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Models for Fermion Generations based on Five Fermionic Coordinates*

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Abstract

We show that a limited range of options for fermion families may be neatly encompassed in a spacetime augmented by five Grassmann internal coordinates if we require that the superfields are self-dual in an SU(5) sense. Amongst the possibilities is a family of just three standard model generations. We consider the nature of Higgs fields in this formalism and the form of possible gauge symmetries.

1. Introduction

The idea that internal space is founded not on bosonic but on fermionic coordinates has many attractions. Perhaps the strongest is the fact that internal multiplet representations are strongly circumscribed by the terminating and antisymmetric character of series expansions in such coordinates. The concept has been advanced in earlier papers by Dondi and Jarvis (1980), Casalbuoni and Gatto (1979, 1980), Delbourgo, *et al.* (1988) and Krolikowski (1989) but elements of freedom in the construction mean that the various formulations have differed to a considerable extent. Nevertheless, each model is subject to strong constraints and has to confront the recently established fact that only three light generations of standard particle families occur in nature. (We refer to the LEP collider experiments counting neutrino species via the Z-decay width. For a review see Denegri *et al.* 1990.)

A model based on five internal complex coordinates θ (Delbourgo 1989) has been advocated as an economical way of ensuring the correct multiplet structure and at the same time being capable of containing at least three families. A scheme was offered there in which superfields were expanded in either even *or* odd powers of θ, with the quarks and lepton fields appearing as coefficients in 5's and 10's, according to the standard SU(5) assignments once the quantum numbers of the coordinates are specified. In the most recent attempt at this unification (Delbourgo and White 1990), a predilection towards bosonic superfields was shown: particle multiplets were tied to *odd* powers in θ, and generations were connected with contracted $\bar{\theta}\theta$ factors, since this does not affect the SU(5) character of the states. As a result three families of quarks and leptons arose; but in addition a neutral singlet plus two

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additional 5's were entrained, the latter causing anomaly discomfiture. The alternative version in which superfields are fermionic was excluded because those expansions in *even* powers in θ, augmented by $\bar{\theta}\theta$, will not admit three generations whatever other attractions they may possess.

In this paper we wish to explore a broader version of this Grassmann coordinate model, including the possibility of utilising both Bose and Fermi superfields to describe fermions, by returning to the basic idea of θ expansion to *all* orders. The additional concept we shall incorporate (to keep the resulting proliferation of fermion fields under control) is the notion of Grassmann self-duality. At present some detailed features of the dual operation remain to be refined, so that we restrict ourselves to a catalogue of the small number of possibilities that we see with our present understanding of duality. It is hoped that the few scenarios we outline may be of interest, and may even strike a chord of familiarity with experienced model-builders. In the near future we hope to narrow the options further, at least with regard to a more refined duality rule. For the present, the consequence is a classification scheme in which we find that it appears possible to construct models with two, three or four standard model generations (with the occasional right-handed neutrino), and the possibility of including either of a pair of nonstandard but anomaly-free fermion families, which contain both normal and exotic SU(3)×SU(2)×U(1) fermion representations. We shall describe in some detail how the Grassmannian SU(5) duality idea can serve to restore the balance between 5's and 10's in the previous models, and how when combined with anomaly considerations it leads us to the other models indicated. We then discuss briefly some aspects of the possible gauging of symmetries in the Grassmann picture, and make some remarks about Higgs fields and interaction Lagrangians.

Table 1. Full set of SU(5) multiplets contained in the coordinate expansion of a superfield Ψ

The column and row numbers s and r correspond to the term $(\bar{\theta})^s(\theta)^r$ in the Taylor series for Ψ

$r\backslash s$	0	1	2	3	4	5
0	1	5	10	$\overline{10}$	$\overline{5}$	1
1	$\overline{5}$	1+24	$5+\overline{45}$	$10+\overline{40}$	$\overline{10}+\overline{15}$	$\overline{5}$
2	$\overline{10}$	$\overline{5}$+45	1+24+75	$5+\overline{45}+\overline{50}$	$10+\overline{40}$	$\overline{10}$
3	10	$\overline{10}$+40	$\overline{5}$+45+50	1+24+75	$5+\overline{45}$	10
4	5	10+15	$\overline{10}$+40	$\overline{5}$+45	1+24	5
5	1	5	10	$\overline{10}$	$\overline{5}$	1

2. Superfield Structure, Fermions and Duality

Our internal space is based on five complex fermionic θ coordinates and their conjugates $\bar{\theta}$. Superfields are functions of these variables and their components emerge through the antisymmetric θ products of the Taylor expansion. In order to clarify the significance of the duality and hermiticity constraints that we have in mind, let us begin by listing the full set of SU(5) multiplets, standard and exotic, contained in such a superfield, before any conditions are imposed. The set is summarised in Table 1, where the column s refers to the power of $\bar{\theta}$ and the row r refers to the power of θ: a totality of 1024 states. We first discuss the question of available (and desirable) fermion representations which has been the major focus of our research to date, postponing to later sections the matter of Higgs fields and gauge symmetries.

We hope to extract our fermion degrees of freedom from this array. It should be noted that an alternative (more simplistic?) way to partially combat the problem of proliferating representations is to disregard mixed tensor representations of SU(5) in the set of monomials, i.e. to discard all the multiplets that are not 1's, 5's and 10's. This is the essential content of the models mentioned in the Introduction where two possibilities of three generations with unwelcome additional 5's, or of two generations plus two additional 10's, could be produced from considering bosonic or fermionic superfields (odd or even numbers of Grassmann powers) respectively. Those representations can be easily seen running diagonally down the table, the additional powers of θ and $\bar{\theta}$ for each diagonal step down the table being contracted so as to leave the SU(5) nature unchanged.

One example of the possibilities of Grassmann duality will be to suggest that we can use it to combine these two unsatisfactory models into one—not with seven generations of 5's and $\overline{10}$'s as might be feared, but with precisely three.

For our first set of models we shall demand that all 5's and $\overline{10}$'s have a fixed chirality (right-handed with our assignments of the θ). Further, we shall insist that the $\overline{5}$'s and 10's have the opposite chirality in order to identify them as the conjugates of the previous multiplets; in other words we shall require that the superfield Ψ has some kind of hermiticity property. This essentially produces a halving of the states, with an alternation of chiralities throughout the upper right triangle (Jarvis and White 1990), and the lower left triangle which represents its conjugate. This consideration is implicit in the previous models since we do not wish at present to generate SU(5) mirror representation fermion models. Because the monomials residing on the main diagonal are self-conjugate, the question of their behaviour under hermiticity will be a point of some subtlety. One may regard these terms as representing fields which would have insufficient degrees of freedom to represent fermions. It would be possible to assume that the superfield is antisymmetric in a block diagonal sense such that components of type $(\bar{\theta})^r\psi(r,r)(\theta)^r$ vanish. Thus all $r=s$ components, like $\bar{\theta}^I\phi_I^J\theta_J$, along the main diagonal, i.e. those 1's, 24's and 75's, would be eliminated.

Clearly, even with some type of hermiticity condition there are too many multiplets for comfort and we need to find some method of reducing them. The previous models carried out this reduction by restricting to fermionic or bosonic superfields, but suffered from the fact that this produced sets of representations which were unacceptable from an anomaly standpoint. If we were to simply take all the 5's and $\overline{10}$'s in Table 1 we would have an excessive seven generation (but anomaly-free) model. Within the scheme of taking all the unmixed tensor products of θ's there would also be a single right-handed neutrino from the θ^5 monomial.

An appropriate cutting tool here, and in models based on the full set of monomials, is Grassmann duality, a concept which was introduced previously (Delbourgo and White 1990) in an effort to construct invariant Lagrangians as Berezinian integrals; here we want to make more extensive use of the idea.

The key point is that taking the dual of a superfield (denoted by a breve sign) does not disturb the SU(5) properties of the superfield components. It is implemented by the following rule: if a superfield is expanded in the form

$$\Phi(\theta) = \sum_{r,s=0}^{5} (\bar{\theta})^s (\theta)^r \phi(r,s),$$

the dual superfield employs the self-same component fields, but assigned to duals of the θ monomials:

$$\breve{\Phi} = \sum_{r,s=0}^{5} (\bar{\theta})^{5-r} (\theta)^{5-s} \phi(r,s),$$

where, to be concrete, for the present work we interpret the dual of the typical multicoordinate term $\bar{\theta}^I \theta_P \theta_Q \theta_R$ to be $\epsilon_{PQRST} \bar{\theta}^S \bar{\theta}^T \epsilon^{IJKLM} \theta_J \theta_K \theta_L \theta_M / 2!4!$, as an example of the general rule

$$(\bar{\theta})^s (\theta)^r \to (\bar{\theta})^{5-r} (\theta)^{5-s}.$$

In particular, we note that polynomials of degree 5 in the Grassmann coordinates $(\theta,\ \bar{\theta})$ are self-dual in the above sense up to possible phase factors. That is to say a superfield of the type $\Phi = (\bar{\theta})^5 N$ is invariant under the duality transformation, i.e. $\Phi = \breve{\Phi}$; whereas the dual of 1 is $(\bar{\theta})^5 (\theta)^5 = (\bar{\theta}\theta)^5/5!$, and so on. In particular, while phase factors associated with duality are likely to be harmless in terms of reducing numbers of degrees of freedom for most entries in the table, they are critical (when combined with the rule for identification with the dual component) for the fate of the self-dual monomials on the cross-diagonal. These phases also interact with the hermiticity conditions, especially on the leading diagonal. They can also be sufficently constraining to eliminate an entire class (e.g. even powers) of monomials.

In the context of Table 1, hermitian conjugation corresponds to a reflection about the main diagonal whereas duality, with its substitution rule $r,s \to (5-s),(5-r)$ corresponds to reflection about the cross diagonal. It can be shown that for a set of monomials of given order, $(\bar{\theta})^s(\theta)^r$, the sign factor associated with taking the dual using the rule above alternates ($\pm$) with the number of $(\bar{\theta}\theta)$ contractions in the monomial.

To obtain the halving of states in the upper right triangle we shall impose a duality condition on the superfield Ψ. We are still refining the exact form of the dual transformation, in search of a compelling group-theoretic or super-geometrical formulation that will prescribe phases. For the present work we will use the dual transformation rule above with its ordering choices etc., and characterise some possible detailed phase assignments by including them in the imposition of the duality condition. When these phases are constructed using operators, such as those counting numbers of θ's etc., in monomials, they can lead to complicated restrictions on degrees of freedom.

This brings us to our first set of models. Taking only the unmixed tensor representations of SU(5) we reconsider the set of fermions contained in the bosonic superfield, i.e. odd monomials mentioned previously (Delbourgo and White 1990), but we impose the duality condition

$$\Psi = (-1)\breve{\Psi}.$$

With the dual rule above this produces a theory with two generations of 5's and $\overline{10}$'s, since it identifies the representations about the cross diagonal and eliminates the 5 and the singlet, but not the $\overline{10}$, on the cross diagonal. Of course by invoking self-duality instead of anti-self-duality we could have produced a model with three 5's and one $\overline{10}$ and one right-handed weak singlet neutrino.

As a third option we could excise all the representations on the cross-diagonal leaving two 5's and one $\overline{10}$. This would be a matter of selecting a slightly different dual identification, such as

$$\Psi = -(-1)^{\nu}\check{\Psi},$$

where the operator ν counts the number of $\bar{\theta}\theta$ contractions in each monomial of the Ψ superfield expansion.

The point of this last case is that when taken together with a fermionic superfield (again over just the completely antisymmetric representations) with some type of dual folding about the cross-diagonal, which will therefore contain one 5 and two $\overline{10}$'s, we have three generations of fermion representations appropriate to the standard model. The exact nature of the dual relation used for the fermionic superfield should be less important as there are no self-dual monomials involved. The option of keeping all three cross-diagonal representations (5, $\overline{10}$, 1) would lead to a fourth generation distinguished by having a right-handed neutrino. It should be noted that the duality relation used above for the bosonic superfield is unsuitable for the fermionic one as the phases introduced by the operator $(-1)^{\nu}$ lead to the field components all being set to zero. Note that using the explicit basic dual rule above on all the monomials in Table 1 directly leads via (anti)-self-duality to various combinations of 5's and $\overline{10}$'s, but none of them is suitable from the standpoint of anomaly cancellation.

It might be thought unaesthetic to use both bosonic and fermionic superfields from the point of view of statistics, or it might be regarded as artificial to restrict consideration to the unmixed tensors of SU(5). Accordingly we now turn to an examination of the possibilities that open up if we extend our analysis to a more democratic acceptance of all the monomials of Table 1, rather than just the maximally contracted set. This gives us more options and as we are still developing our detailed idea of duality we choose, for the present, to be guided by searches for acceptable sets of representations.

To see what may be appropriate, we examine the requirement of anomaly cancellation. In Table 2 we list the contribution to the anomaly (King 1981) for each of the representations in question.

Table 2. Anomaly coefficients for SU(5) representations of fixed chirality

SU(5) representation	1, 24, 75	5, 10	15	40	45	50
Anomaly coefficient	0	1	9	16	6	15

Let us count the left-handed multiplets in the upper triangle remembering that $\bar{N}_R$ is the conjugate of N_L and possesses precisely the same anomaly coefficient. There are 6 singlets, 4×24's and 2×75's along the main diagonal, which do not contribute. This leaves $7\times\bar{5}$'s, 7×10's, 3×45's, 1×15's, 1×50's, $2\times\overline{40}$'s and one singlet off the diagonal ($s > r$). We consider first the possibility of taking both odd and even monomials as we did above. The anomaly coefficients along the middle of the second row cancel out as they consist of a standard SU(5) family plus the exotic combination (King 1981), $\overline{10}_R+\overline{15}_R+40_R+\overline{45}_R$. Hence if we are going to invoke some sort of duality condition to halve the upper triangle in this extended model, we must seek a mechanism whereby, along the cross diagonal, the $(\bar{\theta})^3(\theta)^2$ and the singlet $(\bar{\theta})^5$ are eliminated, without also destroying the $(\bar{\theta})^4(\theta)$ term which provides the 10 and 15; otherwise the anomaly cancellation will be endangered.

Once again the subtle aspects of duality are only really needed for the bosonic part of the superfield (odd monomials). Therefore in order to get an anomaly-free theory we take, for the odd monomials,

$$\Psi = (-1)^{\nu+(|\bar{n}-n|+1)/2}\check{\Psi}$$

as the duality condition on Ψ, where the operators n, $\bar{n}$ and ν count the numbers of θ's, $\bar{\theta}$'s and $\bar{\theta}\theta$ contractions in the terms of the superfield expansion. This has the desired effect on the cross-diagonal (by construction). This dual relation cannot be extended to the even monomials because in combining it with either hermiticity or antihermiticity relations

$$\Psi = \pm\bar{\Psi}$$

the conflict of the two conditions eliminates them all. The situation for the even monomials is as before—they are simply identified across the cross-diagonal.

Table 3. Set of SU(5) multiplets contained in the coordinate expansion of bosonic and fermionic superfields after imposition of the duality condition

Asterisks signify Taylor components related to those explicitly listed by conjugation and/or duality while zeros denote vanishing components. The R and L subscripts signify right- and left-handed chiralities

$r\backslash s$	0	1	2	3	4	5
0	0	5_R	10_L	$\overline{10}_R$	$\bar{5}_L$	0
1	*	0	$5_R+\overline{45}_R$	$10_L+\overline{40}_L$	$\overline{10}_R+\overline{15}_R$	*
2	*	*	0	0	*	*
3	*	*	0	0	*	*
4	*	0	*	*	0	*
5	0	*	*	*	*	0

The only families that survive both these constraints are 3 standard generations of 5's and $\overline{10}$'s, and one exotic family consisting of the SU(5) right-handed multiplets $\overline{10}+\overline{15}+40+\overline{45}$, as shown in Table 3.

The inclusion of this last set appears a small price to pay for an anomaly-free grand supermultiplet, considering how many states we started from! It could of course also be regarded as an extravagant way to use up a surplus $\overline{10}$! For all we know, SU(5) theory (see O'Raifeartaigh 1986 and Ross 1984 for reviews) may need these fields when the accelerators push on to higher energies and new physics opens up. They should do no harm provided one can arrange that they become sufficiently heavy. It should be pointed out that the exotic family above itself (just) leads to a failure of SU(5) asymptotic freedom so that some of its members at least will need to be given masses at the GUT scale, or alternatively the GUT symmetry group to be gauged needs to be larger than SU(5). In this regard it is perhaps worth recalling (see King 1981 for details) that this combination is part of a 144 dimensional vector-spinor representation of SO(10), so that if in some modification of the present scenario it were possible to retain three singlet fermions (e.g. from the leading diagonal) to complete the standard generations as 16's of SO(10), as might be possible with a more subtle dual rule, then there would be a possibility of embedding the present collection of fermions into an asymptotically free set of SO(10) representations as discussed by King (1981).

A third class of models emerges if we persevere in attempting to obtain the fermions only from a bosonic superfield by extending the approach of Delbourgo and White (1990) to the larger set of representations in Table 1. In this case we are only considering the odd monomials and there is another possible assignment of chiralities which involves opposite chirality for alternate rows.

Table 4. Set of SU(5) multiplets contained in the coordinate expansion of fermionic superfields after imposition of the simple duality condition for the alternative chirality assignment

Asterisks signify Taylor components related to those explicitly listed by conjugation and/or duality while zeros denote vanishing components. The R and L subscripts signify right- and left- handed chiralities

$r\backslash s$	0	1	2	3	4	5
0	0	5_R	0	$\overline{10}_R$	0	1_R
1	*	0	$5_L+\overline{45}_L$	0	$\overline{15}_L$	0
2	0	*	0	$5_R+\overline{50}_R$	0	*
3	*	0	*	0	*	0
4	0	*	0	*	0	*
5	*	0	*	0	*	0

Taking this set of representations and using the simple dual rule of this paper we find that the condition

$$\Psi = \check{\Psi}$$

leads to the elimination of the cross-diagonal representations $\overline{45}$ and $\overline{10}$, leaving one standard $5_R+\overline{10}_R$ generation, one singlet, one 5_L, one 5_R, and the anomaly-free combination $\overline{15}_L+\overline{45}_L+\overline{50}_R$ as shown in Table 4. The latter set of representations contains quarks and leptons with conventional quantum numbers as well as exotic particles. Again this set of representations is not asymptotically free in SU(5) and so we will need to see that some of the fermions (preferably the colour octets and sextets) become massive at the unification scale.

While the SU(2) representations of the normal fields are not all simple replicas of the conventional standard model generations, it is interesting that the lepton sector of this model system contains exactly three sets of left- and right-handed neutrinos and charged partners, with the only surplus being a doubly charged lepton and its antiparticle. In the colour sector there are three charge 2/3 and four charge −1/3 triplets, together with one set each of charge 4/3 and 5/3 triplets. In addition there are also sextets and octets.

Amusingly, this model has just enough leptons for three generations of left–right symmetric type and just enough charge 2/3 quarks for three generations, while there is a surplus of 'down' quarks. As indicated above, the SU(2) assignments of the fermions are not all of the standard type and it may be necessary to invoke some mixing before the physical particles emerge. It should be remarked that the 5_L and 5_R are associated with different θ monomials so that they cannot have bare Lagrangian mass terms. One can regard the order of the monomial (or the number ν of $\bar{\theta}\theta$ contractions) as a distinguishing quantum number.

If the third generation of leptons had not been experimentally observed it would have been possible to entertain the idea of a variant of the bosonic superfield model which retained only the self-dual monomials of the cross-diagonal. This would have involved a further change in the chirality assignment compared to that in Table 4, requiring that while the basic alternation should be retained, it would be supplemented by an alternation for successive contractions within each (s, r) set. The resulting model would have particle content similiar to the preceding system except for the absence of the 5_L and 5_R, and the fact that the 5_R, $\overline{45}_L$ and $\overline{10}_R$ would be associated with different θ monomials to the case above. In such a model two essentially conventional left–right symmetric generations could have been found (with the additional doubly charged lepton) together with a third generation of unaccompanied quarks, while the imbalance in numbers of 'up' and 'down' type quarks would also have been avoided. Exotic coloured particles and higher charge quarks and leptons would of course remain.

To summarise this section—we find that within the Grassmann coordinate framework for describing how fermion representations and generations emerge, there are only a small number of models that are acceptable from an anomaly point of view, and that could confront experiment with any chance of success, once duality concepts are incorporated. The simplest choice would appear to be exactly three standard model generations. Except for the possible addition of a fourth generation containing a right-handed neutrino, the other alternatives involve a very non-standard presentation of the candidates for known particles, as well as an assortment of exotic ones.

The issues of mass generation, the representations of Higgs fields and their Lagrangians, mixing of the fermions in the Grassmann model, as well as the question of the exact form of the gauge group for the symmetries of each model (which may well depend on the nature of the decimation of the set of monomials in Table 1) will clearly be of great importance in determining whether any of the present models can be promoted to a full unified scheme, rather than just a motivating tool for selecting representations. The natural SU(5) symmetry that is introduced by the use of the five complex Grassmann variables may be reduced by the identifications made by duality conditions, although it might be part of a larger symmetry group, as we shall discuss in a later section. Another feature of this scheme is that the requirement to produce action terms from Berezin integrations over products of superfields provides a major restriction on the nature of possible interactions.

3. Superfields for Higgs Scalars

The question of how to incorporate the Higgs scalars is far from obvious, whichever approach is adopted for the fermions. It is not clear for example if both Bose and Fermi superfields should be utilised. In any case the fields must be a subset of the monomials in Table 1, but now with opposite statistics fields attached. It is still true that we are faced once again with too many SU(5) multiplets and that we need to pare them down by some means or another, even if we tried to limit ourselves to just the unmixed tensor products of θ's. However, it should be noted that to break SU(5) gauge symmetry, we are obliged to include a Higgs 24, so that terms from the main diagonal of Table 1 are also needed, suggesting that we should consider a hermitian Higgs bosonic superfield Φ at the very least; beyond that it may be self-dual or anti-self-dual—we have no means of telling, since there is as yet no direct evidence for any Higgs bosons! The question of also including a fermionic Higgs superfield (odd monomials) could be regarded as depending on whether the fermions were obtained from both odd and even power monomials or not.

The restriction to a hermitian bosonic superfield leads, under simple duality or anti-duality conditions, to three SU(5) singlets, two 24's, two 10's, one $\overline{40}$, one $\overline{5}$ and one 75. The 24's are welcome while the presence of only a single $\overline{5}$ seems a strong constraint, but most of the remainder appear to be of little immediate utility. If we also include a fermionic superfield once again the details of the duality conditions become important and in the absence of anomaly cancellation constraints the options are quite open.

The most reductive step in a duality condition approach would be to eliminate the entire cross-diagonal; this would provide then two more 5's, one $\overline{10}$, and one $\overline{45}$. As we shall see below the self-dual monomials are in any case inadequate as standard Higgs fields. The $\overline{45}$ should be a bonus for splitting the leptons from the down quarks if the interaction terms resemble usual GUT models, and the extra 5's may be welcome (or not), but from both superfields the rest are a distinct embarrassment of riches. At present we can see no natural way to exorcise them.

One appealing idea, from the point of view of economy (although somewhat outside our main use of duality), would be to require that the fermionic Higgs superfield consists only of the self-dual monomials of θ, an impressive collection of $1+10+15+5+45+50$, but unfortunately this leads to a dead end in terms of conventional Higgs fields, since only the quadratic power of such a superfield (five powers of Grassmann coordinates) can survive Berezin integration—with no possibility of a quartic self-coupling and no classical vacuum expectation values for the scalar fields! However we should point out that such a superfield can couple in Yukawa fashion to the fermions. If some fermions condense out, or if masses are to be produced radiatively for some generations, then this formulation might possess some virtues. It might also have some uses in conjunction with the SU(5) scalar associated with the zeroth power monomial of the bosonic superfield.

A duality choice for the fermionic superfield which might be appropriate to the first of our exotic representation models is to adopt a duality condition for the hermitian Higgs fields,

$$\Phi = (-1)^{\nu+(|\bar{n}-n|-1)/2}\check{\Phi}$$

complementary to that used for the fermions. Compared with the model outlined above where the entire cross-diagonal was eliminated we would now have an extra 5, $\overline{45}$ and $\overline{10}$.

Turning to the simple models where we discard all the representations beyond 1, 5 and 10 [which means abandoning the standard way of breaking SU(5) via 24's] we have the following possible Higgs superfield terms.

A self-dual bosonic superfield choice

$$\Phi = \breve{\Phi}$$

yields three SU(5) scalars,

$$[1 + (\bar{\theta}\theta)^5/5!]\aleph(1), \qquad [(\bar{\theta}\theta) + (\bar{\theta}\theta)^4/4!]\aleph(2), \qquad [(\bar{\theta}\theta)^2/2 + (\bar{\theta}\theta)^3/3!]\aleph(3),$$

one $\overline{5}$,

$$[1 + (\bar{\theta}\theta)]\theta^4\bar{\Delta}(1),$$

and two 10's,

$$[1 + (\bar{\theta}\theta)^3/3!]\theta^2\overline{Y}(1), \qquad [(\bar{\theta}\theta) + (\bar{\theta}\theta)^2/2]\theta^2\overline{Y}(2).$$

If we also include a fermionic Higgs superfield then, taking once more the opposite duality relation to the fermion model, i.e.

$$\Xi = (-1)^\nu \breve{\Xi},$$

we obtain three additional 5's, two $\overline{10}$ and a singlet. These consist of the superfield terms

$$[1 + (\bar{\theta}\theta)^4/4!]\bar{\theta}\Delta(2), \qquad [(\bar{\theta}\theta) + (\bar{\theta}\theta)^3/3!]\bar{\theta}\Delta(3), \qquad [1 + (\bar{\theta}\theta)^2/2!]\bar{\theta}^3 Y(3)$$

and the surviving self-dual monomials which we group as

$$\bar{\theta}^5\aleph(4), \qquad (\bar{\theta}\theta)\bar{\theta}^3\overline{Y}(4), \qquad (\bar{\theta}\theta)^2\bar{\theta}\Delta(4),$$

and their conjugates. The free Lagrangian for these fields, taken together, arises painlessly through the superintegral,

$$\int d^5\bar{\theta}\, d^5\theta\, \partial\Phi^\dagger . \partial\Phi = \sum_{r=1}^{4} [\partial\bar{\aleph}(r) . \partial\aleph(r) + \partial\bar{\Delta}(r) . \partial\Delta(r) + \partial\bar{Y}(r) . \partial Y(r)].$$

It is now quite feasible to construct Φ^4 self-interactions. We come across a fair number of terms although there are restrictions arising from the requirements of Berezin integration. Up to hermitian conjugation, we have listed below all the Φ^4 terms which involve the 5's, Δ's and singlets, $\aleph$'s (neglecting the 10's, Y because they are all charged and thus have zero vacuum expectation values):

$$\aleph^4(1),\quad \aleph^2(1)\aleph^2(2),\quad \aleph^2(1)\aleph^2(3),\quad \aleph^2(1)\aleph^2(4),$$
$$\aleph(1)\aleph(2)\aleph^2(3),\quad \aleph(1)\aleph^2(2)\aleph(3),\quad \aleph^3(2)\aleph(3),$$
$$\bar{\Delta}(2)\Delta(2)\aleph^2(1),\quad \bar{\Delta}(2)\Delta(2)\aleph(1)\aleph(2),\quad \bar{\Delta}(2)\Delta(2)\aleph(2)\aleph(3),\quad \bar{\Delta}(2)\Delta(2)\aleph^2(3),$$
$$\bar{\Delta}(2)\Delta(3)\aleph(1)\aleph(2),\quad \bar{\Delta}(2)\Delta(3)\aleph(1)\aleph(3),\quad \bar{\Delta}(2)\Delta(3)\aleph(2)\aleph(3),$$
$$\bar{\Delta}(3)\Delta(3)\aleph^2(1),\quad \bar{\Delta}(3)\Delta(3)\aleph^2(2),\quad \bar{\Delta}(3)\Delta(3)\aleph(1)\aleph(3),$$
$$\bar{\Delta}(1)\Delta(1)\aleph^2(1),\quad \bar{\Delta}(4)\Delta(4)\aleph^2(1),\quad \bar{\Delta}(1)\Delta(1)\aleph(1)\aleph(2),$$
$$\bar{\Delta}(2)\Delta(2)\bar{\Delta}(2)\Delta(1),\quad \bar{\Delta}(3)\Delta(3)\bar{\Delta}(3)\Delta(2),\quad \bar{\Delta}(1)\Delta(2)\bar{\Delta}(2)\Delta(1).$$

The constraints on couplings of these sets of Higgs fields due to the θ structure of superfields shows that they are far from being trivial clones. It should also be pointed out that some of the terms above appear dangerous, as they involve cubic factors of some fields which are untamed by corresponding quartics. One should note that combining the various Higgs fields into superfields for each SU(5) representation, *or* packing the representations into one or two superfields for the fermions (e.g. sorted by odd or even powers of θ's) and similiarly for all the Higgs fields, would enforce strict relationships between the coefficients of the above self-interactions and also between Yukawa terms.

4. Gauge Symmetries

We now turn to the gauge fields associated with these Grassmann models. The question of what to gauge involves consideration of the symmetries of the matter field sector. By virtue of the formulation with five complex anticommuting coordinates there is an obvious action of SU(5) on the monomials that comprise the superfields, with generators

$$F_I^J \equiv \bar{\theta}^J \partial/\partial\bar{\theta}^I - \theta_I \partial/\partial\theta_J .$$

We have accordingly decomposed the monomials and fields into SU(5) representations throughout our paper. We have also used the SU(5) anomaly conditions, but that can be taken simply as a compact check that an anomaly-free SU(3)×SU(2)×U(1) theory emerges at low energy. Standard SU(5) GUT theories are increasingly under threat in the light of results from proton lifetime experiments, and perhaps one could gauge just the standard model symmetries, regarding the number of θ's as a key to the number of generations, rather than as an invocation of SU(5). The presence of the exotic particle representations in some of our schemes suggests that a larger gauge group than SU(5) might be desirable, or that some of the fields should become massive at the unification scale, or both.

The full set of monomials represented by Table 1 does have further symmetries. There is a standard Clifford type representation of SO(10) that involves the extra generators

$$F_{KL} \equiv \bar{\theta}^K \bar{\theta}^L + \partial^2/\partial\theta_K\partial\theta_L$$

and their Hermitian conjugates together with the trace part

$$\bar{\theta}^J \partial/\partial\bar{\theta}^J - \theta_I \partial/\partial\theta_I .$$

In addition, the full array of superfield monomials, being constructed from products of Grassmann variables, forms a Grassmann algebra and this has a set of continuous automorphisms, generated by even derivations of the form

$$F_J^p \equiv (\bar{\theta})^{p-i}(\theta)^i \partial/\partial\bar{\theta}^J, \quad p = 1,3,5$$

and their conjugates. These are candidate symmetries as they map the degrees of freedom into themselves. There are further transformations on the monomials which we may schematically represent as

$$F_q^p \equiv (\bar{\theta})^{p-i}(\theta)^i \; \partial^q/(\partial\bar{\theta})^{q-j}(\partial\theta)^j .$$

We are presently carrying out a detailed study of the algebra of all these generators. Recently a related treatment of higher derivative operators which uses ordering in polynomials of powers and differentiations in such a way as to ensure that each operator generates interchanges of only one pair of monomials has been presented by Eyal (1990). The question of how many of these transformations should be regarded as symmetries awaits resolution.

It remains at this stage to see whether any of these extra symmetries can be maintained in the face of the decimation of the set of Grassmann monomials. SU(5) naturally acts *within* each monomial and some other symmetries, e.g. some transformations between the eventually sorted out generations, may survive. With regard to SU(5) gauge theories it should be remembered that the Grassmann models have greater constraints on possible Lagrangian terms than standard schemes. Such terms must not only be SU(5) scalars but must also contain the correct number of θ's to survive the Berezin integration. Complete phenomenological models using the fermion and Higgs representations of the current paper have yet to be constructed. Once that is achieved the role of these restrictions, acting like extra quantum numbers, needs further examination before SU(5) Grassmann unified models can be declared untenable. One feature of standard GUT theories which may have links with the Grassmann schemes is the introduction of discrete symmetries into the theory; this may be connected with the identifications made between monomials by the duality conditions. Such issues will be relevant to detailed discussions of asymptotic freedom, radiative effects, etc.

5. For the Future

In order to determine the mass matrices for the sources, one first needs to construct a semiclassical renormalisable potential for the various Higgs generations and the vacuum expectation values for each one of the uncharged fields. A superfield version of this is indicated but it may be necessary to distinguish between superfields with opposite statistics, before coupling them as $\lambda\Phi^4$. Next we must study all the Yukawa interactions between Higgs and fermions (exotica as well), and derive the mass terms and mixings, including the many terms encountered previously for 1's, 5's and 10's, again from an appropriate superfield interaction.

The full nature of the symmetry algebra and possible corresponding gauge fields is being mapped out and the concept of generations has also to be elucidated. Clearly there is a great deal of work confronting us before we can come to any definite conclusions. Still, we feel that the method holds considerable promise, because of the elegant way in which the generations emerge and the drastic diminution of states enforced by SU(5) Grassmann duality.

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References

Casalbuoni, R., and Gatto, R. (1979, 1980). *Phys. Lett.* B **88**, 306; B **90**, 81.
Delbourgo, R. (1989). *Mod. Phys. Lett.* A **4**, 1381.
Delbourgo, R., Twisk, S. E., and Zhang, R. B. (1988). *Mod. Phys. Lett.* A **3**, 1073.
Delbourgo, R., and White, M. (1990). *Mod. Phys. Lett.* A **5**, 355.
Denegri, D., Sadoulet, B., and Spiro, M. (1990). *Rev. Mod. Phys.* **62**, 1.
Dondi, P. H., and Jarvis, P. D. (1980). *Z. Phys.* C **4**, 201.
Eyal, O. (1990). Techniques of using Grassmann variables for realizing some algebras, Karlsruhe (preprint).
Jarvis, P. D., and White, M. (1990). Fermion masses from supersymmetric dynamics in proper time, University of Tasmania preprint, submitted to the XXV International Conference on HEP.
King, R. C. (1981). *Nucl. Phys.* B **185**, 133.
Krolikowski, W. (1989). *Acta Phys. Polon.* B **19**, 599.
O'Raifeartaigh, L. (1986) 'Group Structure of Gauge Theories' (Cambridge Univ. Press).
Ross, G. G. (1984). 'Grand Unified Theories' (Frontiers in Physics Series, Benjamin/Cummings Publishing: Menlo Park).

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