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Electron Outflow in Axisymmetric Pulsar Magnetospheres. II*

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Abstract

In the axisymmetric pulsar magnetosphere model of Mestel *et al.* (1985), electrons, following injection with non-negligible speeds from the stellar surface, flow with moderate acceleration, and with poloidal motion that is closely tied to poloidal magnetic field lines, before reaching a limiting surface, near which rapid acceleration occurs. The present paper continues an analysis of flows which either encounter the limiting surface beyond the light cylinder (between the cones of zero axial magnetic field), or do not meet it at all. The formalism introduced by Mestel *et al.* for the description of the outflow is applied in an extended version which fully incorporates γ_0, the emission Lorentz factor of the particles. This treatment removes the singularity of γ_0 at the stellar poles that occurred in the earlier work: because of a nonuniformity in taking the limit of nonrelativistic injection, full incorporation of γ_0 acts to keep it finite.

1. Introduction

Mestel, Robertson, Wang and Westfold (1985; referred to here as MRW^2) introduced an axisymmetric pulsar magnetosphere model in which electrons leave the star with non-negligible speeds and flow with moderate acceleration, and with poloidal motion that is closely tied to poloidal magnetic field lines, before reaching S_l, a limiting surface near which rapid acceleration occurs. The formalism they introduced to describe these flows can be interpreted in terms of a plasma drift across the magnetic field, following injection along it (Burman 1985*a*). An analysis of such moderately-accelerated outflows (Burman 1984) showed that there is a second class of flows—ones which do not encounter a region of rapid acceleration.

I have extended the basic MRW^2 formalism, and my earlier analysis, so as to incorporate fully γ_0, the emission Lorentz factor of the particles (Burman 1986). The possible need for this extension was suggested by a study (Burman 1985*b*; referred to as Part I) of the solutions of the original formalism which represent flows that encounter S_l either beyond the light cylinder or not at all: the outflow from tiny inner cores of the polar caps is either not of this kind or, if it is, then some extended formalism is needed in order to treat it. It is found here that full incorporation of γ_0 overcomes the difficulty: it removes the singularity of γ_0 that occurred at the stellar poles in the earlier work.

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2. MRW² Formalism

The notation and terminology of Part I will be used, largely without further comment; but, for convenience, a few key points are repeated in the next paragraph. In the remainder of this section, the relevant aspects of the extended MRW² formalism will be summarised.

Charge separation is assumed, so that $j_{\mathrm{p}} = \rho_{\mathrm{e}} V_{\mathrm{p}}$. The poloidal part of Ampère's law reduces to $B_{\phi} = -S/X$. It follows from Gauss's law and the toroidal part of Ampère's law that (Mestel *et al.* 1979, eq. 2.8)

$$\nabla^2 \Phi + 2B_z = -(1-X V_{\phi})\rho_{\mathrm{e}}. \tag{1}$$

The steady rotation constraint implies the existence of an integral of the motion (Endean 1972) which (for electrons) has the dimensionless form (MRW²)

$$G = \gamma(1-X V_{\phi}) - \Phi/\epsilon; \tag{2}$$

the parameter ϵ lies roughly in the range 10^{-6} to 10^{-11} for different pulsars. With the outflow emanating from a small polar cap, $V_0(P)$ may be identified as the speed at which the electrons, travelling along the lines of constant P, leave the star. The Endean integral G is approximated very closely by $\gamma_0(P)$, the Lorentz factor corresponding to $V_0(P)$. When the emission speed is nonrelativistic, G has a constant value, namely one, across the flow. Goldreich–Julian (GJ) flow is defined as flow satisfying the fundamental equations of the MRW² formalism, subject to the additional restriction that the term $\nabla^2 \Phi$ in the Gauss–toroidal Ampère law (1) be negligible.

It follows from axisymmetry and neglect of inertial drift that the flow velocity is related to the magnetic field by

$$\boldsymbol{V} = \kappa \boldsymbol{B} + K(P) X \boldsymbol{t}, \tag{3}$$

where κ is a scalar and K denotes $1+\epsilon G'(P)$, with G' representing $\mathrm{d}G/\mathrm{d}P$; this is equation (2.27) of Mestel *et al.* (1979), with inertial drift neglected and with dimensionless variables. MRW² wrote $\mathrm{d}S/\mathrm{d}P$ as $-2V_0(P)$; hence, it follows from $\boldsymbol{V}_{\mathrm{p}} = \kappa \boldsymbol{B}_{\mathrm{p}}$ and $\boldsymbol{j}_{\mathrm{p}} = \rho_{\mathrm{e}} \boldsymbol{V}_{\mathrm{p}}$, together with the expressions for $\boldsymbol{B}_{\mathrm{p}}$ and $\boldsymbol{j}_{\mathrm{p}}$ in terms of their Stokes stream functions P and S, that $\rho_{\mathrm{e}}\kappa = -2V_0$.

The equations $\rho_{\mathrm{e}}\kappa = -2V_0$, $V_{\phi} - KX = -\kappa S/X$ and $\rho_{\mathrm{e}} = -2B_z/(1-XV_{\phi})$ for GJ flow yield (Burman 1986), on eliminating ρ_{e}, V_{ϕ} and κ in pairs,

$$\kappa S = (1-KX^2)/(\tilde{B}_z - 1), \qquad \rho_{\mathrm{e}} = -2V_0 S(\tilde{B}_z-1)/(1-KX^2), \tag{4a, b}$$

$$V_{\phi} = KX - (1-KX^2)/X(\tilde{B}_z-1), \tag{4c}$$

where $\tilde{B}_z$ denotes $B_z/V_0 S$.

When $G' \neq 0$, the light cylinder plays no special part in the mathematical description of GJ flows: rather, it is the surfaces $W: |K|X = 1$ and, for $K > 0$, $V: K^{1/2}X = 1$ that do so, particularly the latter.

The GJ flows can be divided into Class I, for which the Lorentz factor becomes infinite at some point on a flow line, and Class II, which reach the equatorial plane without encountering such a singularity. The following subdivision of Class I, according to where the GJ flow terminates in the singularity of its Lorentz factor, is convenient in distinguishing flows having different mathematical descriptions: IA and IC flows are those for which the infinity occurs at some point where $\tilde{B}_z > 1$ and $\tilde{B}_z < 1$ respectively; IB and IB′ flows are those with the infinity on the surfaces V and W respectively. Class IC flows terminate outside V.

It is only flows with $\tilde{B}_z = 1$ on V that have the possibility of reaching values of $\tilde{B}_z$ below one (Classes IC and II flows), and therefore the possibility of continuing to the $B_z = 0$ cones and beyond to the equatorial plane.

3. Preliminary Results

As in Part I, the auxiliary variables $\bar{P}$, U and Q, denoting $-P$, $X^{2/3}$ and $\bar{P}^{2/3}$, are used where convenient. It is also convenient to put $T(P) \equiv V_0\, S/2\bar{P}$. It follows from $V_0 \equiv \frac{1}{2}\mathrm{d}S/\mathrm{d}\bar{P}$ and $S(0) = 0$ that $S(P)/2\bar{P} = \langle V_0 \rangle(P)$, the average value of V_0 between 0 and P (MRW^2, §3). Hence $T = V_0\langle V_0 \rangle$, so $0 \leqslant T < 1$. Also, $S/2\bar{P} \to V_0(0)$ as $\bar{P} \to 0$. If V_0 decreases with increasing $\bar{P}$, then so does $\langle V_0 \rangle$ and hence $S/2\bar{P}$ and T; in this case, S increases less rapidly than linearly with $\bar{P}$. If V_0 increases with increasing $\bar{P}$, then so does $\langle V_0 \rangle$ and hence $S/2\bar{P}$ and T; in this case, S increases more rapidly than linearly with $\bar{P}$.

Since G is closely equal to γ_0, it follows that

$$G'(P) = -\gamma_0^3\, V_0\, \mathrm{d}V_0/\mathrm{d}\bar{P} = -V_0\, \mathrm{d}(\gamma_0\, V_0)/\mathrm{d}\bar{P}. \qquad (5\mathrm{a,b})$$

The edge $Q = Q_e$ of zone of GJ outflow, defining the edge of a polar cap, occurs where $V_0 = 0$ with $P \neq 0$. The function $T(P)$ varies from $V_0^2(0)$ on $Q = 0$ to zero on $Q = Q_e$. On $Q = Q_e$, both G' and T vanish. Since $K = 1$ on $Q = Q_e$, the surfaces W and V both cross the light cylinder where the poloidal field line surface $Q = Q_e$ does.

In a first approximation, the poloidal magnetic field in the domain of moderately accelerated outflow can be taken to be dipolar, so $\bar{P} = X^2/R^3$. As in Part I, it is X and P, rather than X and Z, that are regarded as the independent variables. The dipole magnetic field is described by (MRW^2)

$$B_p = (2\bar{P}/X^2)(1-\tfrac{3}{4}QU)^{\frac{1}{2}}, \qquad B_z = (2\bar{P}/X^2)(1-\tfrac{3}{2}QU). \qquad (6\mathrm{a,b})$$

Its field lines, $P =$ constant, have the equations $\bar{P}R = \sin^2\theta = QU$. Their dimensionless radius of curvature is given by

$$\tfrac{3}{4}Q\rho = U^{\frac{1}{2}}(1-\tfrac{3}{4}QU)^{\frac{3}{2}}/(1-\tfrac{1}{2}QU). \qquad (6\mathrm{c})$$

Use of $B_\phi = -S/X$ and (6a) for B_p, with $S/2\bar{P} = \langle V_0 \rangle$, gives

$$-B_\phi/XB_p = \langle V_0 \rangle(P)/(1-\tfrac{3}{4}QU)^{\frac{1}{2}}. \qquad (7)$$

The definitions of γ_{m} and D_{∞}, and hence of F_{∞} and $f^{\pm}$ (Burman 1986, eqs 6, 10, 11, 14), can be expressed in terms of B_{ϕ}/B_{p}:

$$1/\gamma_{\mathrm{m}}^2 \equiv 1 - K^2X^2/(1 + B_{\phi}^2/B_{\mathrm{p}}^2), \tag{8a}$$

$$X^2 D_{\infty}^2 \equiv 1 + (1 - K^2X^2)B_{\mathrm{p}}^2/B_{\phi}^2. \tag{8b}$$

(a) V_{ϕ} at the Star

The quantity $K(P)$, or $1+\epsilon G'(P)$, can be written as $\alpha(S)/\Omega$, which is the ratio of an angular speed associated with a poloidal streamline to that of the star. Near a polar cap $\tilde{B}_z \gg 1$ and, in the dipole approximation, $B_z \approx 2\bar{P}/X^2$; hence, since $|K|X^2 \ll 1$, equations (4a) and (4c) for κS and V_{ϕ} show that $\kappa B_{\phi} \approx -XT$ and

$$V_{\phi}/X \approx 1+\epsilon G' - T = 1 - \epsilon\gamma_0^3 V_0 \,\mathrm{d}V_0/\mathrm{d}\bar{P} - T \tag{9a, b}$$

there; equation (5a) has been used for G'. Near a polar cap, the κB_{ϕ} contribution to V_{ϕ} in (3) has, like the KX part, the form of a function of P multiplied by X; the angular speed of the flow there is $\alpha(S) - T(P)\Omega$ or $\{K(P) - V_0\langle V_0\rangle\}\Omega$. Since G' and T vanish on $Q = Q_{\mathrm{e}}$, equation (9a) shows that $V_{\phi} = X$ at the edge of a polar cap.

The MRW[2] choice $\alpha(S) = \Omega$, that is $K = 1$ or $G'(P) = 0$, corresponds to use of the original MRW[2] formalism, with a non-varying Endean integral. As MRW[2] (in their §3) pointed out, the outflow yields a significant departure of the toroidal part of the motion from corotation with the star right down to the star's surface: with $G' = 0$, equation (9a) gives $V_{\phi}/X \approx 1 - T$ near a polar cap, corresponding to an angular flow speed of $(1 - V_0\langle V_0\rangle)\Omega$. Choosing $\alpha(S) = \Omega$ with the MRW[2] form $1 - \frac{3}{2}Q$ for $T(P)$, corresponding to requiring $\tilde{B}_z = 1$ on $X = 1$, gives $V_{\phi}/X = \frac{3}{2}Q < 1$ near a polar cap.

If $G'(P) < 0$, so V_0 increases away from the axis, then (9a) shows that $V_{\phi}/X < 1 - V_0\langle V_0\rangle < 1$ near a polar cap. If $G'(P) > 0$, so V_0 decreases away from the axis, then (9a) shows that $V_{\phi}/X > 1 - V_0\langle V_0\rangle$ near a polar cap. The flow is subrotating there so long as $K < 1 + V_0\langle V_0\rangle$, corresponding to $\epsilon G' < V_0\langle V_0\rangle$ or $-\epsilon\gamma_0^3 \,\mathrm{d}V_0/\mathrm{d}\bar{P} < \langle V_0\rangle$.

(b) $\tilde{B}_z$ and the Surface $\tilde{B}_z = 1$

I shall now examine the general form of the surface $\tilde{B}_z = 1$; this helps to pin down the behaviour of $\tilde{B}_z$. In the dipole approximation we have

$$\tilde{B}_z = (1 - \tfrac{3}{2}QU)/TX^2. \tag{10}$$

So, the surface $\tilde{B}_z = 1$ satisfies a cubic equation in U, namely $TU^3 = 1 - \frac{3}{2}QU$, which has positive discriminant and just one real solution $U = R(Q)$, where

$$2^{\frac{1}{3}}T^{\frac{1}{2}}R(Q) \equiv \{(T+\tfrac{1}{2}P^2)^{\frac{1}{2}} + T^{\frac{1}{2}}\}^{\frac{1}{3}} - \{(T+\tfrac{1}{2}P^2)^{\frac{1}{2}} - T^{\frac{1}{2}}\}^{\frac{1}{3}}. \tag{11}$$

Equation (10) for $\tilde{B}_z$ itself can be written as

$$U^3(\tilde{B}_z - 1) = (R - U)(TU^2 + TRU + 1/R). \tag{10$'$}$$

For small Q and not-so-small values of T—more precisely for $P^2/2T \ll 1$—the definition (11) of R shows that the surface $\tilde{B}_z = 1$ (i.e. $U = R$) is given by

$$T^{\frac{1}{3}}U = 1 - Q/2T^{\frac{1}{3}} + P^2/24T + \dots. \tag{12}$$

This demonstrates the asymptotic approach, as $Z^2 \to \infty$ (corresponding to intersections with poloidal field/flow lines having $Q \to 0$), of the surface $\tilde{B}_z = 1$ to the cylinder $X = 1/V_0(0)$.

The surface $\tilde{B}_z = 1$ crosses the light cylinder where it intersects the poloidal field line surface $Q = Q_c$ or $P = P_c$ defined by

$$\tfrac{3}{2}Q_c = 1 - T(P_c); \tag{13}$$

that is, $\tilde{B}_z = 1$ moves inside $X = 1$ provided $Q_c < Q_e$.

Since, for $x \gg 1$,

$$\{(1+x)^{\frac{1}{2}} \pm 1\}^{\frac{1}{3}} = x^{\frac{1}{6}}(1 \pm 1/3x^{\frac{1}{2}} + 1/18x \mp 4/81x^{\frac{3}{2}} \dots),$$

the definition (11) of R shows that for small T and not-so-small values of Q—more precisely for $P^2/2T \gg 1$—the surface $\tilde{B}_z = 1$ is given by

$$\tfrac{3}{2}QU = 1 - 8T/27P^2 + \dots. \tag{14}$$

This demonstrates the approach of the surface $\tilde{B}_z = 1$ to intersection with a $B_z = 0$ cone (i.e. $QU = \frac{2}{3}$) at the edge of a GJ outflow zone, where $V_0 = 0 = T$ with $P \neq 0$.

To summarise: the surface $\tilde{B}_z = 1$ is asymptotic to the cylinder $X = 1/V_0(0)$ for $Q \to 0$, crosses the light cylinder for $Q = Q_c$, provided $Q_c < Q_e$, and intersects a $B_z = 0$ cone at the edge $Q = Q_e$ of the outflow zone.

(c) Surfaces V and W

These surfaces arise naturally in the MRW^2 formalism, as extended to incorporate a varying Endean integral, in a way that the light cylinder, which arises in a similar manner in the original formalism, does not; that surface appears merely as the common limit of V and W as $G' \to 0$.

The surface V is defined, for $K > 0$ ($\epsilon G' > -1$), by $K^{\frac{1}{2}}X = 1$. It may be characterised as the surface on which the functions F and F_∞, defined by equations (9) and (11) of my paper on the extended MRW^2 formalism (Burman 1986), are equal to one, or as the surface on which, for $K \neq 1$ ($G' \neq 0$), the functions f^+ and f^-, defined by equation (14) of that paper, are both one; none of these functions can equal unity anywhere else. (When $G' = 0$, the functions F, F_∞ and f^+ equal one on $X = 1$, but $f^- = -1$ there.)

The surface W may be characterised as the surface on which one of the functions $f^\pm$ becomes infinite. For $K > 0$ ($\epsilon G' > -1$) but $G' \neq 0$, it is f^- that is singular there; for $K < 0$ ($\epsilon G' < -1$), it is f^+. For flows of practical interest, since ϵ is such a small parameter, $|\epsilon G'| < 1$ so K is positive: it is f^- that is singular, for $G' \neq 0$, on W.

For $0 < K < 1$ ($-1 < \epsilon G' < 0$), W lies beyond V, which is outside the light cylinder; the negative G' means that V_0 increases away from the pole. For $K > 1$

($\epsilon G' > 0$), corresponding to V_0 decreasing from the pole, W lies within V, which is within the light cylinder.

If $\tilde{B}_z \neq 1$ on V, then $1/\gamma^2 = -\epsilon G'$ there (Burman 1986, §3). For $0 < K < 1$ ($-1 < \epsilon G' < 0$), the GJ Lorentz factor is real and finite on V if $\tilde{B}_z > 1$ there, but it becomes infinite somewhere between V and W; if $\tilde{B}_z = 1$ on V with γ real and finite there, then $\tilde{B}_z$ is below f^+ infinitesimally beyond V; if it is still below f^+ on W, then γ is real and finite inside and on W. For $K > 1$ ($\epsilon G' > 0$), if $\tilde{B}_z$ is still above f^+ on W, then γ is real and finite inside and on W; $\tilde{B}_z = 1$ on V is a necessary condition for γ to be finite on V. So, for all $K > 0$, $\tilde{B}_z = 1$ on V is a necessary condition for the GJ outflow to penetrate both V and W.

It follows from the fact that $F = 1$ on V that one of the fundamental equations of the MRW^2 formalism (fundamental in the sense of not involving neglect of $\nabla^2 \Phi$), namely the MRW^2 form of the Gauss–toroidal Ampère law obtained from (1) above (MRW^2, eq. 3.18; Burman 1986, eq. 8), shows that $\tilde{B}_z = 1 - \nabla^2 \Phi / 2 V_0 S$ on V. Thus, if $\nabla^2 \Phi$ is still negligible on V—more precisely if $\nabla^2 \Phi \ll 2 V_0 S$ on V—then $\tilde{B}_z \approx 1$ on V.

4. Flows with $\tilde{B}_z = 1$ on V

(a) The Condition

As mentioned above, the condition $\tilde{B}_z = 1$ on V is a necessary one for the flow to cross both V and W with the GJ Lorentz factor still finite. In the dipole approximation $\tilde{B}_z = (K - J\frac{3}{2}Q)/T$ on V, with $J \equiv K^{2/3}$, so this condition becomes

$$K - J\tfrac{3}{2}Q = T. \tag{15}$$

This is the same as the cubic equation in U for the surface $\tilde{B}_z = 1$ (as discussed in Section 3*b*), with U replaced by $1/I$, where $I \equiv K^{1/3}$: it is the condition for the two surfaces $\tilde{B}_z = 1$ and V to coincide. The surface V now has the behaviour described for the surface $\tilde{B}_z = 1$ in Section 3*b*. For $G' = 0$, equation (15) gives the previously used form $1 - \frac{3}{2}Q$ for $T(P)$.

By continuity, γ^{-2} cannot jump from a positive value inside V to a negative value on V: given that γ^{-2} is positive inside V and is nonzero on V, it follows that $\gamma^{-2} > 0$ on V; the requirement that the GJ Lorentz factor be still finite on V ensures that it is also real there.

As $Q \to 0$, the J term in the condition (15) drops out, leaving $K(0) = T(0) = V_0^2(0)$, corresponding to $-\epsilon G'(0) = 1 - T(0) = 1 - V_0^2(0) \equiv 1/\gamma_0^2(0)$. [The form $1 - \frac{3}{2}Q$ for $T(P)$, describing IC/II flow when $G' = 0$, though giving $T(0) = 1$, leads (inconsistently) to $G'(0)$ infinite.] Since $G'(0) < 0$, equation (5a) for $G'(P)$ shows that $V_0(P)$ initially increases from a nonzero value on the axis.

Since $T = 0$ defines the edge, $Q = Q_e$, of a polar cap and since, from (5), G' must also vanish there, the condition (15) shows that $Q_e = \frac{2}{3}$, exactly as found when the variation of the Endean integral is neglected (MRW^2). Since, from its definition (13), $Q_c < \frac{2}{3}$, it follows that V crosses the light cylinder.

Putting $K = 0$ in the condition (15) gives $T = 0$; but $G' = 0$ and $K = 1$ when $T = 0$; hence, K cannot vanish anywhere for these flows. It follows, since $K(0) = V_0^2(0) > 0$, that $K(P) > 0$ and $-\epsilon G'(P) < 1$. Since $J > 0$ and $T \geqslant 0$, equation (15) shows that $\frac{3}{2}Q \leqslant I$, with equality holding on $Q = \frac{2}{3}$.

Since $K(0) = V_0^2(0)$, the surfaces V and W are asymptotic to the cylinders $X = 1/V_0(0)$ and $X = 1/V_0^2(0)$, respectively, as $Z^2 \rightarrow \infty$. As Q increases from zero, initially $G' < 0$, $0 < K < 1$ and V_0 increases: V and W move in toward $X = 1$, with W remaining outside V until V_0 reaches its maximum value at $P = P_c$, satisfying (13); there $G' = 0$ and V and W meet on the light cylinder. At this intersection, the condition (15) reduces to (13) defining Q_c, corresponding to the previously used form for T. For Q larger than this, $G' > 0$, $K > 1$ and V_0 decreases as Q increases; W and V lie inside the light cylinder, with W inside V, until V_0 reaches zero, where G' again vanishes and W and V again meet on $X = 1$, this time intersecting the surface $Q = \frac{2}{3}$ and the $B_z = 0$ cone.

The condition (15) to have $\tilde{B}_z = 1$ on V can be regarded as a cubic equation for $I(P)$. Except for $T = 0$, when there is also a double root of zero which, as seen above, can be excluded since $G' = 0$ and $K = 1$ on $Q = Q_e$, there is just one real root; thus,

$$I(P) = A_+ + A_- + \tfrac{1}{2}Q, \tag{16}$$

where

$$2A_\pm^3 \equiv T + \tfrac{1}{4}P^2 \pm \{(T + \tfrac{1}{4}P^2)^2 - (\tfrac{1}{4}P^2)^2\}^{\frac{1}{2}}. \tag{17}$$

The quantities A_+ and A_- vary from $V_0^{2/3}(0)$ and 0 on $Q = 0$ to $\frac{1}{3}$ for both on $Q = Q_e$; A_+ never vanishes; A_- does so on $Q = 0$ only. It is readily seen that $\frac{1}{2}Q \leqslant A_+ \leqslant (T + \frac{1}{4}P^2)^{1/3}$ and $0 \leqslant A_- \leqslant \frac{1}{2}Q$. Since $I \geqslant \frac{3}{2}Q$, equation (16) shows that $A_+ + A_- \geqslant Q$; the equality holds on $Q = \frac{2}{3}$.

The condition to have $\tilde{B}_z = 1$ on V, with a dipolar poloidal magnetic field, now in the form (16), and with $G'(P)$ expressed by (5a), is a first-order integro-differential equation for V_0 or a second-order quasilinear differential equation for S. The earlier neglect of G' gave (incompletely) the condition as $T = 1 - \frac{3}{2}Q$, an integral equation for V_0 or a first-order differential equation for S.

(b) Implications

By using the work of Part I as a guide, I shall now examine how far the formalism for GJ flows satisfying the condition $\tilde{B}_z = 1$ on V, in a dipolar poloidal magnetic field, can be developed without solving their fundamental equation (16).

For these flows, the poloidal motion is along lines of constant Q with Q bounded by zero and $\frac{2}{3}$; these limits correspond to particles emitted from the poles and to particles which cross the surface V on the circles ($X = 1$, $Z^2 = \frac{1}{2}$), where V crosses both the light cylinder and the $B_z = 0$ cone. That is, the outflow comes from inner polar cap regions, bounded by colatitudes given by $\sin\theta_1 = (\frac{2}{3})^{3/4}\sin\theta_0$, or $\theta_1 = 0{\cdot}74\theta_0$, where θ_0 denotes the boundary of the GJ polar cap, and symmetrically in the other hemisphere. (The GJ polar cap is bounded by the feet of the dipole magnetic field lines which are tangential to the light cylinder, corresponding to $\bar{P} = 1 = Q$.)

On using (16) for $I(P)$, equation (9a) shows that

$$V_\phi/X \approx (A_+ + A_- + \tfrac{1}{2}Q)^3 - V_0\langle V_0\rangle \tag{18}$$

near a polar cap. So the ratio V_ϕ/X of the flow's angular speed to that of the star varies from zero at the pole to one at the edge of the polar cap, as was found for Class IC/II flow with $G' = 0$.

For convenience and clarity in writing formulas to be derived below, the following quantities will be used:

$$\tilde{Q} \equiv Q/I, \qquad \tilde{U} \equiv IU, \qquad H(\tilde{U}) \equiv 1/(1+\tilde{U}+\tilde{U}^2). \tag{19a, b, c}$$

The surface V is expressed in terms of the renormalised cylindrical radial coordinate $\tilde{U}$ by $\tilde{U} = 1$; the quantity $1-KX^2$ is $(1-\tilde{U})/H$.

Using the condition (15) to substitute for T in (10) for $\tilde{B}_z$ gives

$$\tilde{B}_z = (1-\tfrac{3}{2}QU)/\tilde{U}^3(1-\tfrac{3}{2}\tilde{Q}), \tag{20}$$

from which follows

$$(\tilde{B}_z-1)/(1-KX^2) = \{1-(1-H)\tfrac{3}{2}\tilde{Q}\}/\tilde{U}^3(1-\tfrac{3}{2}\tilde{Q}). \tag{21}$$

The ratio on the left-hand side of (21) recurs throughout the extended MRW^2 formalism—cf. equations (4) for κS, ρ_e and V_ϕ. Equation (21) expresses this ratio as a function of the coordinates $\tilde{U}$ and $\tilde{Q}$ for the case of GJ flow satisfying $\tilde{B}_z = 1$ on V in a dipolar poloidal magnetic field. In particular, (21) shows that

$$\lim_{\tilde{U}\to 1}\{(\tilde{B}_z-1)/(1-KX^2)\} = (1-\tilde{Q})/(1-\tfrac{3}{2}\tilde{Q}). \tag{22}$$

Since $\tfrac{3}{2}\tilde{Q} \leqslant 1$, with equality only for $Q = \tfrac{2}{3}$, the necessary condition (15) for the GJ flow in a dipolar poloidal magnetic field to penetrate V at least ensures that the ratio $(\tilde{B}_z-1)/(1-KX^2)$ is finite and nonzero for $0 \leqslant Q < \tfrac{2}{3}$ on V.

On using (21) for $(\tilde{B}_z-1)/(1-KX^2)$, equations (4) can be written as

$$\kappa S = \tilde{U}^3(1-\tfrac{3}{2}\tilde{Q})/\{1-(1-H)\tfrac{3}{2}\tilde{Q}\}, \tag{23a}$$

$$\rho_e = -(4\bar{P}/X^2)\{1-(1-H)\tfrac{3}{2}\tilde{Q}\} \tag{23b}$$

$$= -2B_z\{1-(1-H)\tfrac{3}{2}\tilde{Q}\}/(1-\tfrac{3}{2}QU), \tag{23b$'$}$$

$$V_\phi/X = KH\tfrac{3}{2}\tilde{Q}/\{1-(1-H)\tfrac{3}{2}\tilde{Q}\}; \tag{23c}$$

the definition $V_0\,S/2\bar{P}$ of T and the condition (15) have been used in obtaining (23b); equation (6b) for B_z has been used in getting (23b′). Substituting (23a) for κS and (6a) for B_p into $V_p = \kappa B_p$ gives

$$V_p/V_0 = (1-\tfrac{3}{4}QU)^{\frac{1}{2}}/\{1-(1-H)\tfrac{3}{2}\tilde{Q}\}; \tag{24}$$

the definition of T and the condition (15) have been used.

Substituting (23c) and (24) for V_ϕ and V_p into the definition of the Lorentz factor shows that

$$\gamma = \{1-(1-H)\tfrac{3}{2}\tilde{Q}\}/d, \tag{25}$$

where

$$d^2 \equiv \{1-(1-H)\tfrac{3}{2}\tilde{Q}\}^2-(\tfrac{3}{2}\tilde{Q}HKX)^2-V_0^2(1-\tfrac{3}{4}QU). \tag{26}$$

The quantity d varies from $1/\gamma_0(0)$ on $Q = 0$ to $H(1-X^2)^{1/2}$ on $Q = \tfrac{2}{3}$.

Equation (23c) for V_ϕ shows that

$$1 - X V_\phi = (1 - \tfrac{3}{2} Q U)/\{1 - (1 - H)\tfrac{3}{2}\tilde{Q}\}\,. \tag{27}$$

Since $\frac{3}{2}\tilde{Q} \leqslant 1$ and $H > 0$, the denominator is always positive; $1 - X V_\phi$ changes from positive to negative on the $B_z = 0$ cones beyond the light cylinder. Also, (23c) shows that $X V_\phi$ increases monotonically away from the star along the poloidal field/flow lines.

(c) Inertial Effects

The MRW[2] treatment of GJ flow does not involve complete neglect of the noncorotational electric potential, merely neglect of the $\nabla^2 \Phi$ term in the combined Gauss–toroidal Ampère law (1). The Endean integral (2) can be used to calculate Φ in GJ flow, so long as the result is consistent with neglect of the $\nabla^2 \Phi$ term in (1).

Inserting (25) and (27) for γ and $1 - X V_\phi$ into the Endean integral (2) with G replaced by γ_0 yields

$$\Phi/\epsilon = (1 - \tfrac{3}{2} Q U)/d - \gamma_0\,. \tag{28}$$

This gives $\Phi = 0$ on the symmetry axis: the noncorotational electric field component along the axis vanishes and the particles are unaccelerated: V_ϕ vanishes, V_p remains equal to $V_0(0)$ and γ to $\gamma_0(0)$. At the edge $Q = \frac{2}{3}$ of the outflow zone, where $V_p = 0$ and $V_\phi = X$, equations (25) and (28) reduce, as they must, to the forms appropriate to purely corotational flow: $\gamma = 1/(1 - X^2)^{1/2}$ and $\Phi/\epsilon = (1 - X^2)^{1/2} - 1$.

So long as the flow is only moderately accelerated, appropriate dimensionless length scales for variation of γV_p and $\gamma V_\phi t$, in the sense of forming the quantities $\nabla \times (\gamma V_p)$ and $\nabla \times (\gamma V_\phi t)$, are ρ and R. Thus, the ratio of inertial to magnetic terms in the toroidal and poloidal parts of the magnetoidal field B^* may be estimated by the 'magnetic Rossby numbers' ϵ_M^ϕ and ϵ_M^p, defined as $\epsilon\gamma V_p/(-B_\phi \rho)$ and $\epsilon\gamma V_\phi/B_p R$ (cf. Wright 1978).

Using (25) for γ with (24) and (23c) for V_p and V_ϕ gives

$$\gamma V_p = (1 - \tfrac{3}{4} Q U)^{\frac{1}{2}} V_0/d\,, \qquad \gamma V_\phi = K X H 3\tilde{Q}/2d\,. \tag{29a, b}$$

Using $-B_\phi = 2\bar{P}\langle V_0\rangle/X$—a form of $B_\phi = -S/X$—and (6c) for ρ shows that

$$-B_\phi \rho = (8 Q^{\frac{1}{2}}/3 U)\langle V_0\rangle(1 - \tfrac{3}{4} Q U)^{\frac{3}{2}}/(1 - \tfrac{1}{2} Q U)\,. \tag{29c}$$

The relations $\bar{P} = X^2/R^3$ and (6a) for B_p give

$$B_p R = (2Q/U^2)(1 - \tfrac{3}{4} Q U)^{\frac{1}{2}}\,. \tag{29d}$$

Equations (29) yield

$$\epsilon_M^\phi/\epsilon = (3/8d) R (V_0/\langle V_0\rangle)(1 - \tfrac{1}{2} Q U)/(1 - \tfrac{3}{4} Q U)\,, \tag{30a}$$

$$\epsilon_M^p/\epsilon = (3/4d) X/(1 + 1/\tilde{U} + 1/\tilde{U}^2)(1 - \tfrac{3}{4} Q U)^{\frac{1}{2}}\,; \tag{30b}$$

in (30a), the relation $U/Q^{1/2} = R$—a form of $\bar{P} = X^2/R^3$—has been used. On the axis $Q = 0 = U$, where $d = 1/\gamma_0(0)$, equations (30) show that $\epsilon_M^\phi/\epsilon = 3\gamma_0(0)Z/8$ and $\epsilon_M^p = 0$. On $Q = \frac{2}{3}$, where $d = H(1-X^2)^{1/2}$, equations (30) show that $\epsilon_M^\phi = 0$ and $\epsilon_M^p/\epsilon = \frac{3}{4}X\,U^2/(1-X^2)^{1/2}(1-\frac{1}{2}U)^{1/2}$.

5. On V, W and Beyond

In this section, the behaviour of GJ outflow satisfying $\tilde{B}_z = 1$ on V is demonstrated by evaluating quantities on certain surfaces, namely V, W, the $B_z = 0$ cones (beyond the light cylinder) and the equatorial plane. The condition (15) for $\tilde{B}_z$ to equal 1 on V can be written as

$$1-\tfrac{3}{2}\tilde{Q} = T/K. \tag{15'}$$

A discussion of inertial effects is included at the end of the section.

(a) On V: $K^{\frac{1}{2}}X = 1$

This surface satisfies $U = 1/I$ or $\tilde{U} = 1$. From $\bar{P} = X^2/R^3$ and the definition (19c) of H, it follows that

$$QU = \tilde{Q} = 1/(KZ^2+1), \qquad H = \tfrac{1}{3} \quad \text{on} \quad V. \tag{31}$$

Equations (8) give, after substituting $1+\epsilon G'$ for K,

$$\gamma_m^2 = (1+B_p^2/B_\phi^2)/(1-\epsilon G' B_p^2/B_\phi^2), \tag{32a}$$

$$D_\infty^2/K = 1-\epsilon G' B_p^2/B_\phi^2 \quad \text{on} \quad V. \tag{32b}$$

Equation (7) gives

$$\begin{aligned} -B_\phi/B_p &= \langle V_0\rangle/K^{\frac{1}{2}}(1-\tfrac{3}{4}\tilde{Q})^{\frac{1}{2}} \\ &= \{(KZ^2+1)/(KZ^2+\tfrac{1}{4})\}^{\frac{1}{2}}\langle V_0\rangle/K^{\frac{1}{2}} \quad \text{on} \quad V. \end{aligned} \tag{33}$$

Since $K(0) = V_0^2(0)$, it follows that $-B_\phi/B_p$ approaches 1 at large Z^2 on V. On the circles ($X = 1$, $Z^2 = \frac{1}{2}$) where V, W and the $B_z = 0$ cones meet the light cylinder, $-B_\phi/B_p = \sqrt{2}\langle V_0\rangle_e$, with $\langle V_0\rangle_e$ denoting $V_0(P)$ averaged right across the outflow. For GJ flows satisfying the condition $\tilde{B}_z = 1$ on V, in a dipolar poloidal magnetic field, the toroidal magnetic field, which is vanishingly small near the star, varies on V from 100% of the poloidal field as $Z^2 \to \infty$ to $(141\langle V_0\rangle_e)$% at $Z^2 = \frac{1}{2}$.

Equation (22) states that

$$\lim_{\tilde{U}\to 1}\{(1-KX^2)/(\tilde{B}_z-1)\} = (1-\tfrac{3}{2}\tilde{Q})/(1-\tilde{Q}) = 1-1/2KZ^2; \tag{22'}$$

the ratio $(1-KX^2)/(\tilde{B}_z-1)$ is finite on V, varying from 1 as $Z^2 \to \infty$ to 0 on the circles ($X = 1$, $Z^2 = \frac{1}{2}$).

Equation (4a) gives, because of (22′),

$$\kappa S = (1-\tfrac{3}{2}\tilde{Q})/(1-\tilde{Q}) = 1-1/2KZ^2 \quad \text{on} \quad V; \tag{34a}$$

so κS on V varies from one as $Z^2 \to \infty$ to zero at $Z^2 = \frac{1}{2}$. Equations (23b) and (23b′) give

$$\rho_e = -4K\bar{P}(1-\tilde{Q}) = -4/Z^3(1+1/KZ^2)^{\frac{5}{2}} \quad \text{on} \quad V, \tag{34b}$$

$$\rho_e/(-2B_z) = (1-\tilde{Q})/(1-\tfrac{3}{2}\tilde{Q}) = 1/(1-1/2KZ^2)^{\frac{1}{2}} \quad \text{on} \quad V; \tag{34b′}$$

so ρ_e on V approaches the fiducial (GJ) value $-2B_z$ as $Z^2 \to \infty$, but remains nonzero (equal to $-\frac{8}{9}\sqrt{\frac{2}{3}}$ or $-0\cdot726$) as $Q \to \frac{2}{3}$. Equation (23c) gives

$$V_\phi/X = \tfrac{1}{2}K\tilde{Q}/(1-\tilde{Q}) = 1/2Z^2 \quad \text{on} \quad V; \tag{34c}$$

the angular speed of the flow, normalised to that of the star, varies on V from 0 to 1 as Z^2 goes from ∞ to $\frac{1}{2}$.

Equation (24) shows that

$$V_p/V_0 = (1-\tfrac{3}{4}\tilde{Q})^{\frac{1}{2}}/(1-\tilde{Q}) = (1+5/4KZ^2+1/4K^2Z^4)^{\frac{1}{2}} \quad \text{on} \quad V. \tag{35}$$

This ratio, expressing the acceleration received by the poloidal flow between the star and the surface V, varies from 1 for particles emitted from the poles to $3/\sqrt{2}$ or $2\cdot12$ for those emitted from the edge of a polar cap.

Equations (25) and (26) for γ and d show that

$$\gamma = (1-\tilde{Q})/\{(1-\tilde{Q})^2 - \tfrac{1}{4}K\tilde{Q}^2 - V_0^2(1-\tfrac{3}{4}\tilde{Q})\}^{\frac{1}{2}} \quad \text{on} \quad V \tag{36a}$$

$$= 1/\{1-1/4KZ^4 - V_0^2(1+1/KZ^2)(1+1/4KZ^2)\}^{\frac{1}{2}} \quad \text{on} \quad V. \tag{36b}$$

Equation (28) for the noncorotational potential gives

$$\Phi/\epsilon = (1-\tfrac{3}{2}\tilde{Q})/\{(1-\tilde{Q})^2 - \tfrac{1}{4}K\tilde{Q}^2 - V_0^2(1-\tfrac{3}{4}\tilde{Q})\}^{\frac{1}{2}} - \gamma_0 \quad \text{on} \quad V \tag{37a}$$

$$= (1-1/2KZ^2)/\{1-1/4KZ^4 - V_0^2(1+1/KZ^2)(1+1/4KZ^2)\}^{\frac{1}{2}} - \gamma_0 \quad \text{on} \quad V. \tag{37b}$$

As Z^2 goes from ∞ to $\frac{1}{2}$, γ varies from $\gamma_0(0)$ to ∞, while Φ varies from 0 to $-\epsilon$; this is true for any surface stretching from $Z^2 = \infty$ to a circle ($X = 1$, $Z^2 = \frac{1}{2}$). [The right-hand sides of (37) are indeterminate when $Q = \frac{2}{3}$ or $Z^2 = \frac{1}{2}$, but the fact that Φ takes the form appropriate to purely corotational motion on $Q = \frac{2}{3}$ shows that $\Phi = -\epsilon$ on the circles ($X = 1$, $Z^2 = \frac{1}{2}$).]

(b) On W: $|K|X = 1$

This surface satisfies $U = 1/J$ or $\tilde{U} = 1/I$. From $\bar{P} = X^2/R^3$ and the definition (19c) of H, it follows that

$$QU = Q/J = 1/(K^2Z^2+1), \qquad H = J/(J+I+1) \quad \text{on} \quad W. \tag{38}$$

Equations (8) give

$$\gamma_{\mathrm{m}}^2 = 1 + B_{\mathrm{p}}^2/B_{\phi}^2, \qquad D_{\infty}^2 = K^2 \quad \text{on} \quad W. \tag{39a, b}$$

For the flows under consideration, $K > 0$.

Equation (7) shows that

$$\begin{aligned} -B_{\phi}/B_{\mathrm{p}} &= \langle V_0 \rangle / K(1-3Q/4J)^{\frac{1}{2}} \\ &= \{(K^2Z^2+1)/(K^2Z^2+\tfrac{1}{4})\}^{\frac{1}{2}} \langle V_0 \rangle / K \quad \text{on} \quad W. \end{aligned} \tag{40}$$

The ratio $-B_{\phi}/B_{\mathrm{p}}$ on W approaches $1/V_0(0)$ at large Z^2, corresponding to $Q \to 0$; on the circles where V, W and the $B_z = 0$ cones meet the light cylinder, $-B_{\phi}/B_{\mathrm{p}} = \surd 2 \langle V_0 \rangle_{\mathrm{e}}$. For GJ flows satisfying $\tilde{B}_z = 1$ on V, in a dipolar poloidal magnetic field, the toroidal magnetic field varies on W from $(100/V_0(0))\%$ of the poloidal field as $Z^2 \to \infty$ to $(141 \langle V_0 \rangle_{\mathrm{e}})\%$ at $Z^2 = \frac{1}{2}$.

Equation (20) shows that

$$\tilde{B}_z = K(1-3Q/2J)/(1-\tfrac{3}{2}\tilde{Q}) \quad \text{on} \quad W; \tag{41}$$

so $\tilde{B}_z$ on W varies from $V_0^2(0)$ as $Z^2 \to \infty$ to 1 at $Z^2 = \frac{1}{2}$. It follows from (21) that

$$(\tilde{B}_z-1)/(1-KX^2) = K\{1-(1-H)\tfrac{3}{2}\tilde{Q}\}/(1-\tfrac{3}{2}\tilde{Q}) \quad \text{on} \quad W; \tag{42}$$

the ratio $(1-KX^2)/(\tilde{B}_z-1)$ on W varies from $1/V_0^2(0)$ as $Z^2 \to \infty$ to 0 on the circles $(X = 1,\ Z^2 = \frac{1}{2})$.

Equation (4a) gives, because of (42),

$$\kappa S = (1-\tfrac{3}{2}\tilde{Q})/K\{1-(1-H)\tfrac{3}{2}\tilde{Q}\} \quad \text{on} \quad W; \tag{43a}$$

so κS on W varies from $1/V_0^2(0)$ as $Z^2 \to \infty$ to 0 at $Z^2 = \frac{1}{2}$. Equations (23b) and (23b′) show that

$$\rho_{\mathrm{e}} = -4K^2\bar{P}\{1-(1-H)\tfrac{3}{2}\tilde{Q}\} \quad \text{on} \quad W \tag{43b}$$

$$\rho_{\mathrm{e}}/(-2B_z) = \{1-(1-H)\tfrac{3}{2}\tilde{Q}\}/(1-3Q/2J) \quad \text{on} \quad W; \tag{43b′}$$

so ρ_{e} on W approaches the fiducial value $-2B_z$ as $Z^2 \to \infty$, but remains nonzero (equal to $-\frac{8}{9}\surd\frac{2}{3}$ or $-0{\cdot}726$) as $Q \to \frac{2}{3}$. Equation (23c) shows that

$$V_{\phi} = \tfrac{3}{2}\tilde{Q}H/\{1-(1-H)\tfrac{3}{2}\tilde{Q}\} \quad \text{on} \quad W; \tag{43c}$$

the angular speed of the flow, normalised to that of the star, varies on W from 0 to 1 as Z^2 goes from ∞ to $\frac{1}{2}$.

Equation (24) shows that

$$V_{\mathrm{p}}/V_0 = (1-3Q/4J)^{\frac{1}{2}}/\{1-(1-H)\tfrac{3}{2}\tilde{Q}\} \quad \text{on} \quad W. \tag{44}$$

This ratio, expressing the acceleration received by the poloidal flow between the star and the surface W, varies from one for particles emitted from the poles to $3/\surd 2$ or $2{\cdot}12$ for those emitted from the edge of a polar cap.

On W, as for any surface stretching from $Z^2 = \infty$ to a circle ($X = 1$, $Z^2 = \frac{1}{2}$), γ varies from $\gamma_0(0)$ to ∞ as Z^2 goes from ∞ to $\frac{1}{2}$, while Φ varies from 0 to $-\epsilon$.

(c) On the $B_z = 0$ Cones

Equation (6b) for the dipolar B_z, the condition (15′) for $\tilde{B}_z$ to equal 1 on V and the definition (19c) of H show that

$$QU = \tfrac{2}{3}, \quad T/K = 1 - 1/\tilde{U}, \quad 1-(1-H)\tfrac{3}{2}\tilde{Q} = \tilde{U}^2 H \quad \text{on} \quad B_z = 0. \quad (45\text{a, b, c})$$

Equation (7) shows that

$$-B_\phi/B_p = \surd 2\, X\langle V_0\rangle = 4\langle V_0\rangle/3^{\frac{3}{2}}\bar{P} \quad \text{on} \quad B_z = 0. \tag{46}$$

The ratio $-B_\phi/B_p$ on $B_z = 0$ varies from $\surd 2\langle V_0\rangle_e$ to ∞ as X goes from 1 to ∞.

Equation (21) shows that

$$(1-KX^2)/(\tilde{B}_z-1) = \tilde{U}^3-1 \quad \text{on} \quad B_z = 0, \tag{47}$$

which varies from 0 to ∞ as X goes from 1 to ∞.

Equation (4a) now shows that

$$\kappa S = \tilde{U}^3-1 \quad \text{on} \quad B_z = 0; \tag{48a}$$

so κS on $B_z = 0$ varies from 0 to ∞ as X goes from 1 to ∞. Equation (23b) shows that

$$\rho_e = -6J(2/3\,U)^{\frac{5}{2}} H(\tilde{U}) \quad \text{on} \quad B_z = 0; \tag{48b}$$

so $-\rho_e$ on the $B_z = 0$ cones varies from $\frac{8}{9}\surd\frac{2}{3}$ or $0\cdot726$ on $X = 1$ to 0 as $X \to \infty$.

Equation (23c) shows that

$$V_\phi = 1/X = (\tfrac{3}{2}Q)^{\frac{3}{2}} \quad \text{on} \quad B_z = 0; \tag{48c}$$

the angular speed of the flow, normalised to that of the star, varies on $B_z = 0$ from 1 to 0 as X goes from 1 to ∞.

Equation (24) shows that

$$V_p/V_0 = (1+1/\tilde{U}+1/\tilde{U}^2)/\surd 2 = \{1+\tfrac{3}{2}\tilde{Q}+(\tfrac{3}{2}\tilde{Q})^2\}/\surd 2 \quad \text{on} \quad B_z = 0. \tag{49}$$

This ratio, expressing the acceleration received by the poloidal flow between the star and the $B_z = 0$ cones, varies from $1/\surd 2$ or $0\cdot707$ for particles emitted from the poles to $3/\surd 2$ or $2\cdot12$ for those emitted from the edge of a polar cap. For $\tilde{U} = 1$, $\frac{4}{3}$ and ∞, corresponding to $\tilde{Q} = \frac{2}{3}$, $\frac{1}{2}$ and 0, the ratio (49) is $2\cdot12$, $1\cdot64$ and $0\cdot707$; it has fallen to 1 at $\tilde{U} = 3\cdot17$, corresponding to $\tilde{Q} = 0\cdot210$; for smaller values of $\tilde{Q}$ than this, there is net deceleration of the poloidal flow between the star and these cones. Comparison of (49) with (35) for V_p/V_0 on the surface V at corresponding values of $\tilde{Q}$ shows that, for $\tilde{Q} < 0\cdot41$, the poloidal flow receives net deceleration between V and the $B_z = 0$ cones; for $0\cdot41 < \tilde{Q} < \frac{2}{3}$, it is very slightly accelerated, by no more than a few per cent.

Equations (48c) and (49) for V_ϕ and V_p on $B_z = 0$ give

$$1/\gamma^2 = 1 - 1/U^3 - \tfrac{1}{2} V_0^2(1 + 1/\tilde{U} + 1/\tilde{U}^2)^2 \quad \text{on} \quad B_z = 0; \tag{50}$$

so γ on $B_z = 0$ varies from ∞ to $1/\{1 - \frac{1}{2} V_0^2(0)\}^{1/2}$ as X goes from 1 to ∞.

Since $X V_\phi = 1$ on $B_z = 0$, the Endean integral (2), with G replaced by γ_0, shows that Φ is negative, equal to $-\epsilon\gamma_0$, on the $B_z = 0$ cones. Equations (30) show that

$$\epsilon_M^\phi/\epsilon = \tfrac{1}{2}\gamma R(V_0/\langle V_0\rangle)(1 + 1/\tilde{U} + 1/\tilde{U}^2), \tag{51a}$$

$$\epsilon_M^p/\epsilon = 3(\tfrac{1}{2})^{\frac{3}{2}}\gamma X \quad \text{on} \quad B_z = 0. \tag{51b}$$

(d) On the Equatorial Plane

For a dipole poloidal magnetic field, the relation $\bar{P} = X^2/R^3$ shows that $\bar{P}X = 1$ on this plane; it follows that

$$QU = 1, \quad 1 - (1-H)^{\frac{3}{2}}\tilde{Q} = (2 - \tilde{Q} - \tilde{Q}^2)/2(1 + \tilde{Q} + \tilde{Q}^2) \quad \text{on} \quad Z = 0. \tag{52a, b}$$

In the work with G' neglected, it is the flow lines with $Q < \frac{1}{2}$ that reach the equatorial plane. Near $Q = \frac{1}{2}$, the quantity $\epsilon G'$ must be very small, and K is correspondingly close to 1; so the maximum value of Q for which the GJ flow reaches the equatorial plane is very close to $\frac{1}{2}$. I shall take $\tilde{Q} = \frac{1}{2}$, corresponding to $\tilde{U} = 2$ on $Z = 0$, as a convenient reference value at which to make illustrative numerical evaluations of some quantities of interest.

Equation (7) shows that

$$-B_\phi/B_p = 2\langle V_0\rangle/\bar{P} \quad \text{on} \quad Z = 0. \tag{53}$$

In the calculations neglecting G', the ratio $-B_\phi/B_p$ on $Z = 0$ was found to increase monotonically with X from the value 3·7 on $U = 2$. Since $\bar{P}X = \sin^3\theta$ and $QU = \sin^2\theta$ for a dipole field, (7) shows that $-B_\phi/B_p \propto \langle V_0\rangle/\bar{P}$ on any cone of constant θ, including the limiting case $Z = 0$; since $\bar{P}\mathrm{d}\langle V_0\rangle/\mathrm{d}\bar{P} = V_0 - \langle V_0\rangle$, it is clear that $-B_\phi/B_p$ on any of these cones increases with distance from the star so long as $V_0 < 2\langle V_0\rangle$.

Equation (21) gives

$$(\tilde{B}_z - 1)/(1 - KX^2) = \tilde{Q}^3(2 - \tilde{Q} - \tilde{Q}^2)/(2 - 3\tilde{Q})(1 + \tilde{Q} + \tilde{Q}^2) \quad \text{on} \quad Z = 0; \tag{54}$$

this varies from 5/28 to 0 as $\tilde{U}$ goes from 2 to ∞.

Equation (4a) now shows that

$$\kappa S = (2 - 3\tilde{Q})(1 + \tilde{Q} + \tilde{Q}^2)/\tilde{Q}^3(2 - \tilde{Q} - \tilde{Q}^2) \quad \text{on} \quad Z = 0; \tag{55a}$$

this varies from 28/5 to ∞ as $\tilde{U}$ goes from 2 to ∞. Equations (23b) and (23b′) give

$$\rho_e = -(2/X^3)(2 - \tilde{Q} - \tilde{Q}^2)/(1 + \tilde{Q} + \tilde{Q}^2) \quad \text{on} \quad Z = 0, \tag{55b}$$

$$\rho_e/2B_z = (2 - \tilde{Q} - \tilde{Q}^2)/(1 + \tilde{Q} + \tilde{Q}^2) \quad \text{on} \quad Z = 0. \tag{55b′}$$

So ρ_e on $Z = 0$ is negative and of the order of the fiducial value $2B_z$; more precisely, $\rho_e/2B_z$ varies from 5/7 to 2 as $\tilde{U}$ goes from 2 to ∞. Equation (23c) shows that

$$V_\phi/X = 3Q^3/(2-\tilde{Q}-\tilde{Q}^2) \quad \text{on} \quad Z = 0; \tag{55c}$$

the angular speed of the flow, normalised to that of the star, varies on the equatorial plane from approximately 3/10 to 0 as $\tilde{U}$ goes from 2 to ∞.

Equation (24) shows that

$$V_p/V_0 = (1+\tilde{Q}+\tilde{Q}^2)/(2-\tilde{Q}-\tilde{Q}^2) \quad \text{on} \quad Z = 0. \tag{56}$$

Thus, on the equatorial plane, V_p/V_0 is 7/5 for $\tilde{U} = 2$ and tends to $\frac{1}{2}$ as $X \to \infty$. Equations (35), (49) and (56) for V_p/V_0 on the surfaces V, $B_z = 0$ and $Z = 0$ show that the poloidal flow suffers net deceleration both between V and the equatorial plane and between the $B_z = 0$ cones and the equatorial plane.

Equations (55c) and (56) for V_ϕ and V_p on $Z = 0$ give

$$\gamma = (2-\tilde{Q}-\tilde{Q}^2)/\{(2-\tilde{Q}-\tilde{Q}^2)^2-9Q^3-V_0^2(1+\tilde{Q}+\tilde{Q}^2)^2\}^{\frac{1}{2}} \quad \text{on} \quad Z = 0; \tag{57}$$

the Lorentz factor on $Z = 0$ varies between ∞, for a value of $\tilde{U}$ very close to 2 (cf. Part I), and $1/\{1-\frac{1}{4}V_0^2(0)\}^{1/2}$ as $X \to \infty$.

Equation (55c) for V_ϕ on $Z = 0$ shows that $1-XV_\phi < 0$ there; hence, the Endean integral (2), with G replaced by γ_0, implies that $\Phi/\epsilon < 0$; more precisely,

$$\Phi/\epsilon = -(1+\tilde{Q}+\tilde{Q}^2)/\{(2-\tilde{Q}-\tilde{Q}^2)^2-9Q^3 - V_0^2(1+\tilde{Q}+\tilde{Q}^2)^2\}^{\frac{1}{2}}-\gamma_0 \quad \text{on} \quad Z = 0. \tag{58}$$

Equations (30) show that

$$\epsilon_M^\phi/\epsilon = \tfrac{3}{2}\gamma X(V_0/\langle V_0\rangle)(1+\tilde{Q}+\tilde{Q}^2)/(2-\tilde{Q}-\tilde{Q}^2) \quad \text{on} \quad Z = 0, \tag{59a}$$

$$\epsilon_M^p/\epsilon = 3\gamma X/(2-\tilde{Q}-\tilde{Q}^2) \quad \text{on} \quad Z = 0. \tag{59b}$$

(e) Inertial Effects

These manifest themselves in two ways: through the existence of the noncorotational electric potential and through the occurrence of inertial drift. For the flows under consideration, the former is given by

$$\Phi/\epsilon = \gamma(1-\tfrac{3}{2}QU)/\{1-(1-H)\tfrac{3}{2}\tilde{Q}\}-\gamma_0, \tag{60}$$

while the latter effect is estimated by the magnetic Rossby numbers:

$$\epsilon_M^\phi/\epsilon = \tfrac{3}{8}\gamma R(V_0/\langle V_0\rangle)(1-\tfrac{1}{2}QU)/\{1-(1-H)\tfrac{3}{2}\tilde{Q}\}(1-\tfrac{3}{4}QU), \tag{61a}$$

$$\epsilon_M^p/\epsilon = \tfrac{3}{4}\gamma X/\{1-(1-H)\tfrac{3}{2}\tilde{Q}\}(1+1/\tilde{U}+1/\tilde{U}^2)(1-\tfrac{3}{4}QU)^{\frac{1}{2}}. \tag{61b}$$

Far from the star, $\epsilon_M^\phi/\epsilon \sim \gamma RV_0/\langle V_0\rangle$ and $\epsilon_M^p/\epsilon \sim \gamma X$.

It is clear that Φ in these flows is a very small quantity, of order ϵ, except near the surface S_l on which the GJ Lorentz factor diverges. So, the noncorotational potential deduced from the MRW^2 treatment of GJ flow (which neglects the $\nabla^2 \Phi$ contribution to the Gauss–toroidal Ampère law) is consistent with the assumptions underlying that approach. The magnetic Rossby numbers indicate that inertial drift becomes important near S_l and also at large distances, $R \sim 1/\epsilon$, from the star.

6. Concluding Remarks

Flows that cross the surface V without encountering the MRW^2 limiting surface S_l are emitted from an inner polar cap region, having about three-quarters of the radius of the standard GJ polar cap. The requirement that their GJ Lorentz factor be finite on V, together with the assumption of a dipolar form for the poloidal magnetic field, lead to a well-defined mathematical description of these flows. The higher-latitude (Class II) flows, emanating from the inner 80% by radius of the MRW^2 polar cap, do not encounter S_l. The flows (Class IC) emanating from the remaining ring of the MRW^2 cap reach S_l beyond V. The surface S_l extends outwards from the circles where $B_z = 0$ on the light cylinder, crossing the equatorial plane at $X \approx 2^{3/2}$.

It is inherent in this treatment of the outflow that it fails in the vicinity of S_l, through buildup of Φ and inertial drift. It appears that it also fails, because of inertial drift, far from the star, but only at such large distances—of order $1/\epsilon$ in units of the light cylinder radius—as to be probably a matter of little real concern; failure of the dipolar approximation for the poloidal magnetic field is likely to be of more significance.

In the earlier work, with G' neglected, simple exact solutions for $V_0(P)$ and $S(P)$ were available (MRW^2). In the present analysis, an approximate technique of solving for these quantities remains to be developed, proceeding from either a first-order integro-differential equation for $V_0(P)$ or a second-order quasilinear differential equation for $S(P)$.

The techniques developed here have shown that allowing for variation of the Endean integral eliminates the singularity of the emission Lorentz factor that occurred at the stellar poles in the earlier treatment of these flows. There might be other ways of eliminating that singularity. It could simply be, as mentioned in the Introduction, that the flow from tiny inner cores of the polar caps cannot be of this (Class IC/II) kind: that is, there could be another branch of S_l which the flow along lines of tiny Q intersects inside the light cylinder, meaning that the flow is of Class IA. Setting aside this possibility, the occurrence of the singularity points to some missing element in the earlier work. The neglected quantities are G', dealt with here, $\nabla^2 \Phi$, which must become significant before S_l is reached, and inertial drift.

A referee has argued that, for poloidal field/flow lines emanating from very near a pole, $\nabla^2 \Phi$ is non-negligible in the immediate vicinity of the star, meaning that V_0 cannot, for tiny Q, be identified with the emission speed. This has the consequences that it is non-negligible $\nabla^2 \Phi$ that is relevant to eliminating the singularity, and that G will remain close to unity. The techniques that I have developed incorporate G' fully, and show that the singularity is thus removed in a self-consistent way. Since I have no knowledge of work in which $\nabla^2 \Phi$ is incorporated, I am not in a position fully to assess the relative significance of $\nabla^2 \Phi$ and G'.

It has also been argued that my extended version of the MRW^2 formalism (with G' incorporated) is relevant only if there is non-electric injection of particles; for example, by spallation caused by particles returning to the star. It is the case that the question of how the electrons are accelerated to their emission speeds has been left unanswered by the work presented here, as well as by the earlier work with G' neglected (MRW^2 and Part I). Treating the stellar surface as that of a perfect conductor means that any description of this process has been abandoned: the electrons are taken, in this approximation, as simply appearing at the surface of the star with the appropriate speeds. (Here, by 'surface', I refer to the geometrical surface of the star, as distinct from the physical surface layer of a neutron star.)

More realistically, the finite conductivity of the surface layer will allow Φ to penetrate into the star. It is the corresponding component $E_{\|}$ of the electric field parallel to the magnetic field that accelerates the electrons before they reach the geometrical surface of the star. If the flow is to be of Class IC/II, then the near-surface regions of the magnetosphere must adjust, with the right latitude dependence, so as to match to the leakage $E_{\|}$ required to yield the appropriate emission speed as a function of latitude.

There is an interesting nonuniformity in taking the limit of nonrelativistic emission: ignoring G' by putting the emission Lorentz factor equal to one before determining the emission speed leads to $V_0(P)$ for Class IC/II flows which equals c at the stellar poles. But fully incorporating G'—leaving it to the mathematics to determine γ_0 through V_0—leads to an emission speed that does not have this problem. Thus, fully incorporating γ_0—allowing the emission to be arbitrarily relativistic—has the effect of showing that the emission speed required for Class IC/II flows is, in fact, more moderately relativistic than appeared to be the case when γ_0 was initially approximated by unity. This occurs because G' increases by one the order of the differential, or integro-differential, equation for $V_0(P)$.

Thus, inclusion of G' is a matter of self-consistent determination of the function $V_0(P)$ required for Class IC/II flow. Description of the injection process, by which the particles might be accelerated to the required emission speed, is another problem.

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