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Resonances in Few-body Systems*

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Abstract

I will show how the resonance poles of the scattering amplitude correspond to eigenstates of the Hamiltonian that is analytically continued into the complex r-space via the dilatation group transformation, and how the corresponding eigenstates satisfy a modified form of the orthogonality condition. As an illustration of the results, I will consider examples of resonances in $\pi - d$, and $\alpha - d$ scattering as a three-body system.

1. Introduction

One of the many things I have learned from Ian over the years is that ideas and methods developed in one branch of physics often lead to new insight when applied to other fields of physics. Ian has applied this methodology with considerable skill and great success. His use of nuclear reaction theory to atomic and molecular physics is just one example of this approach. In an attempt to make a modest contribution to the unifying spirit of theoretical physics on this special occasion of Ian's sixtieth birthday, I will try to demonstrate how the methods used in nuclear few-body problems for the study of resonances are basically the same as those developed in atomic and molecular physics.

Resonances in quantum system have a long history dating back to the early years of quantum mechanics with the work of Gamow (1928), who used the concept to give a description of α-decay of heavy nuclei. To date, resonances have been observed in most scattering experiments from low energy scattering off atomic and molecular systems, to the recent determination of the width of the Z^0 at the highest energies achieved in the laboratory. In all cases the experimental observation is an enhancement in the cross section when plotted as a function of energy. A more detailed analysis of such data often reveals that this enhancement in the cross section is due to one specific partial wave. To that extent, resonances have a definite set of quantum numbers just like bound states, the only difference being the fact that these states have a finite lifetime and thus correspond to complex energy eigenstates. This raises the question of how a Hermitian Hamiltonian could give rise to a complex eigenvalue and do we include these resonance states in our Hilbert space?

* Dedicated to Professor Ian McCarthy on the occasion of his sixtieth birthday.

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The study of resonances in few-body systems goes back to a set of lectures given by Lovelace (1964) on the Faddeev equation for three-particle scattering. Lovelace made use of the method of contour rotation in momentum space to show that: (i) A rotation of the contour by an angle ϕ can extend the domain, over which the kernel of the two- and three-body integral equations is a Schmidt operator. (ii) The rotation of the contour can be used to analytically continue the two- and three-body amplitude through the unitarity cuts, onto the second Riemann sheet of the complex energy plane. This second result could reveal poles in the scattering amplitude which, if close to the real energy plane, can be considered as resonances. This method has since been employed to study resonances in two- and three-body systems.

On the other hand, the method of 'contour rotation' in coordinate space, often referred to as the dilatation group transformation (Aguilar and Combes 1971; Balslev and Combes 1971; Simon 1973) was developed to examine the spectrum of Green's function on the second energy sheet. This method has been adapted extensively in atomic and molecular physics to the determination of properties of resonances using the variational method (Ho 1983; Moiseyev 1984).

I will show the equivalence of the contour rotation method, in both coordinate and momentum representation, and how each gives a complementary insight into the concept of a resonances being an eigenstate of the Hamiltonian for that system. In particular I will demonstrate that in momentum space, where the boundary conditions are included in the integral equation, the deformation of the contour can be constrained by singularities that result from the boundary condition on the system. I will also demonstrate, in coordinate representation, that the wavefunction for a resonance is both an eigenstate of the Hamiltonian, and normalisable, and could be used as a basis for variational calculations or perturbation theory. Although most of the results presented have been reported in the literature, I have found the situation is such that in atomic and molecular physics the method predominantly used is the deformation of the contour in coordinate space, while in the nuclear few-body problem, the emphasis is on contour rotation in momentum representation. A union of these two methods would give a better understanding of the concept of resonances in quantum systems.

In Section 2, I will demonstrate how the poles of the scattering amplitude on the second energy sheet are exposed by the method of rotation of the contour in momentum space ($k \rightarrow k e^{-i\phi}$), and how the corresponding integral equation, when converted to coordinate representation, leads to the Schrödinger equation with the coordinate transformation $r \rightarrow r e^{i\phi}$, where ϕ is the angle of rotation of the contour. This allows me to establish the fact that the solutions of the Schrödinger equation for a resonance are normalisable and can be use as basis for either a perturbation expansion or a variational calculation. I will then proceed in Section 3 to the analysis of the method of contour rotation in momentum space for the Faddeev equation assuming a separable two-body interaction. In particular, I will concentrate on the case when two of the particles form a bound state, while any of these two can form a resonance state with the third particle. Examples of such a system are the πNN and αNN systems. Here, I will show how to analytically continue my equations to a considerable portion of the second Riemann sheet and part of the third sheet

of the energy plane. To illustrate the power of this method and how it can help in the analysis of resonances above the threshold for the production of three-body final state, I consider in Section 4 both $\pi-d$ and $\alpha-d$ scattering, and in particular how the energy dependence in the cross section is not necessarily all due to resonances, but could partly be due to the square root threshold resulting from the fact that two of the three particles in the system can form a resonance. This raises a question regarding the analysis of the data based solely on two-body scattering theory. Finally, In Section 5 I present some concluding remarks.

2. Contour Rotation Method

Experimentally, structures are often observed in the cross section as a function of energy, which are referred to as resonances. These resonances have quantum numbers similar to bound states and correspond to poles of the partial wave scattering amplitude on the second Riemann sheet of the complex energy plane.

In this section I demonstrate in what sense these resonances are solutions of the Schrödinger equation, and how the properties of these resonances can be calculated, i.e. their energy and wavefunction, when given the Hamiltonian for the system.

For the present analysis I consider a two-body system with the Hamiltonian

$$H = H_0 + V, \tag{1}$$

where $H_0 = k^2/2\mu$. The corresponding Lippmann–Schwinger equation, after partial wave expansion, is then given by

$$T_\ell(k,k';E^+) = V_\ell(k,k') + \int_0^\infty dk''\, k''^2\, V_\ell(k,k'')\, G_0(E^+,k'')\, T_\ell(k'',k';E^+). \tag{2}$$

Here the free particle Green's function is $G_0(E^+,k) = (E^+ - k^2/2\mu)^{-1}$, with $E^+ = E + i\varepsilon$. The condition $\varepsilon > 0$ and infinitesimal provides the necessary boundary condition for an outgoing scattering wave. In operator form, equation (2) is written as

$$T_\ell(E^+) = V_\ell + K_\ell(E^+)\, T_\ell(E^+), \tag{3}$$

where $K_\ell(E) = V_\ell\, G_0(E)$. It is clear from the integral on the right-hand side of (2), that the T-matrix $T_\ell(E^+)$ has a square root branch point at $E = 0$, with the corresponding branch cut along the positive real axis in the complex E-plane. This branch cut, the unitarity cut, determines the sheet structure over which the T-matrix is defined. To get onto the second sheet of the energy plane, and determine the position of the poles of the T-matrix that correspond to resonances, the contour of integration must be rotated by performing the transformation $k'' \to k'' e^{-i\phi}$, with $\phi > 0$ in (2). This procedure rotates the branch cut from $\arg E = 0$ to $\arg E = -2\phi$, and in the process exposes a part of the second energy sheet that might have the poles of the T-matrix that correspond to resonances. In other words, the rotation of the contour of integration in (2) defines the partial wave T-matrix in the new domain $-2\phi < \arg E < 2\pi - 2\phi$. The maximum angle of rotation is determined in this case by the singularities of the potential $V_\ell(k,k'')$ (Nuttall 1967).

In this energy domain, the kernel of the Lippmann–Schwinger equation is compact, and the T-matrix can be written in terms of the solution of the homogeneous equation as (see the Appendix)

$$T_\ell(E) = \sum_n \frac{\Phi_\ell^{(n)}}{1-\lambda_n(E)} a_n \quad \text{with} \quad a_n = \langle \Phi_\ell^{(n)} | V_\ell \,, \tag{4}$$

where $\lambda_n(E)$ and $\Phi_\ell^{(n)}$ satisfy the eigenvalue equation

$$\lambda_n \Phi_\ell^{(n)} = K_\ell(E)\, \Phi_\ell^{(n)} \,. \tag{5}$$

It is clear from equations (4) and (5) that the energy at which any of the eigenvalues of the kernel $K_\ell(E)$ is one, the partial wave T-matrix has a pole, and the residue at this pole is proportional to the corresponding eigenstate. In other words, the poles of the T-matrix are determined by the equation

$$\Phi_\ell = V_\ell\, G_0(E)\, \Phi_\ell \,, \tag{6}$$

for $-2\phi < \arg E < 2\pi - 2\phi$. Here, ϕ should be chosen such that the energy of the resonance is in the energy domain on which (6) is defined. This equation can be written in the form of the Schrödinger equation

$$(E - H_0)\, \psi_\ell = V_\ell\, \psi_\ell \,, \tag{7}$$

where

$$\psi_\ell = G_0(E)\, \Phi_\ell \,, \tag{8}$$

and the free Hamiltonian in momentum space is given by $H_0 = k^2/2\mu$, with k now along the rotated contour, i.e. $k = |k|\, e^{-i\phi}$, and $\phi > 0$. Here I would like to emphasise that (7), for $\phi = 0$, i.e. no contour rotation, is the usual Schrödinger equation for a Hermitian Hamiltonian, and the only solutions are for real energies, i.e. on the real E-axis. However, on the rotated contour, the Hamiltonian $H = H_0 + V_\ell$ is not Hermitian, and (7) admits solutions with complex E. In order to find a resonance at $E = E_R = R_r - iE_i$, the contour must be rotated by an angle ϕ such that $\tan 2\phi > E_i/E_r$, and in this way the position of the resonance is in the energy domain $-2\phi < \arg E < 2\pi - 2\phi$. Furthermore, since the kernel of the integral equation is a Schmidt operator, the residue at the pole is a projection operator (Lovelace 1964), which implies that the wavefunction Φ_ℓ is normalisable.

To write the Schrödinger equation in coordinate space, I need to first consider (5) in momentum space for $\lambda = 1$, i.e.

$$\Phi_\ell(k) = \int dk'\, k'^2\, V_\ell(k,k')\, G_0(E,k')\, \Phi_\ell(k') \,, \tag{9}$$

or, the corresponding equation for $\psi_\ell(k)$, which is given by

$$\left(E - \frac{k^2}{2\mu}\right) \psi_\ell(k) = \int dk'\, k'^2\, V_\ell(k,k')\, \psi_\ell(k') \,. \tag{10}$$

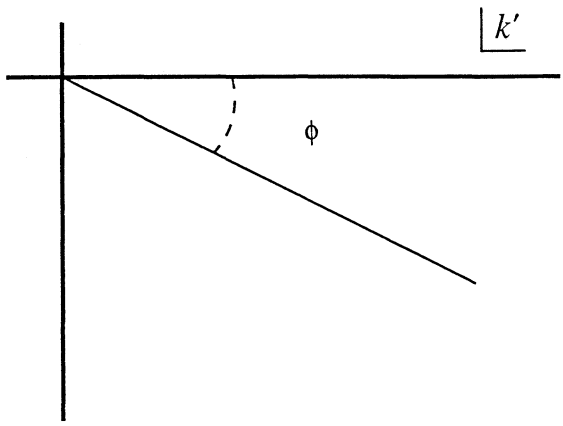

Fig. 1. Contour of integration in momentum space.

Here k and k' are along the ray defined by the angle $-\phi$ (see Fig. 1). In other words, this equation can be written as

$$\left(E-\frac{k^2}{2\mu}e^{-2i\phi}\right)\psi_\ell(ke^{-i\phi})=\int_0^\infty dk'\,k'^2\,e^{-3i\phi}\,V_\ell(ke^{-i\phi},k'e^{-i\phi})\,\psi_\ell(k'e^{-i\phi}), \quad (11)$$

where k and k' are now real and the partial wave potential V_ℓ is given by

$$V_\ell(ke^{-i\phi},k'e^{-i\phi})=\frac{2}{\pi}\int_0^\infty dr\,r^2\,j_\ell\left(kre^{-i\phi}\right)V(r)\,j_\ell\left(k're^{-i\phi}\right), \quad (12)$$

with $j_\ell(kr)$ being the spherical Bessel function. A change of variables to $\xi=re^{-i\phi}$ allows me to rewrite (11) as

$$\left(E-\frac{k^2}{2\mu}e^{-2i\phi}\right)\psi_\ell(ke^{-i\phi})=\frac{2}{\pi}\int_0^\infty d\xi\,\xi^2\,j_\ell(k\xi)\,V(\xi e^{i\phi})\,\psi_\ell^\phi(\xi), \quad (13)$$

where

$$\psi_\ell^\phi(\xi)=\int_0^\infty dk\,k^2\,j_\ell(k\xi)\,\psi_\ell(ke^{-i\phi}). \quad (14)$$

If I now multiply (13) by $k^2 j_\ell(kr)$, integrate over k, make use of the orthogonality of the spherical Bessel function, and the differential equation it satisfies, I get, after some algebra,

$$\left\{\frac{e^{-2i\phi}}{2\mu}\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)-\frac{\ell(\ell+1)}{r^2}\right]+V\left(re^{i\phi}\right)\right\}\psi_\ell^\phi(r)=E\,\psi_\ell^\phi(r). \quad (15)$$

This is basically the radial Schrödinger equation in coordinate representation with the substitution $r\rightarrow re^{i\phi}$, and gives a non-Hermitian Hamiltonian, which implies that the corresponding eigenvalues, or energies, may in general be complex. Furthermore, from the above analysis in momentum space, it is clear that a determination of a resonance at $E=E_r-iE_i$ on the second energy sheet will require that I rotate the contour by an angle ϕ, where $\tan 2\phi > E_i/E_r$. This procedure of 'contour rotation' in coordinate space is commonly referred to

as a dilatation group transformation on the Hamiltonian (Aguilar and Combes 1971; Balslev and Combes 1971; Simon 1973), and has been used in atomic and molecular physics for the determination of properties of resonances (Ho 1983; Moiseyev 1984). On the other hand, the use of contour rotation in momentum space as first proposed by Lovelace (1964) has been used in nuclear two- and three-body systems.

To impose resonance boundary conditions on (15), the solutions of this equation for $r \to \infty$ must be determined. This can be achieved by taking $z = re^{i\phi}$, realising that the resultant equation is the Schrödinger equation in the variable z. If the potential $V(z)$ goes to zero as $r \to \infty$, then the possible solutions to (15) are spherical Hankel functions of z, i.e. the most general solution is of the form

$$\psi_\ell(z) = A\, h_\ell^{(+)}(kz) + B\, h_\ell^{(-)}(kz), \tag{16}$$

where $k^2 = 2\mu E$. For a resonance, the energy is given by $E = E_r - iE_i$, and $k = |k| e^{-i\theta}$ with $\tan 2\theta = E_i/E_r$. In this case, for the wavefunction to be normalisable, $B = 0$ and $\phi > \theta$. Both of these conditions are required if the resonance states are to belong to the Hilbert space. I should point out at this stage that the condition on the angle ϕ for this dilatation group transformation is identical to the condition I obtain in momentum space on the rotation of the contour to expose the resonance pole of the T-matrix on the second energy sheet.

From the above analysis of resonances in both momentum and coordinate representation, I can conclude that the inclusion of resonance states in our Hilbert space will require the rotation of the contour as defined above, and in this case the Schrödinger equation is given by

$$H_\phi |\psi_n^\phi\rangle = E_n |\psi_n^\phi\rangle, \tag{17}$$

where $-2\phi < \arg E < 2\pi - 2\phi$. For $\phi > 0$, the Hamiltonian H_ϕ is non-Hermitian, and (17) admits solutions with complex energies on the second Riemann sheet of the energy plane. In the event that these solutions correspond to eigenenergies that are close to the real energy axis, they will be considered as resonances to the extent that they will effect the cross section. Since the Hamiltonian in this new energy domain is non-Hermitian, and satisfies the condition that

$$H_\phi^* = H_{-\phi}, \tag{18}$$

I can rewrite (17) as

$$H_{-\phi} |\psi_{n'}^{-\phi}\rangle = E_{n'}^* |\psi_{n'}^{-\phi}\rangle \tag{19}$$

This implies that for every state we have on the second energy sheet at energy E_R, there is another state at energy E_R^* also on the second Riemann sheet. Furthermore, by multiplying (17) by $\langle\psi_{n'}^{-\phi}|$ and the complex conjugate of (19) by $|\psi_n^\phi\rangle$ and subtracting, I get

$$\langle\psi_{n'}^{-\phi}| H_\phi |\psi_n^\phi\rangle - \langle\psi_{n'}^{-\phi}| H_\phi |\psi_n^\phi\rangle = (E_n - E_{n'}) \langle\psi_{n'}^{-\phi}|\psi_n^\phi\rangle = 0. \tag{20}$$

This gives the orthonormality of the eigenstates of the Hamiltonian in the energy domain $-2\phi < \arg E < 2\pi - 2\phi$ and with proper choice of ϕ will include resonance states. I should point out at this stage that the normalisation of the resonance eigenstate with energy E_R is not with its complex conjugate, but with the complex conjugate of the state corresponding to the energy E_R^*. This is expected, considering the fact that the Hamiltonian in the new energy domain is not Hermitian. For the case when the eigenvalues of the Hamiltonian H_ϕ are real, i.e. a bound state, the wavefunctions $|\psi_n^\phi\rangle$ and $|\psi_n^{-\phi}\rangle$ are identical, and the orthonormality is identical to that for a Hermitian Hamiltonian.

Although the above analysis was for the special case of a two-body system, there are no inherent problems in extending the arguments to a three-body system provided that the integral equation in momentum space has a compact kernel. In other words, the Faddeev equations need to be used in either coordinate space or momentum space. In fact most of the results presented in the next section will be for systems that can be approximated by a three-body Hamiltonian. The only modification that might be required for the three-body system is that the contour deformation to reveal the second sheet of the energy plane might prove to be more than a simple rotation of the contour, or alternatively, only parts of the second Riemann sheet of the energy plane will be accessible. This is mainly due to the fact that the kernel of the Faddeev equation includes the two-body T-matrix, which gives rise to moving singularities. These moving singularities are basically the boundary conditions on the three-body system.

3. Application to Three-body Systems

Having established the fact that the method of contour rotation in either momentum or coordinate representation exposes that part of the second Riemann sheet of the energy plane that has resonance poles, I turn my attention to the application of this method to light nuclear systems which can be modeled in terms of a three-body Hamiltonian. Here the rotation of contour in momentum representation has the advantage that the complicated boundary conditions associated with the three-body nature of the problem are included in the kernel and do not have to be imposed on the wavefunction. The applications I would like to examine in detail are: (i) The question of dibaryon resonances in the πNN system. (ii) The spectrum of ^{6}Li in an αNN model. Both of these systems have similar structures in that the cross section for the scattering of a boson, π or α, from the deuteron, is dominated by enhancement in the cross section when examined as a function of energy. Furthermore, the subsystems in both cases have P-wave resonances in the form of the $\Delta(1232)$ for the πN system, and the $P_{3/2}$ resonance, i.e. the ground state of ^{5}He, in the $\alpha - N$ system. The question that has often been raised in the analysis of the data for $\pi - d$ and $\alpha - d$ scattering is whether the observed structure is a resonance in the three-body system, or a result of the threshold for the production of a resonant sub-system, e.g. $\Delta(1232)$ or ^{5}He. The analysis of the data in terms of two-body scattering theory fails because these structures are above the threshold for three-body final states. Furthermore, a fit to the phase shift with a Breit-Wigner form could be possible

even though the only singularity in the amplitude is a square root branch point due to the presence of a resonant sub-system rather than a pole.

Since in both of these problems the dominant dynamics are the deuteron in the $N-N$ subsystem and the P-wave resonance in the $\pi-N$ or $\alpha-N$ subsystem, it is a good approximation to represent the $N-N$, $\pi-N$, and $\alpha-N$ interaction by a separable potential. Such separable potentials give a good description of the experimental phase shift at the relevant energies, and preserve the important features of the two-body amplitude, i.e. the resonance structure. From a practical point of view, the use of a separable potential reduces the Faddeev equation, after partial expansion, to a one-dimensional integral equation of the form

$$X_{\alpha,\beta}(k,k';E) = Z_{\alpha,\beta}(k,k';E) + \sum_{\gamma} \int_0^\infty dk'' \, k''^2 \, Z_{\alpha,\gamma}(k,k'';E) \, \tau_\gamma \left(E - \frac{k''^2}{2\mu_\gamma} \right) X_{\gamma,\beta}(k'',k';E), \quad (21)$$

where μ_γ is the reduced mass of the spectator particle and the interacting pair that form the quasi-particle γ. Here, α, β and γ label the possible two-body channels, e.g. for the πNN system the possible two-body clustering could be $(NN)\pi$ and $(N\pi)N$. Furthermore, for the present analysis, I will assume there is one partial wave in the $N-N$ interaction, i.e. the deuteron, and one channel in the $\pi-N$ interaction corresponding to the $\Delta(1232)$. In this approximation, (21) consists of two coupled integral equations. The Born amplitudes $Z_{\alpha,\beta}$ correspond to one particle exchange, while in the kernel, $\tau_\gamma(\epsilon)$ is the quasi-particle propagator. Thus for $\gamma = d$, $\tau_d(\epsilon)$ has a pole at the deuteron binding energy, i.e. $\epsilon = -2 \cdot 2246\,\text{MeV}$, while $\tau_\Delta(\epsilon)$ has a pole at $\epsilon = M_\Delta - i\Gamma/2 = (1232 - 55i)\,\text{MeV}$, the position of the $\Delta(1232)$ resonance.

To determine the position of resonance poles for these two three-body systems, I need to consider the homogeneous integral equation corresponding to (21) as an eigenvalue problem, i.e.

$$\lambda_n \, \Phi^n_\alpha(k;E) = \sum_{\gamma} \int_0^\infty dk' \, k'^2 \, Z_{\alpha,\gamma}(k,k';E) \, \tau_\gamma \left(E - \frac{k'^2}{2\mu_\gamma} \right) \Phi^n_\gamma(k',E). \quad (22)$$

To gain access to the region of the energy plane where the resonance poles reside, I need to deform the contour of integration to allow me an analytic continuation of (22) in the energy variable. This can be achieved by the contour rotation $k \to k\,e^{-i\phi}$ and $k' \to k'\,e^{-i\phi}$ in (22). In carrying through this contour rotation, I need to insure that there are no singularities from the Born amplitude $Z_{\alpha,\gamma}$ and the quasi-particle propagator τ_γ in the k'-plane that are crossed in the transformation. In general, $Z_{\alpha,\gamma}$ is of the form

$$Z_{\alpha,\gamma}(k,k';E) \propto \int_{-1}^{+1} dx \, \frac{f(k,k',x) \, P_\ell(x)}{E - ak^2 - bk'^2 - ckk'x}, \quad (23)$$

where $f(k,k',x)$ depends on the form of the separable potential, and $P_\ell(x)$ is the Legendre polynomial of order ℓ, while a, b and c depend on the masses of the three particles. In general, $Z_{\alpha,\gamma}$ develops logarithmic branch points in the k'-plane when the poles of the integrand in (23) coincide with the end points of the integration, i.e. $x = \pm 1$. By taking $k \to k\,e^{-i\phi}$ and $k' \to k'\,e^{-i\phi}$ and $|\arg E| < 2\phi$, $Z_{\alpha,\gamma}$ can have branch points only from the poles of $f(k,k',x)$, and

these are, for a Yamaguchi-type separable potential, along the imaginary k'-axis (Stelbovics 1977). Thus, as far as $Z_{\alpha,y}$ is concerned, I can analytically continue (22) to cover the full energy domain where the resonance poles might reside.

Having established that the Born amplitude does not create any problem in rotating the contour of integration provided *both* k and k' are rotated by the same angle, I turn my attention to the quasi-particle propagator $\tau_y(E-k'^2/2\mu_y)$. This quasi-particle propagator can be written as

$$\tau\left(E-\frac{k'^2}{2\mu_y}\right)=\begin{cases}\dfrac{S_y\left(E-\dfrac{k'^2}{2\mu_y}\right)}{E-\dfrac{k'^2}{2\mu_y}+B} & \text{for } N-N\\[2ex] \dfrac{S_y\left(E-\dfrac{k'^2}{2\mu_y}\right)}{E-\dfrac{k'^2}{2\mu_y}-M_r+\frac{1}{2}i\Gamma} & \text{for } \pi-N \text{ or } \alpha-N,\end{cases} \tag{24}$$

where M_r is the mass of the resonance, i.e. the mass of the Δ for the $\pi-N$ system, or the mass of ^{5}He for the $\alpha-N$ system, and Γ is the width of the resonances. Here, S_y at the pole is the residue of τ_y, and in general has a square root branch point at $E-k'^2/2\mu_y=0$, which gives rise to the unitarity cut in the two-body subsystem and determines the three-body threshold. From the above expression for τ_y it is clear that as I analytically continue (22) into the second Riemann sheet of the E-plane, there is a corresponding change in the position of the poles and branch point of τ_y in the k'-plane. These poles and branch point, which are the result of including the two- and three-body boundary condition in the integral equation, will constrain the domain in the energy plane that I can access using this procedure (Pearce and Afnan 1984).

To determine how far I can analytically continue (24) into the second Riemann sheet of the energy plane, I need to examine how these singularities from τ_y effect the contour rotation. In particular, to guarantee that the singularities of $Z_{\alpha,y}$ remain on the imaginary k'-axis I need to have both k and k' in (24) on the same ray, making an angle of ϕ with the real k'-axis and in the fourth quadrant. The branch points in τ_y are at

$$k'_b=\pm\sqrt{2\mu_y E}, \tag{25}$$

which for $E=E_r-iE_i$, $E_i>0$, have the pole of τ_y, in the fourth quadrant of the k'-plane, at an angle θ_b given by

$$\tan 2\theta_b=\frac{E_i}{E_r}. \tag{26}$$

Thus as far as these branch points are concerned, I need to take $\phi>\theta_b$ to avoid these singularities. I now turn to the deuteron pole in τ_y. In this case the pole in the k'-plane is at

$$k'_d=\pm\{2\mu_y(E+B)\}^{\frac{1}{2}}. \tag{27}$$

This pole subtends an angle of ϕ_d in the fourth quadrant of the k'-plane and is determined by the equation

$$\tan 2\theta_d = \frac{E_i}{E_r + B}. \tag{28}$$

Clearly, this angle is smaller than that due to the branch point, i.e. θ_b, and to that extent I need not worry about its constraint on the angle of rotation of the contour ϕ.

The resonance pole in τ_y gives rise to a square root branch point in the scattering amplitude and the solution of (22) in the E-plane at $E = M_r - \frac{1}{2}i\Gamma$. This branch point can, if close to the real axis, effect the energy dependence of the cross section to the extent that it could be mistaken for a resonance in the three-body system. More important, at this stage, is the fact that this branch point and associated branch cut divides the fourth quadrant of the E-plane where the resonances might reside into four regions as illustrated in

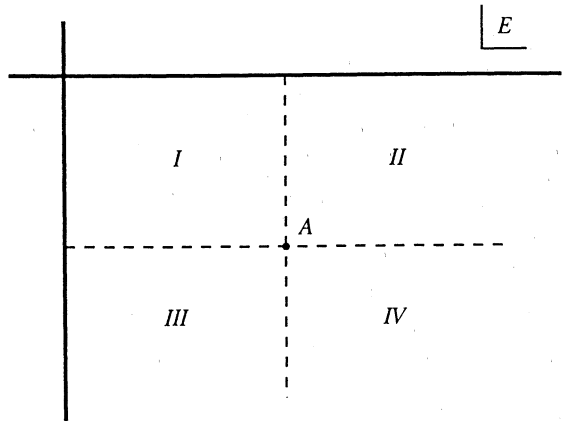

Fig. 2. Region of the energy plane that can be accessed via contour rotation. The point A at $E = M_r - \frac{1}{2}i\Gamma$ corresponds to the branch point resulting from the resonance in the two-body subsystem.

Fig. 2. Here the two-body resonance pole in τ_y leads to two poles in the k'-plane at

$$k'_r = \pm\left\{2\mu_y(E - M_r + \tfrac{1}{2}i\Gamma)\right\}^{\frac{1}{2}}. \tag{29}$$

The corresponding angle for the resonance pole in the k'-plane is θ_r, and is given by

$$\tan 2\theta_r = -\frac{E_i - \frac{1}{2}\Gamma}{E_r - M_r}, \tag{30}$$

where I have taken $E = E_r - iE_i$. For $E = 0$, the angle $2\theta_r$ is in the second quadrant, and therefore, $\pi/4 < \theta_r < \pi/2$. As I proceed along the real energy axis to the point $E = M_r$, θ_r gets to a value of $\pi/4$, while proceeding along the imaginary axis to $E = -\frac{1}{2}i\Gamma$, θ_r gets to a value of $\pi/2$. If I now carry

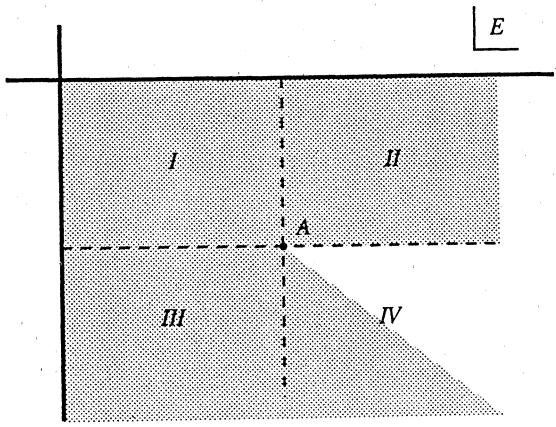

Fig. 3. The shaded area is the domain of the second Riemann sheet of the energy plane to which I can analytically continue (22), while maintaining the contour deformation along a ray in the fourth quadrant of the k'-plane.

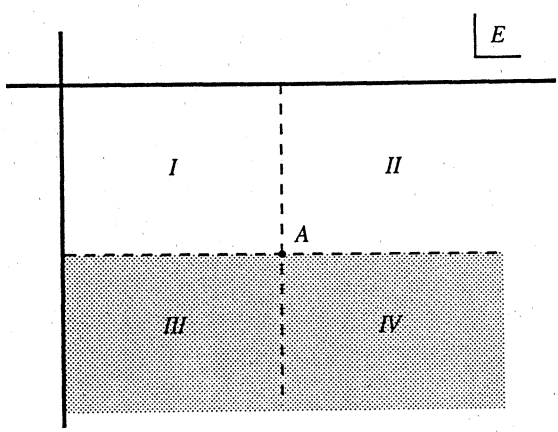

Fig. 4. The shaded area is the domain of the third Riemann sheet of the energy plane to which I can analytically continue (22), while maintaining the contour deformation along a ray in the fourth quadrant of the k'-plane. Access to this sheet is via the square root branch cut resulting from the resonance pole in τ_γ.

this analysis through, I find that as I analytically continue my equation in the energy variable from the real axis through region *I* to region *III* and then to region *IV* (see Fig. 2), one of the resonance poles in the k'-plane gets into the region $-\pi/4 < \theta_r < 0$ approaching from $\theta_r = -\pi/4$. At this stage the two-body unitarity branch point is moving towards an angle $\theta = \pi/4$. These two singularities could force the contour to deviate from being along the ray, and this in turn will introduce logarithmic branch points from the Born amplitude $Z_{\alpha,\beta}$. Thus the energy domain on the second Riemann sheet, to which I can analytically continue (22) without introducing elaborate contours of integration, is shown as the shaded area in Fig. 3 (Pearce and Afnan 1984).

In addition to the above energy domain, I can analytically continue (22) onto the third Riemann sheet through the branch cut generated by the resonance pole in τ_γ, i.e. start on the real axis in region *II*, then proceed through the branch cut to region *IV* on the third Riemann sheet, and then to region *III* on the third Riemann sheet (see Fig. 2). In this case as I proceed from region *II* to region *IV*, the resonance pole in the k'-plane crosses the real axis into the fourth quadrant, and I can analytically continue the equation into region *IV* and then *III* of the third Riemann sheet. However, if I attempt to go to region *I* of the third energy sheet I find that the contour of integration is forced onto the negative imaginary k'-axis by the resonance pole, and here I encounter the singularities of the Born amplitude $Z_{\alpha,\gamma}$. Thus the only part of the third Riemann energy sheet I can access is the region with $\Im(E) < -\frac{1}{2}\Gamma$ (see Fig. 4). In this way I have gained access to most of the second and third Riemann sheet of the energy plane where a resonance pole could influence the scattering amplitude on the real axis, and thus produce rapid variation in the measured cross section. To go beyond this energy domain I need to resort to contours that are not a simple ray and trace the motion of the logarithmic branch points resulting from the Born amplitude $Z_{\alpha,\gamma}$.

4. Numerical Results

The motivation for this study has been to demonstrate that structures in the energy dependence of the cross section above the threshold for three-body final state do not have to correspond to resonances only. To illustrate this I will consider the two systems discussed above. In many ways the πNN and αNN are similar in that they correspond to the scattering of a spin zero particle from the deuteron. In both systems the spin zero particle and the single nucleon form a well known resonance. Finally, the cross sections for both $\alpha - d$ and $\pi - d$ scattering show an enhancement close to the threshold for the production of a resonance subsystem.

(a) $\pi - d$ Scattering

The question of resonances in the πNN system is an old one which is still of interest. In the $N-N$ sector of this problem there was a suggestion that the structure observed in the polarisation data is due to a dibaryon resonance. However, it is now accepted that this structure is in fact due to the branch point resulting from the Δ resonance. In the $\pi - d$ sector, the cross section is dominated by a peak which could be considered a resonance if the data are analysed in terms of two-body scattering theory with an amplitude that is the sum of a Breit–Wigner resonance and a background term. If I consider $\pi - d$ scattering in terms of the three-body model discussed in the last section, I find that if the pion is treated relativistically, which is required to get the right total cross section, then the resonance pole is on the third energy sheet at $E = (118 - 141i)$ MeV (Pearce and Afnan 1984). This is much further from the real energy axis than the branch point at $E = (152 - 55i)$ MeV or the position of the resonance as predicted from an analysis of the cross section, using a Breit–Wigner form, of $E = (143 - 90i)$ MeV. This suggests very strongly that the extraction of resonance parameters from the data, using standard two-body

scattering theory, can be misleading. In fact, the cross section seems to be partially dominated by the branch point due to Δ resonance, which is not included in the standard method of analysis of the data.

(b) Resonances in ^{6}Li

If I consider ^{6}Li as an αNN system, which seems to be a good approximation for $\alpha-d$ scattering below the threshold for ^{3}He–^{3}H production of $14\cdot32$ MeV, then I can study the low energy resonances in ^{6}Li using the above three-body equations. In particular, ^{6}Li has a 2^+ resonance state at $0\cdot61$ MeV above the threshold for $\alpha-n-p$, with a half-width of $0\cdot85\pm0\cdot2$ MeV. In this case the square root branch point at $E=(0\cdot777-0\cdot320i)$ MeV is due to the $\alpha-N$ resonance, i.e. ^{5}He. Here again, the resonance is close to the square root branch point and could effect the analysis of the $\alpha-d$ scattering data, and in particular the extraction of resonance parameters. To get a realistic result that can be compared with experiment, I have included in addition to the $^3S_1$ $N-N$ and $P_{3/2}$ $\alpha-N$ interactions, the $S_{1/2}$ and $P_{1/2}$ $\alpha-N$ interactions. In this case I find that the resonance pole is on the second Riemann sheet of the energy plane at $E=(0\cdot15-0\cdot4i)$ MeV (Eskandarian and Afnan 1990). This discrepancy with experiment can be reduced by changing the input two-body $\alpha-N$ interactions. In fact if I fit the low energy $\alpha-N$ phase shifts, the position of this resonance can be shifted to $E=(0\cdot53-0\cdot65i)$ MeV (Eskandarian and Afnan 1990). In either case, the closeness of the branch point to the pole suggests that any analysis of the data should take into consideration the fact that there is a branch point close enough to the real axis that it could influence the energy dependence of the cross section.

5. Conclusions

In the present analysis I have shown that resonances can be considered as part of the spectrum of the Hamiltonian if I extend the energy domain to include the second Riemann sheet of the energy plane where resonance poles reside. The eigenstates corresponding to the resonance states are part of the Hilbert space and form an orthonormal basis with a minor modification of the definition of normalisation. To achieve this analytic continuation of the Schrödinger equation in the energy variable it was necessary to perform a contour rotation in either coordinate or momentum representation.

To illustrate the above result I have considered two three-body problems in which one of the pairs form a bound state, while the other two pairs have resonance states. To include the three-body boundary condition I have carried out the analysis in momentum representation. Here, I demonstrate how the different boundary conditions restrain the energy domain to which I can analytically continue my equation.

For such a three-body system it is often observed that near the threshold for the production of a resonance subsystem, the cross section has rapid energy dependence. This energy dependence is often attributed to the existence of resonances in the full system. By considering as examples the πNN (Pearce and Afnan 1984) and the αNN systems (Eskandarian and Afnan 1990), I have demonstrated that the extraction of resonance parameters from scattering data

will require the inclusion of the threshold for the production of the resonance subsystem. This is often not included in the analysis of experimental data.

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Appendix: Solution of Integral Equation

When writing the solution of the inhomogeneous integral equation in terms of the eigenvalues and eigenstates of the kernel in (4), it is commonly assumed that the kernel is Hermitian. However, on deforming the contour of integration, the kernel ceases to be Hermitian. The present appendix is devoted to the generalisation of (4) for the case when the kernel is not Hermitian.

Let me consider the inhomogeneous integral equation

$$f(x) = g(x) + \int_0^\infty dy\, K(x,y)\, f(y). \tag{A1}$$

The corresponding homogeneous equation is then given as an eigenvalue problem of the form

$$\lambda_n \Phi_n(x) = \int_0^\infty dy\, K(x,y)\, \Phi_n(y). \tag{A2}$$

I now can write the solution of the inhomogeneous equation in terms of the solution of the homogeneous equation as

$$f(x) = \sum_n b_n \Phi_n(x). \tag{A3}$$

With this result in hand I can write (A1) as

$$\sum_n b_n \Phi_n(x) = g(x) + \sum_n b_n \lambda_n \Phi_n(x), \tag{A4}$$

which can be rewritten as

$$\sum_n (1 - \lambda_n)\, b_n \Phi_n(x) = g(x). \tag{A5}$$

To solve this equation for the coefficient b_n, the orthogonality of the basis $\Phi_n(x)$ needs to be determined. For the case of a Hermitian kernel, i.e. $K^*(x,y) = K(y,x)$, the orthogonality is given by

$$\int_0^\infty dx\, \Phi_m^*(x)\, \Phi_n(x) = \delta_{mn}\,. \tag{A6}$$

In this case the coefficients b_n are given by

$$b_n = \frac{1}{1-\lambda_n} \int_0^\infty dx\, \Phi_n^*(x)\, g(x)\,. \tag{A7}$$

In other words, the solution of the integral equation, (A1) is given by

$$f(x) = \sum_n \frac{\Phi_n(x)\, a_n}{1-\lambda_n} \quad \text{with} \quad a_n = \int_0^\infty dx\, \Phi_n^*(x)\, g(x)\,, \tag{A8}$$

which is the result given in (4).

When the contour of integration is rotated into the complex plane to reveal the energy domain that has the resonance poles, the kernel of the integral equation along the rotated contour is not Hermitian, and the above results need to be generalised. In this case, the kernel of the equation depends on the angle of rotation of the contour, and satisfies the condition

$$K_{-\phi}(x,y) = K_{\phi}^*(y,x)\,. \tag{A9}$$

The homogeneous equation can now be written for both K_ϕ and $K_{-\phi}$ as

$$\lambda_n^{\phi}\, \Phi_n^{\phi}(x) = \int_0^\infty K_{\phi}(x,y)\, \Phi_n^{\phi}(y)\,, \tag{A10}$$

and

$$\lambda_m^{-\phi}\, \Phi_m^{-\phi}(x) = \int_0^\infty K_{-\phi}(x,y)\, \Phi_m^{-\phi}(y)\,. \tag{A11}$$

If I now multiply (A10) by $\Phi_m^{-\phi^*}$ and integrate over x, and multiply the complex conjugate of (A11) by $\Phi_n^{\phi}(x)$ and integrate over x, then subtract the resultant two equations making use of the symmetry of the kernel, i.e. (A9), I get

$$\left(\lambda_n^{\phi} - \lambda_m^{-\phi^*}\right) \int_0^\infty dx\, \Phi_m^{-\phi^*}(x)\, \Phi_n^{\phi}(x) = 0\,. \tag{A12}$$

For this equation to be satisfied, I will need to take

$$\lambda_n^{\phi} = \lambda_n^{-\phi^*} \quad \text{and} \quad \int_0^\infty dx\, \Phi_n^{-\phi^*}(x)\, \Phi_n^{\phi}(x) = 1 \quad \text{for} \quad n = m\,, \tag{A13}$$

and

$$\int_0^\infty dx\, \Phi_m^{-\phi^*}(x)\, \Phi_n^{\phi}(x) = 0 \quad \text{for} \quad n \neq m\,. \tag{A14}$$

This result is identical to that stated in (19) regarding the orthonormality of the resonance wavefunctions. Making use of the orthonormality of the eigenstates of the homogeneous integral equation, I can write the solution of the inhomogeneous equation as

$$f(x) = \sum_n \frac{\Phi_n^{\phi}(x)\, a_n}{1-\lambda_n} \quad \text{with} \quad a_n = \int_0^{\infty} dx\, \Phi_n^{-\phi^*}(x)\, g(x). \tag{A15}$$

This result, when applied to the Lippmann–Schwinger equation or the Faddeev equations, gives the position of the poles of the scattering amplitude to correspond to those energies where one of the eigenvalues of the kernel is unity.

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