$ are the generalised Laguerre polynomials (e.g. Abramowitz and Stegun 1965, p. 775). Some useful properties of these functions are detailed in the Appendix. The solution of equation (13) can now be written in the form

$$g_n^s(r, \phi) = C' J_{n-s}^s(\tfrac{1}{2} \mid q \mid Br^2) \exp\{i(s-n)\phi \mid q \mid /q\}, \tag{15}$$

with

$$s = n + \ell \mid q \mid /q, \tag{16}$$

and a normalisation constant C'.

The energy eigenvalues (7) depend only on the parallel component of momentum and not explicitly on either of the other components. In this second solution we did not define a p_y quantum number as we did in the first solution and this highlights the fact that p_y is not to be regarded as a physical momentum quantum number. Rather we see in Section 2*d* that it defines the centre of gyration.

(c) Covariant Normalisation

The formal procedure we use in normalising the wavefunctions involves calculating the (0,0) component of the energy–momentum tensor (e.g. BLP, p. 32)

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^*)} \partial^\nu \Psi^* + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} \partial^\nu \Psi - g^{\mu\nu} \mathcal{L}, \tag{17}$$

from the Lagrangian, $\mathcal{L}$, for the field. This gives us the energy density of the field and the normalisation condition we use is

$$\int_V d^3\mathbf{x}\, T^{00} = \mathcal{E}, \tag{18}$$

where the integral is to be performed over a suitable normalisation volume, V. This choice of normalisation is manifestly covariant and corresponds to having a single particle or antiparticle, of unsigned energy $\mathcal{E}$, in the normalisation volume. The normalisation chosen by WKH is effectively normalisation to plus or minus unity.

The Lagrangian for the Klein–Gordon field with an ambient electromagnetic field is (e.g. IZ, p. 282)

$$\mathcal{L} = (\partial^\mu + iqA^\mu)\Psi(\partial_\mu - iqA_\mu)\Psi^* - m^2 \Psi\Psi^*. \tag{19}$$

For the wavefunctions (5) with (6), (7) and (9) one finds

$$\mathcal{L} = |q| B\{(2n+1-\xi^2)f_n(\xi)^2 - f_n'(\xi)^2\}, \tag{20}$$

$$T^{00} = \{2\mathcal{E}^2 - (2n+1)|q|B + |q|B\xi^2\}f_n(\xi)^2 + |q|Bf_n'(\xi)^2 . \tag{21}$$

Applying condition (18) and making use of the recursion and integration relations for the Hermite polynomials given in the Appendix gives

$$C = \left\{\frac{(|q|B)^{\frac{1}{2}}}{2\mathcal{E}L_yL_z}\right\}^{\frac{1}{2}}, \tag{22}$$

where L_y and L_z are normalisation lengths in the y and z directions respectively. The corresponding normalisation constant for plane-wave solutions to the spin-0 Klein–Gordon equation in the unmagnetised case is (BLP, p. 32)

$$C_{\text{unmag}} = \frac{1}{(2\mathcal{E}V)^{\frac{1}{2}}}, \tag{23}$$

and this suggests that the natural unit of length in the x-direction in our case is $(|q|B)^{-\frac{1}{2}}$.

(d) Expectation Values

Due to our choice of normalisation the wavefunctions have dimensions of $(\mathcal{E}V)^{-\frac{1}{2}}$ and thus $|\Psi|^2$ is not a probability density. We define the expectation value of an operator $\hat{O}$ to be

$$\langle\Psi_a|\hat{O}|\Psi_b\rangle = 2\int d^3\mathbf{x}\, \mathcal{E}_a^{\frac{1}{2}}\Psi_a^*(\mathbf{x})\hat{O}\mathcal{E}_b^{\frac{1}{2}}\Psi_b(\mathbf{x}). \tag{24}$$

The WKH normalisation takes the factor of two into account and gives a probability density interpretation to $|\Psi|^2$ but it requires an alteration to the usual density of states which appears in summing over quantum numbers. As an illustration of (24) we calculate the expectation value of the position vector $\mathbf{r}$, the centre of gyration, by substituting the wavefunctions (5) with (9) and (22) into equation (24) to obtain

$$\langle\mathbf{r}\rangle = (\epsilon p_y/qB, 0, 0). \tag{25}$$

This gives the physical interpretation of p_y as the scaled centre of gyration of a particle and a change in this quantum number represents a drift across magnetic field lines.

3. Interaction Hamiltonian

The Lagrangian density for magnetised spin-0 particles is (19), viz.

$$\mathcal{L}_M = \partial_\mu\Psi^*\partial^\mu\Psi + iqA^\mu\Psi\partial_\mu\Psi^* - iqA^\mu\Psi^*\partial_\mu\Psi + q^2A^2\Psi^*\Psi - m^2\Psi^*\Psi .$$

We add in the effect of a photon field, described by the operator $\hat{A}^\mu$, via the replacement

$$A^\mu \to A^\mu + \hat{A}^\mu, \tag{26}$$

thus obtaining the total interaction Lagrangian density, $\mathcal{L}_{\rm tot}$,

$$\mathcal{L}_{\rm tot} = \mathcal{L}_M + \mathcal{L}_{\rm int}, \tag{27}$$

with

$$\mathcal{L}_{\rm int} = iq\hat{A}^\mu \Psi \overleftrightarrow{\partial}_\mu \Psi^* + q^2\hat{A}^2\Psi^*\Psi + 2q^2 A\hat{A}\Psi^*\Psi, \tag{28}$$

being the interaction Lagrangian density. The canonical momenta for the particle field are

$$\pi = \frac{\partial \mathcal{L}_{\rm tot}}{\partial(\partial_0\Psi)} = \partial^0\Psi^* - iq(A^0 + \hat{A}^0)\Psi^*, \tag{29}$$

$$\pi^* = \frac{\partial \mathcal{L}_{\rm tot}}{\partial(\partial_0\Psi^*)} = \partial^0\Psi + iq(A^0 + \hat{A}^0)\Psi, \tag{30}$$

and in analogy with classical mechanics, the Hamiltonian density is defined by

$$\mathcal{H}_{\rm tot}(\Psi, \Psi^*, \pi, \pi^*) = \pi\dot{\Psi} + \pi^*\dot{\Psi}^* - \mathcal{L}_{\rm tot}, \tag{31}$$

where $\mathcal{H}_{\rm tot}$ must be written out in terms of Ψ, Ψ^*, π and π^* only, giving

$$\begin{aligned}\mathcal{H}_{\rm tot} = {} & \pi\pi^* - \partial_k\Psi^*\partial^k\Psi - iq(A^0 + \hat{A}^0)(\Psi\pi - \Psi^*\pi^*) + iq(A^k + \hat{A}^k)\Psi^*\overleftrightarrow{\partial}_k\Psi \\ & - q^2(A^k + \hat{A}^k)(A_k + \hat{A}_k)\Psi\Psi^* + m^2\Psi\Psi^*. \end{aligned}\tag{32}$$

Omitting from this the parts which are independent of the photon fields yields the interaction Hamiltonian

$$\mathcal{H}_{\rm int} = iq\hat{A}^0(\Psi^*\pi^* - \Psi\pi) + iq\hat{A}^k\Psi^*\overleftrightarrow{\partial}_k\Psi - 2q^2A^k\hat{A}_k\Psi\Psi^* - q^2\hat{A}^k\hat{A}_k\Psi\Psi^*, \tag{33}$$

which can be rewritten, in noncanonical form

$$\mathcal{H}_{\rm int} = q\hat{A}^\mu\Psi^*(i\overleftrightarrow{\partial}_\mu - 2qA_\mu)\Psi - q^2\hat{A}^2\Psi\Psi^* - q^2\hat{A}_0^2\Psi\Psi^*. \tag{34}$$

The term coupling the photon field to the ambient field is the only difference between the magnetised and unmagnetised cases and an equivalent term does not arise in the fermion case, where the explicit form of the interaction Hamiltonian density is unchanged on inclusion of an ambient magnetic field.

Equation (34) is derived in the Schrödinger picture. In order to derive the Feynman rules, we need to transform to the interaction picture, which involves changing the sign of the final term (IZ, p. 283). This is a result of the fact that in the interaction picture the canonical momenta do not include the terms involving the interacting field, $\hat{A}^0$. The subtraction of these terms from (33) changes (34) by a term $2q^2\hat{A}_0^2\Psi\Psi^*$. The final term is obviously not gauge invariant, however, as is shown below when considering the $\hat{S}$-matrix, it cancels with other non-covariant terms arising from contractions involving the differential operator ∂_μ.

4. Vertex Function

In the magnetised case there are two difficulties in constructing Feynman rules in momentum space: (i) the particle propagator depends separately on the coordinates of the two space–time points associated with it and cannot be Fourier transformed in the usual way, and (ii) the operator which appears in $\mathcal{H}_{\text{int}}$ involves the ambient magnetic field. Defining a vertex function (MPI) allows us to surmount these difficulties by grouping the wavefunctions and vertex operators together, outside the particle propagator (which we modify in Section 5). We define

$$\begin{aligned}[y^{\epsilon'\epsilon}_{q'q}(\mathbf{k})]^\mu &= d^{\epsilon'\epsilon}_{q'q}(\mathbf{k})[\Gamma^{\epsilon\epsilon}_{q'q}(\mathbf{k})]^\mu \\ &= \int d^3\mathbf{x}[\Psi^{\epsilon'*}_{q'}(\mathbf{x})\{\overleftrightarrow{p}^\mu - 2qA^\mu(x)\}\Psi^{\epsilon}_{q}(\mathbf{x})]\exp(-i\mathbf{k}\cdot\mathbf{x}), \end{aligned} \tag{35}$$

where $\hat{p}^\mu$ is the usual momentum operator and $d^{\epsilon'\epsilon}_{q'q}(\mathbf{k})$ is chosen to contain all gauge dependence. The wavefunctions in (35) have no time dependence (all time dependences are grouped separately and lead to conservation of energy at vertices) so we require the added definition

$$\Psi^{\epsilon'*}_{q'}(\mathbf{x})\overleftrightarrow{p}^0\Psi^{\epsilon}_{q}(\mathbf{x}) = (\epsilon\mathcal{E}_q + \epsilon'\mathcal{E}_{q'})\Psi^{\epsilon'*}_{q'}(\mathbf{x})\Psi^{\epsilon}_{q}(\mathbf{x}). \tag{36}$$

In the magnetised spin-$\frac{1}{2}$ case the definition of the vertex function involves a Dirac matrix rather than our vertex operator in (35).

The vertex function is an important quantity since it is associated with vertices in Feynman diagrams and because it contains most of the mathematics required in transforming to momentum space. Finally, we note that it has the useful property

$$[y^{\epsilon'\epsilon}_{q'q}(\mathbf{k})]^{\mu*} = [y^{\epsilon\epsilon'}_{qq'}(-\mathbf{k})]^\mu. \tag{37}$$

(a) Landau Gauge

In order to derive an explicit form for the vertex function in the Landau gauge we use the wavefunctions (5) with (9) and (22) in the definition (35). Calculation is reduced to the consideration of two explicitly occurring integrals. The first integral is the same as one used by MPI in their treatment of the spin-$\frac{1}{2}$ magnetised plasma, apart from changes in notation and the introduction of a signed charge. The integral is

$$\begin{aligned} I_{n,n'} &= \int_{-\infty}^{\infty} dx H_{n'}(\xi')H_n(\xi)\exp(-\tfrac{1}{2}\xi^2 - \tfrac{1}{2}\xi'^2 - ik_x x) \\ &= (\pi^{\frac{1}{2}}2^{n'}n'!)^{\frac{1}{2}}(\pi^{\frac{1}{2}}2^n n!)^{\frac{1}{2}}(|q|B)^{-\frac{1}{2}}\exp\{-ik_x(\epsilon p_y + \epsilon' p'_y)/2qB\} \\ &\quad \times(-ie^{i\psi|q|/q})^{n'-n}J^n_{n'-n}(k_\perp^2/2|q|B). \end{aligned} \tag{38}$$

The variable, ξ, is given by (6) and ξ' by the same, with primed quantum numbers (ϵ', q') replacing the unprimed quantum numbers. The x-component of the wavevector $\mathbf{k}$ is k_x and the angle ψ is defined by

$$\mathbf{k} = (k_\perp \cos\psi, k_\perp \sin\psi,\ k_z). \tag{39}$$

The functions $J^n_{n'-n}$ are defined by equation (14). A second integral which appears can be integrated by parts and the result expressed in terms of the first integral $I_{n,n'}$:

$$\begin{aligned} K_{n,n'} &= \int_{-\infty}^{\infty} dx\, x H_{n'}(\xi') H_n(\xi) \exp\left(-\tfrac{1}{2}\xi^2 - \tfrac{1}{2}\xi'^2 - ik_x x\right) \\ &= \frac{n}{(|q|B)^{\frac{1}{2}}} I_{n-1,n'} + \frac{n'}{(|q|B)^{\frac{1}{2}}} I_{n,n'-1} + \frac{\epsilon p_y + \epsilon' p'_y - ik_x}{2|q|B} I_{n,n'}. \end{aligned} \tag{40}$$

The y and z integrals in the vertex function simply yield delta functions, representing conservation of linear momentum in those directions. The resulting explicit expressions for $d^{\epsilon'\epsilon}_{q'q}(\mathbf{k})$ and $[\Gamma^{\epsilon'\epsilon}_{q'q}(\mathbf{k})]^\mu$ are

$$[\Gamma^{\epsilon'\epsilon}_{q'q}(\mathbf{k})]^\mu = \frac{1}{2(2m\mathcal{E}_q\mathcal{E}_{q'})^{\frac{1}{2}}} \{-i \exp(ia)\}^{n'-n} \times \begin{pmatrix} (\epsilon\mathcal{E}_q + \epsilon'\mathcal{E}_{q'}) J^n_{n'-n} \\ -i(|q|/q)\{k_y J^n_{n'-n} - (2n|q|B)^{\frac{1}{2}} \exp(ia) J^{n-1}_{n'-n+1} \\ \qquad - (2n'|q|B)^{\frac{1}{2}} \exp(-ia) J^n_{n'-n-1}\} \\ i(|q|/q)\{k_x J^n_{n'-n} + (2n|q|B)^{\frac{1}{2}} \exp(ia) J^{n-1}_{n'-n+1} \\ \qquad - (2n'|q|B)^{\frac{1}{2}} \exp(-ia) J^n_{n'-n-1}\} \\ (\epsilon p_z + \epsilon' p_z') J^n_{n'-n} \end{pmatrix}, \tag{41}$$

where $a = \psi|q|/q$ and

$$d^{\epsilon'\epsilon}_{q'q}(\mathbf{k}) = \frac{(2\pi)^2}{V(|q|B)^{\frac{1}{2}}} \exp\left\{\frac{-ik_x(\epsilon p_y + \epsilon' p'_y)}{2qB}\right\} \delta(\epsilon p_y - \epsilon' p'_y - k_y)\delta(\epsilon p_z - \epsilon' p'_z - k_z) \tag{42}$$

with $V = (|q|B)^{-\frac{1}{2}} L_y L_z$ being the normalisation volume and where we omit the argument, $k_\perp^2/2|q|B$, of the J functions. The form of the vertex function in the spin-0 case differs from that of the spin-$\frac{1}{2}$ case (MPI) only in the detailed form of the four vector quantity $[\Gamma^{\epsilon'\epsilon}_{q'q}(\mathbf{k})]^\mu$.

Note that the vertex function is defined here to be dimensionless. It has the property that reversing the sign of the charge is equivalent to reversing the y direction and changing the sign on the angle ψ. Alternatively (as can be seen from our choice of gauge 4) we can reverse the sense of the magnetic field rather than change the sign of the charge.

(b) Cylindrical Gauge and Gauge Invariance

Here we use the gauge (10) and the wavefunctions (11) with (15). We expect, since our theory is manifestly gauge invariant, that the vertex function involves a gauge independent four-vector, all gauge dependence being contained in a premultiplying factor. In cylindrical coordinates the z integrals yield delta functions for conservation of momentum along the field and the $\mu=0$ and $\mu=3$ components of the vertex function involve a modified form of the integral found in MPI (equation 51)

$$\begin{aligned} L_{n,s} &= \int_0^{2\pi} d\phi \int_0^{\infty} dr\, r \exp\{-ik_\perp r\cos(\phi+\psi\,|\,q\,|\,/q)\} J^{s'}_{n'-s'} J^{s}_{n-s} \exp\{i(n-s-n'+s')\phi\} \\ &= -\frac{2\pi}{qB}\{-i\exp(i\psi\,|\,q\,|/q)\}^{s-n+n'-s'} J^{s}_{s'-s} J^{n}_{n'-n}\,, \end{aligned} \tag{43}$$

where we omit the argument, $k_\perp^2/2\,|\,q\,|\,B$, of the J functions. The $\mu=1$ and $\mu=2$ components are more difficult to evaluate since the respective components of the vertex operator involve derivatives with respect to both r and ϕ. However, given that we expect the vertex function to have a gauge independent four-vector and a premultiplying factor, we can use the $\mu=0$ and $\mu=3$ results, compared with vertex function in the Lorentz gauge to anticipate that in the cylindrical gauge

$$\begin{aligned} [\gamma^{\epsilon'\epsilon}_{q'q}(\mathbf{k})]^{\mu} &= -\frac{2(2\pi)^2}{qB} C'C(\mathcal{E}_q\mathcal{E}_{q'})^{\frac{1}{2}}\{-i\exp(i\psi\,|\,q\,|\,/q)\}^{s-s'} \\ &\quad \times J^{s}_{s'-s}\delta(\epsilon p_z-\epsilon' p'_z-k_z)[\Gamma^{\epsilon'\epsilon}_{q'q}(\mathbf{k})]^{\mu}\,, \end{aligned} \tag{44}$$

where again we omit the argument of the J functions. The constants C' and C are normalisation constants of the cylindrical gauge wavefunctions, which we do not need to evaluate here. One can verify the result for the $\mu=1$ and $\mu=2$ components by inverse Fourier transforming (which is easier than working forwards since the integrals which appear in inverse Fourier transforming can easily be done by referring to the Lorentz gauge results). One finds after much tedious algebra and with the help of the recursion relations for the J functions (see the Appendix), that the result (44) is indeed correct.

The above verifies, at least for our two choices of gauge, that the gauge invariance of the theory manifests itself in the form of the gauge invariant part of the vertex function. Moreover, since we expect this to be so, the calculations in the two gauges constitute a check on the explicit form of the vertex function. The vertex function contains most of the information about the particle/antiparticle, photon interactions, since it involves the particle wavefunctions as well as the ambient magnetic field, and so a detailed comparison with the corresponding fermion quantity (MPI) would be useful. One superficial difference is that the spin-0 vertex function involves a single J function in its $\mu=0$ and $\mu=3$ components and three in its $\mu=1$ and $\mu=2$ components. The corresponding fermion quantity contains two J functions in each component (it should be noted that in no way can the use of recursion relations reduce our three functions to two).

5. Propagators

The particle and photon wavefunctions are second quantised as

$$\hat{\Psi}(x) = \sum_{q\epsilon} \hat{a}_q^\epsilon \Psi_q^\epsilon(\mathbf{x}) \exp(-i\epsilon \mathcal{E}_q t), \tag{45}$$

$$\hat{A}^\mu(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \hat{A}^\mu(k), \tag{46}$$

with (Melrose 1983—hereinafter MII)

$$\hat{A}^\mu(k) = \sum_M a_M(\mathbf{k})\{e_M^\mu(\mathbf{k})\hat{c}_M(\mathbf{k})(2\pi)^4\delta(k-k_M) + e_M^{\mu*}(\mathbf{k})\hat{c}_M^*(\mathbf{k})(2\pi)^4\delta(k+k_M)\}, \tag{47}$$

where

$$a_M(\mathbf{k}) = \left\{\frac{R_M(\mathbf{k})}{V\epsilon_0\omega_M(\mathbf{k})}\right\}^{\frac{1}{2}} \tag{48}$$

is the electric amplitude of the photon field in mode M. The polarisation vector is $e_M^\mu(\mathbf{k})$ and we are considering an elemental range $Vd^3\mathbf{k}/(2\pi)^3$ [our normalisation has been chosen to give a single photon in this range (e.g. MII)]. The operators, $\hat{a}_q^\epsilon$ and its conjugate, are the annihilation and creation operators for a particle ($\epsilon = +1$) with quantum numbers q, or the creation and annihilation operators for an antiparticle ($\epsilon = -1$) with quantum numbers q, respectively. The operators $\hat{c}_M$ and its conjugate are the annihilation and creation operators for a photon in mode M respectively. The wavefunctions $\Psi_q^\epsilon(\mathbf{x})$ are (5) with (9) and (22), written without their time dependence (which has been included explicitly in 45).

The particle propagator in vacuo is defined by

$$G(x,x') = i\langle 0 \mid \hat{\tau}\{\hat{\Psi}(x)\hat{\Psi}^*(x')\} \mid 0\rangle, \tag{49}$$

where $\hat{\tau}$ is the boson time-ordering operator

$$\hat{\tau}\{\hat{\Psi}(x)\hat{\Psi}^*(x')\} = \theta(t-t')\hat{\Psi}(x)\hat{\Psi}^*(x') + \theta(t'-t)\hat{\Psi}^*(x')\hat{\Psi}(x), \tag{50}$$

and $\theta(t-t')$ is the Heaviside step function. The step function may be written in terms of its Fourier transform

$$\theta(t) = \int \frac{d\omega}{2\pi} \frac{i}{\omega + i0} e^{-i\omega t}, \tag{51}$$

and then the propagator can be written in the form, as in the spin-$\frac{1}{2}$ case (MPIII),

$$G(x,x') = \sum_{\epsilon q} \Psi_q^\epsilon(\mathbf{x})\Psi_q^{\epsilon*}(\mathbf{x}') \int \frac{dE}{2\pi} e^{-iE(t-t')} \bar{g}_q^\epsilon(E), \tag{52}$$

with

$$g_q^\epsilon(E) = \frac{1}{E + \epsilon(\mathcal{E}_q - i0)} - \frac{1}{E - \epsilon(\mathcal{E}_q - i0)} \tag{53}$$

defining a modified propagator. This modified propagator is different from the spin-$\frac{1}{2}$ propagator. The form (52) involves particle and antiparticle wavefunctions which we include in our definition of the vertex function (35); the appropriate form for the development of the Feynman rules in momentum space is (53).

Similarly, the photon propagator is defined to be

$$D^{\mu\nu}(x-x') = i\langle 0 \mid \hat{\tau}\{\hat{A}^{\mu}(x)\hat{A}^{\nu}(x')\} \mid 0\rangle . \tag{54}$$

There is no difficulty in transforming this to momentum space to get (MII)

$$D^{\mu\nu}_{\text{res}}(k) = \sum_{M} \frac{i\pi\mu_0 R_M(\mathbf{k})}{\omega_M(\mathbf{k})} e^{\mu}_{M}(\mathbf{k}) e^{\nu *}_{M}(\mathbf{k}) \delta\{\omega - \omega_M(\mathbf{k})\}, \tag{55}$$

for the resonant part of the propagator.

6. $\hat{S}$-Matrix Expansion

The interaction Hamiltonian (34) can be divided into a linear and a quadratic part in the photon field $\hat{A}^{\mu}$ and, after normal ordering (denoted:),

$$\hat{\mathcal{H}}^{(1)}_{\text{int}} = q : \hat{A}_{\mu}(x)\hat{\Psi}^{*}(x)\{\overleftrightarrow{\hat{p}}^{\mu} - 2qA^{\mu}(x)\}\hat{\Psi}(x) :, \tag{56}$$

$$\hat{\mathcal{H}}^{(2)}_{\text{int}} = q^2 : \hat{\Psi}^{*}(x)\{\hat{A}^2_0(x) - \hat{A}^2(x)\}\hat{\Psi}(x) : \tag{57}$$

[here we are in the interaction picture with the change of sign of the $\hat{A}^2_0(x)$ term in (34) implied]. The $\hat{S}$-matrix expansion is then

$$\hat{S} = \sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{(-i)^n}{r!(n-r)!}\int dx^4_1 \dots dx^4_n \; \hat{\tau}\{\hat{\mathcal{H}}^{(1)}_{\text{int}}(x_1)\dots\hat{\mathcal{H}}^{(1)}_{\text{int}}(x_r)\hat{\mathcal{H}}^{(2)}_{\text{int}}(x_{r+1})\dots\hat{\mathcal{H}}^{(2)}_{\text{int}}(x_n)\}, \tag{58}$$

where $\hat{\tau}$ is the boson time ordering operator (50).

In order to determine the Feynman rules we need to consider low order diagrams which isolate a vertex, a propagator or a closed loop. Consider first a vertex with a single photon line, corresponding to gyromagnetic emission (or the crossed processes: gyromagnetic absorption, pair creation and pair annihilation). The $\hat{S}$-matrix term is

$$\hat{S}^{(1)} = -i\int dx^4 \hat{\mathcal{H}}^{(1)}_{\text{int}}(x), \tag{59}$$

which, upon using the definition of $\hat{\mathcal{H}}^{(1)}_{\text{int}}$ and our second quantised wavefunctions, can be expressed in the form (as in the spin-$\frac{1}{2}$ case, MPIII),

$$\hat{S}^{(1)} = -iq \sum_{\epsilon' q', \epsilon q} \int \frac{d^4k}{(2\pi)^4} : [\hat{G}^{\epsilon'\epsilon}_{q'q}(-k)]^{\mu}\hat{A}_{\mu}(k) :, \tag{60}$$

with

$$[\hat{G}^{\epsilon'\epsilon}_{q'q}(k)]^{\mu} = \hat{a}^{\epsilon' *}_{q'}\hat{a}^{\epsilon}_{q} D^{\epsilon'\epsilon}_{q'q}(k)[\Gamma^{\epsilon'\epsilon}_{q'q}(\mathbf{k})]^{\mu}, \tag{61}$$

where

$$D^{\epsilon'\epsilon}_{q'q}(k) = d^{\epsilon'\epsilon}_{q'q}(\mathbf{k}) 2\pi\delta(\epsilon\mathcal{E}_q - \epsilon'\mathcal{E}_{q'} - \omega). \tag{62}$$

The fact that the result takes the same form as in the spin-$\frac{1}{2}$ case is not surprising, since the result comes from the first order Hamiltonian (56) which differs from the spin-$\frac{1}{2}$ case only by the form of the operator between the wavefunctions. In both cases the operator is absorbed into a vertex function.

In the spin-0 case there is an additional vertex, with two photon lines. This has been noted in the unmagnetised case (e.g. Scadron 1979, p. 200; Williams and Melrose 1989) and has no counterpart in the spin-$\frac{1}{2}$ theory.The associated $\hat{S}$-matrix term involves a single second order interaction Hamiltonian

$$\hat{S}^{(2)} = -i\int dx^4 \hat{\mathcal{H}}^{(2)}_{\rm int}(x). \tag{63}$$

The non-covariant $\hat{A}^2_0(x)$ term in $\hat{\mathcal{H}}^{(2)}_{\rm int}$ cancels out of the $\hat{S}$-matrix expansion (as verified below) and with our ambient field in the temporal gauge $[A^0(x) = 0]$ we can write $\hat{S}^{(2)}$ in a similar form to that of $\hat{S}^{(1)}$

$$\hat{S}^{(2)} = iq^2 \sum_{\epsilon' q', \epsilon q} \int \frac{d^4k}{(2\pi)^4}\int \frac{d^4k'}{(2\pi)^4} : \hat{H}^{\epsilon'\epsilon}_{q'q}(-k-k')\hat{A}_{\mu}(k)\hat{A}^{\mu}(k') :, \tag{64}$$

with

$$\hat{H}^{\epsilon'\epsilon}_{q'q}(k) = 2\hat{a}^{\epsilon' *}_{q'}\hat{a}^{\epsilon}_{q} D^{\epsilon'\epsilon}_{q'q}(k)\frac{[\Gamma^{\epsilon'\epsilon}_{q'q}(k)]^0}{\epsilon\mathcal{E}_q + \epsilon'\mathcal{E}_{q'}}. \tag{65}$$

For a general choice of gauge for our ambient field, specifically if we wanted to include an electric field by taking A^0 to be a function of space, we could not write this last expression. It comes from the integral in (63) which reduces to a factor of the form

$$\int d^3x \Psi^*(x)\Psi(x)e^{-ikx},$$

and this is only simply related to the vertex function defined by (35) for $A^0 = 0$.

The next diagrammatic element we consider is a particle/antiparticle line between two vertices. Taking the case of two factors of $\hat{\mathcal{H}}^{(1)}_{\rm int}$ in the $\hat{S}$-matrix expansion, we have

$$\hat{S}^{(1,1)} = -\tfrac{1}{2}\int dx^4_1\, dx^4_2\; \hat{\tau}\{\hat{\mathcal{H}}^{(1)}_{\rm int}(x_1)\hat{\mathcal{H}}^{(1)}_{\rm int}(x_2)\}. \tag{66}$$

Here we have a time ordered product of normal ordered Hamiltonian densities, each with two particle/antiparticle operators and one photon operator. To put this in normal order we employ Wick's theorem (e.g. IZ, p. 180), taking every possible combination of contractions between operators of the same kind (i.e. particle/antiparticle or photon) and leaving the rest in normal order.

The contraction between particle/antiparticle operators gives the propagator

$$\langle 0 |\ \hat{T}\{\hat{\Psi}(x_1)\hat{\Psi}^*(x_2)\}|\ 0\rangle = -iG(x_1,x_2), \tag{67}$$

but we also have particle/antiparticle wavefunctions which are operated on by $\hat{p}^\mu$. For these we use the rules (IZ, p. 284)

$$\langle 0 |\ \hat{T}\{\hat{p}^\mu_{x_1}\hat{\Psi}(x_1)\hat{\Psi}^*(x_2)\}|\ 0\rangle = -i\hat{p}^\mu_{x_1}G(x_1,x_2), \tag{68}$$

$$\langle 0 |\ \hat{T}\{\hat{p}^\mu_{x_1}\hat{\Psi}(x_1)\hat{p}^\nu_{x_2}\hat{\Psi}^*(x_2)\}|\ 0\rangle = -i\hat{p}^\mu_{x_1}\hat{p}^\nu_{x_2}G(x_1,x_2) - ig_{\mu 0}g_{\nu 0}\delta^4(x_1-x_2), \tag{69}$$

where we specifically label the momentum operators with the variables on which they act in the propagators. The non-covariant term in (69) cancels out the non-covariant terms contributed by $\mathcal{H}^{(2)}_{\text{int}}$, throughout the $\hat{S}$-matrix expansion. Thus we are justified in ignoring the non-covariant term in $\mathcal{H}^{(2)}_{\text{int}}$ and naively commuting derivatives and contractions. Contracting over photon wavefunctions gives the photon propagator

$$\langle 0 |\ \hat{T}\{\hat{A}^\mu(x_1)\hat{A}^\nu(x_2)\}|\ 0\rangle = -iD^{\mu\nu}(x_1-x_2). \tag{70}$$

From the definition of the particle/antiparticle propagator (52) we see that a particle/antiparticle contraction introduces a sum over the quantum numbers (ϵ, q) of the intermediate state and a Fourier transform of the energy denominators in the modified propagator. The integrals over four-coordinates at each vertex in (66) then give vertex functions as before. In terms which give photon propagators we need to Fourier transform the photon propagator

$$D^{\mu\nu}(x_1-x_2) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x_1-x_2)} D^{\mu\nu}(k), \tag{71}$$

and the integrals over four-coordinates also give vertex functions. Terms with one particle/antiparticle propagator are of the form

$$-\tfrac{1}{2}q^2 \sum_{\epsilon' q', \epsilon q} \int \frac{d^4k'}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} : [\hat{G}^{\epsilon'\epsilon}_{q'q}(-k',-k)]^{\mu\nu}\hat{A}_\mu(k')\hat{A}_\nu(k) :,$$

with

$$[\hat{G}^{\epsilon'\epsilon}_{q'q}(k',k)]^{\mu\nu} = \hat{a}^{\epsilon' *}_{q'}\hat{a}^{\epsilon}_{q} D^{\epsilon'\epsilon}_{q'q}(k',k) \sum_{\epsilon'' q''}\{-ig^{\epsilon''}_{q''}(\epsilon\mathcal{E}_q - \omega)[\Gamma^{\epsilon'\epsilon''}_{q'q''}(k')]^\mu[\Gamma^{\epsilon''\epsilon}_{q''q}(k)]^\nu\}, \tag{72}$$

$$D^{\epsilon'\epsilon}_{q'q}(k',k) = d^{\epsilon'\epsilon''}_{q'q''}(\mathbf{k}')d^{\epsilon''\epsilon}_{q''q}(\mathbf{k})2\pi\delta(\epsilon\mathcal{E}_q - \epsilon'\mathcal{E}_{q'} - \omega - \omega). \tag{73}$$

This result may also be obtained by a direct analogy with the spin-$\frac{1}{2}$ case, and it can be generalised to the case of n factors of $\hat{\mathcal{H}}^{(1)}_{\text{int}}$ and $(n-1)$ particle/antiparticle contractions (MPIII) by writing

$$\frac{(-i)^n}{n!}\int dx^4_n \dots dx^4_1\ \hat{\mathcal{H}}^{(1)}_{\text{int}}(x_n)\dots\hat{\mathcal{H}}^{(1)}_{\text{int}}(x_1) \rightarrow \tag{74}$$

$$\frac{(-iq)^n}{n!}\sum_{\epsilon' q', \epsilon q}\int \frac{d^4k_n}{(2\pi)^4}\cdots\frac{d^4k_1}{(2\pi)^4} : [\hat{G}^{\epsilon'\epsilon}_{q'q}(-k_n, \dots -k_1)]^{\mu_n \dots \mu_1}\hat{A}_{\mu_n}(k_n)\dots\hat{A}_{\mu_1}(k_1) :,$$

with

$$[\hat{G}^{\epsilon'\epsilon}_{q'q}(k_n,...,k_1)]^{\mu_n...\mu_1} = \hat{a}^{\epsilon'*}_{q'}\hat{a}^{\epsilon}_{q}D^{\epsilon'\epsilon}_{q'q}(k_n,...,k_1)$$
$$\times \sum_{Q_1...Q_{n-1}} (-i)^{n-1}g^{\epsilon_{n-1}}_{q_{n-1}}(E_{n-1})...g^{\epsilon_1}_{q_1}(E_1)[\Gamma^{\epsilon'\epsilon_{n-1}}_{q'q_{n-1}}(\mathbf{k}_n)]^{\mu_n}...[\Gamma^{\epsilon_1\epsilon}_{q_1q}(\mathbf{k}_1)]^{\mu_1}, \tag{75}$$

where we have energy conservation at each vertex so that

$$E_r = \epsilon\mathcal{E}_q - \sum_{i=1}^{n}\omega_i, \tag{76}$$

and Q_r denotes (ϵ_r, q_r).

The quantity $D^{\epsilon'\epsilon}_{q'q}(k_n,...,k_1)$ is a generalisation of (73) including a factor $d^{\epsilon'\epsilon}_{q'q}(\mathbf{k})$ for each vertex and an energy conservation delta function. This quantity is actually independent of intermediate states and can be written (MPIII)

$$D^{\epsilon'\epsilon}_{q'q}(k_n,...,k_1) =$$
$$2\pi\delta\left(\epsilon\mathcal{E}_q - \epsilon'\mathcal{E}_{q'} - \sum_{i=1}^{n}\omega_i\right)2\pi\delta\left(\epsilon p_y - \epsilon' p_{y'} - \sum_{i=1}^{n}k_{iy}\right)2\pi\delta\left(\epsilon p_z - \epsilon' p_{z'} - \sum_{i=1}^{n}k_{iz}\right)$$
$$\times \frac{1}{V(|q|B)^{\frac{1}{2}}}\exp\left[\frac{i}{2qB}\left\{\sum_{j<i}(\mathbf{k}_i\times\mathbf{k}_j)_z - \sum_{i=1}^{n}k_{ix}(\epsilon p_y + \epsilon' p_{y'})\right\}\right]. \tag{77}$$

The result (74) applies to Feynman diagrams involving only particle/antiparticle contractions and single photon vertices. The generalisation to diagrams with two photon vertices and photon propagators requires the replacement of some particle/antiparticle propagators with photon propagators and of single photon vertex factors with two photon vertex factors. The details are given in Section 7.

The last elements we need to consider are closed loop diagrams in which we have n vertices and n contractions. Once again the result is analogous to the spin-$\frac{1}{2}$ case. One has (MPIII)

$$\frac{(-i)^n}{n!}\int dx_n^4...dx_1^4\hat{\mathcal{H}}^{(1)}_{\text{int}}(x_n)...\hat{\mathcal{H}}^{(1)}_{\text{int}}(x_1) \rightarrow$$
$$\frac{(-iq)^n}{n!}\int\frac{d^4k_n}{(2\pi)^4}...\frac{d^4k_1}{(2\pi)^4}L^{\mu_n...\mu_1}(-k_n,...,-k_1)\hat{A}_{\mu_n}(k_n)...\hat{A}_{\mu_1}(k_1):, \tag{78}$$

with

$$L^{\mu_n...\mu_1}(k_n,...,k_1) = -\frac{qB}{2\pi}\int\frac{dE}{2\pi}\frac{dp_z}{2\pi}\exp\left\{\frac{i}{2qB}\sum_{j<i}(\mathbf{k}_i\times\mathbf{k}_j)_z\right\}$$
$$\times\sum_{Q_1...Q_n}(-i)^n g^{\epsilon_n}_{q_n}(E_n)...g^{\epsilon_1}_{q_1}(E_1)[\Gamma^{\epsilon'\epsilon_n}_{q'q_n}(\mathbf{k}_n)]^{\mu_n}...[\Gamma^{\epsilon_1\epsilon}_{q_1q}(\mathbf{k}_1)]^{\mu_1}, \tag{79}$$

where the integral is over the undetermined loop energy and parallel momentum. This calculation is valid for Feynman diagrams with only single photon vertices and particle/antiparticle propagators. Once again the generalisation to diagrams with two photon vertices and photon propagators is obvious.

7. Feynman Rules

The Feynman rules in the magnetised spin-0 case can be developed from the $\hat{S}$-matrix expansion. The rules are the same as for the spin-$\frac{1}{2}$ case except for the addition of a two photon vertex, with its own unique vertex factor, the explicit form of the vertex function $[\Gamma^{\epsilon'\epsilon}_{q'q}(\mathbf{k})]^{\mu}$ and the form of the particle propagator. The existence of a two photon vertex also adds new variety to the Feynman diagrams as diagrams with adjacent photon propagators and closed loops with only photon propagators are now possible. In considering a particular process, the inclusion of such diagrams introduces terms in the corresponding Feynman amplitude for which there is no counterpart in the spin-$\frac{1}{2}$ case.

The rules are as follows:

- Feynman diagrams are written with the initial state on the right and final state on the left. A given Feynman amplitude can be written in one of two forms, (74) or (78).
- Incoming and outgoing particle/antiparticle lines give no contribution to a diagram, they merely serve to indicate the initial and final particle/antiparticle states.
- An incoming photon line labelled with four-vector index μ and with wavevector $\mathbf{k}$ in mode M is associated with the factor $a_M(\mathbf{k})e^{\mu}_M(\mathbf{k})$. An outgoing photon line labelled with four-vector index ν and with wavevector $\mathbf{k}'$ in mode M' contributes $a^*_{M'}(\mathbf{k}')e^{\nu *}_{M'}(\mathbf{k}')$.
- A single photon vertex with photon line labelled with four-vector index μ and wavevector $\mathbf{k}$ directed out of the vertex, and with initial and final particle/antiparticle states (ϵ_i, q_i) and (ϵ_j, q_j), respectively, contributes a vertex function

$$[\Gamma^{\epsilon_j\epsilon_i}_{q_jq_i}(\mathbf{k})]_{\mu_j}$$

to $[\hat{G}^{\epsilon'\epsilon}_{q'q}(k_n, ..., k_1)]^{\mu_n ... \mu_1}$ or $L^{\mu_n ... \mu_1}(k_n, ..., k_1)$. The vertex also contributes a factor $-iq$ to the Feynman amplitude.
- A two photon vertex with outgoing photon lines labelled with indices μ_i and μ_j and wavevectors $\mathbf{k}_i$ and $\mathbf{k}_j$ respectively and with initial and final particle/antiparticle states (ϵ_i, q_i) and (ϵ_j, q_j) and corresponding particle/antiparticle energies E_{q_i} and E_{q_j} contributes

$$2g^{\mu_j\mu_i}\frac{[\Gamma^{\epsilon_j\epsilon_i}_{q_jq_i}(\mathbf{k_j}+\mathbf{k_i})]^0}{\epsilon_j\mathcal{E}_{q_j}+\epsilon_i\mathcal{E}_{q_i}}\,.$$

A factor $-iq$ is also contributed to the amplitude.

- An internal particle/antiparticle line with intermediate state (ϵ_r, q_r) and energy E_r is associated with a modified propagator,

$$-ig^{\epsilon_r}_{q_r}(E_r)\,.$$

- An internal photon line contributes a factor

$$-iD^{\mu_j\mu_i}(k)$$

instead of the particle/antiparticle propagator with k directed from vertex μ_j to vertex μ_i.

8. Conclusions

The aim of this paper has been to derive the Feynman rules for magnetised spin-0 bosons. The major results of this paper are the explicit forms of the spin-0 vertex function (41) with (42) and (44), the modified particle propagator (53) and the Feynman rules for the magnetised spin-0 particles in Section 7.

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Appendix: Properties of the Functions H_n and J_ν^n

Properties of the Hermite Polynomials, H_n

(i) Recursion relation

$$H'_n(x) = 2nH_{n-1}(x) \,. \tag{A1}$$

(ii) Orthogonality

$$\int_{-\infty}^{\infty} dx\, H_l(x)H_m(x)e^{-x^2} = \pi^{\frac{1}{2}}2^l l!\delta_{l,m} \,. \tag{A2}$$

(iii) Useful integrals

$$\int_{-\infty}^{\infty} xH_n(x)H_m(x)e^{-x^2}dx = (\pi 2^{n+m}n!m!)^{\frac{1}{2}} \begin{cases} \{(n+1)/2\}^{\frac{1}{2}} & m = n+1 \\ \{n/2\}^{\frac{1}{2}} & m = n-1 \\ 0 & \text{otherwise}\,, \end{cases} \tag{A3}$$

$$\int_{-\infty}^{\infty} x^2 H_n^2(x) e^{-x^2} dx = \pi^{\frac{1}{2}}(2n+1)2^{n-1} n!, \tag{A4}$$

$$\int_{-\infty}^{\infty} H_n(x) H'_m(x) e^{-x^2} dx = n^2 \pi^{\frac{1}{2}} 2^{n+1}(n-1)! \delta_{n,m}. \tag{A5}$$

Properties of J_ν^n

(i) J_ν^n was defined in Section 2 and has the property

$$J_\nu^n(x) = (-)^\nu J_{-\nu}^{n+\nu}(x). \tag{A6}$$

(ii) Recursion relations

$$J_\nu^{n+1}(x) = \left(\frac{n+\nu+1}{n+1}\right)^{\frac{1}{2}} J_\nu^n(x) - \left(\frac{x}{n+1}\right)^{\frac{1}{2}} J_{\nu+1}^n(x) \tag{A7}$$

$$= \frac{-x+\nu+1}{\{(n+1)(n+\nu+1)\}^{\frac{1}{2}}} J_\nu^n(x) + \left\{\frac{x(n+\nu)}{(n+1)(n+\nu+1)}\right\}^{\frac{1}{2}} J_{\nu-1}^n(x), \tag{A8}$$

$$J_\nu^{n-1}(x) = \left(\frac{n+\nu}{n}\right)^{\frac{1}{2}} J_\nu^n(x) - \left(\frac{x}{n}\right)^{\frac{1}{2}} J_{\nu-1}^n(x) \tag{A9}$$

$$= \frac{-x+n}{\{n(n+\nu)\}^{\frac{1}{2}}} J_\nu^n(x) + \left\{\frac{x(n+\nu+1)}{n(n+\nu)}\right\}^{\frac{1}{2}} J_{\nu+1}^n(x), \tag{A10}$$

$$(x+\nu) J_\nu^n(x) = \{x(n+\nu)\}^{\frac{1}{2}} J_{\nu-1}^n(x) + \{x(n+\nu+1)\}^{\frac{1}{2}} J_{\nu+1}^n(x). \tag{A11}$$

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