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Finite Larmor Radius Modifications to the Magnetoionic Wave Modes and Their Effects on Electron Cyclotron Maser Emission

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Abstract

Auroral kilometric radiation, Jupiter's decametric and Saturn's kilometric radio emissions, solar microwave spike bursts and microwave emissions from some flare stars have all been attributed to the electron cyclotron maser instability. The maser instability is usually assumed to involve the generation of magnetoionic waves. We investigate the modifications to the magnetoionic wave modes due to finite Larmor radius (FLR) corrections arising from a 'warm' background electron plasma with a Maxwellian distribution. We then consider the effects of these modifications on maser emission at frequencies near the fundamental of the electron cyclotron frequency Ω_e. The FLR effects are found to be small; the maximum temporal growth rate generally differs by $\lesssim 10\%$ from that for emission occurring in the magnetoionic modes. Small shifts occur in the frequencies and propagation angles corresponding to the maximum growth rates.

1. Introduction

Electron cyclotron maser emission (ECME) has been proposed as the mechanism for a variety of intrinsically very bright and highly polarized radio bursts from planetary and astrophysical sources. The most widely accepted proposal is that auroral kilometric radiation (AKR) is due to such maser emission, driven by electrons with a loss cone anisotropy (Wu and Lee 1979; Melrose *et al.* 1982; Omidi and Gurnett 1982; Wu *et al.* 1982). The process also provides favourable explanations of decametric radio emission (DAM) from Jupiter (Hewitt *et al.* 1981), of kilometric radio emission (SKR) from Saturn, of microwave emission from some flare stars (Melrose and Dulk 1982; Dulk *et al.* 1983), and of microwave spike bursts from the solar corona (Holman *et al.* 1980; Melrose and Dulk 1982; Sharma *et al.* 1982).

The AKR is the most widely studied of these emissions because it has been possible to make various *in situ* measurements. Energetic ('inverted V') electrons precipitating along auroral magnetic field lines are associated with large potential drops in an auroral plasma cavity very deficient in the 'cold' background plasma of ionospheric origin. This reduces the plasma frequency ω_p to a value much less than the electron cyclotron frequency Ω_e. Some of the precipitating electrons are absorbed by the ionosphere while others are reflected by the converging magnetic field lines to produce a 'hot' electron distribution with an upward directed loss cone. It is the 'hot' anisotropic, loss cone distribution which supposedly drives the electron cyclotron maser (Wu and Lee 1979; Benson and Calvert 1979). The radiation is emitted predominantly with x-mode (fast extraordinary) polarization at frequencies just above

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the local electron cyclotron frequency. Weaker bursts of o-mode (ordinary) and z-mode (slow extraordinary) radiation have also been observed. The other planetary emissions DAM and SKR display similar characteristics but the mechanisms differ in detail.

Calculations of growth rates and other quantities for ECME have generally involved the assumption that there are two essentially independent plasma distributions in the emission region. The dispersive properties of the region are assumed to be determined solely by a relatively dense, cold, background plasma component. Thus emission occurs in the wave modes (x, o and z) determined by magnetoionic theory. The cyclotron instability causing maser action is assumed to be driven by an anisotropic distribution of 'hot' electrons, at a typical temperature of $\sim 10^8$ K and an average density very much less than that of the background plasma. These assumptions are consistent with the environments in the low and intermediate altitude source regions of AKR and probably with those of DAM and SKR. For the solar corona and the atmospheres of flare stars, however, the background plasma is probably at a temperature between 10^6 and 10^7 K. Some authors have considered the situation of a single hot plasma with no background distribution (Winglee 1983, 1985; Wu *et al.* 1982). This model is relevant to high altitude source regions of AKR and will not be considered in this paper.

Fundamental ECME driven by a single sided loss cone distribution occurs for the z and x modes at frequencies close to and just above Ω_e. At these frequencies, thermal effects are likely to be important because the elements of the cold plasma dielectric tensor exhibit singularities at $\omega = \Omega_e$. Loss cone driven ECME at higher harmonics of Ω_e also involves the generation of radiation at frequencies just above the relevant multiple of Ω_e and so thermal effects are likely to be important in these situations too. The thermal effects arise in two ways—from the finite Larmor radius of the electron and from the relativistic mass dependence of the electron gyrofrequency. Together, these effects can lead to significant modifications to the cold plasma wave mode structure. In addition to the probable importance of thermal effects at frequencies near Ω_e, the positions of the cutoffs (where $n = 0$) and resonances (where $|n^2|$ becomes infinite) are changed when thermal effects are included.

In this paper we relax the assumption that the background plasma in the source region is cold. As in most of the papers cited above, we assume that the electron distribution may be described by the sum of a relatively dense background distribution and an anisotropic energetic distribution. The former distribution dominates the hermitian part of the dielectric tensor ϵ and determines the (time-reversible) dispersive properties of the plasma. We neglect thermal effects which arise from the relativistic mass increase of the electrons, but we do investigate the frequency range in which such effects are likely to be important. Our method of approximation of the dielectric tensor for the thermal background plasma is suitable only for frequencies near the electron cyclotron frequency Ω_e and hence we consider only finite Larmor radius (FLR) modifications to maser emission at the fundamental. The antihermitian part of ϵ is dominated by the anisotropic distribution and gives rise to growth (or damping).

In Section 2 we calculate the dispersive properties for a background distribution which is taken to be Maxwellian at a temperature of 2×10^6 K (corresponding to a typical temperature of the solar corona). We also estimate the parameter regimes in which our approximations break down, and in which a full relativistic treatment is required. In Section 3 we calculate the growth properties of the thermal modes due to maser action driven by a 'hot' loss cone distribution and compare our results with the corresponding results for growth in the magnetoionic modes as given by Hewitt *et al.* (1982), Hewitt and Melrose (1983) and Melrose *et al.* (1984). The results are summarized in Section 4.

2. Dispersive Properties of the Warm Background Plasma

(a) Dielectric tensor

The dispersive properties of the warm background plasma may be described in terms of the hermitian part ϵ^{h} of the dielectric tensor for a Maxwellian electron distribution. Analytic expressions for ϵ^{h} may be obtained if purely relativistic effects are neglected by setting the Lorentz factor $\gamma = (1-v^2/c^2)^{-\frac{1}{2}}$ equal to 1 (see for example Melrose 1980*b*; p. 264). The expressions obtained for the elements of ϵ^{h} involve products of the real part of the plasma dispersion function of Fried and Conte (1961) with argument

$$z = (\omega - s\Omega_{\mathrm{e}})\tan\theta/\Omega_{\mathrm{e}}(2\lambda)^{\frac{1}{2}},$$

and the modified Bessel functions $I_s(\lambda)$ and their derivatives. The parameter λ may be written as

$$\lambda = n^2(v/c)^2(\omega/\Omega_{\mathrm{e}})^2\sin^2\theta,$$

where $n = kc/\omega$ is the refractive index, v is the thermal velocity of the electrons given by $k_{\mathrm{B}}\,T = m_{\mathrm{e}}\,v^2$ and θ is the angle between the wavevector $\boldsymbol{k}$ and the magnetic field $\boldsymbol{B}$.

Calculations for maser emission at the fundamental—in the wave modes of a cold plasma—indicate that growth occurs in parameter regimes satisfying $\omega \approx \Omega_{\mathrm{e}}$ and $\sin^2\theta \sim 1$ (Lee *et al.* 1980; Omidi and Gurnett 1982; Hewitt and Melrose 1983). In this case $\lambda \approx n^2(v/c)^2$ and for a 'warm' plasma $(v/c)^2 \ll 1$ so λ will generally be small. It is therefore appropriate to expand ϵ^{h} as a power series in λ retaining terms to first order, provided $\lambda \ll 1$ and $|z^2| \gg 1$ (Sitenko and Stepanov 1957). The resulting expression may be written in the form

$$\epsilon^{\mathrm{h}} = \epsilon^{\mathrm{c}} + n^2\epsilon^{\mathrm{T}}, \tag{1}$$

where ϵ^{c} is the cold plasma dielectric tensor and is independent of both n and v/c whilst ϵ^{T} is independent of n and proportional to $(v/c)^2$ (Akhiezer *et al.* 1975; p. 232). So $n^2\epsilon^{\mathrm{T}}$ is the first-order nonrelativistic thermal correction (giving finite Larmor radius effects) to the cold plasma dielectric tensor.

(i) *Conditions for Validity*

The approximations made here place some constraints on the validity of equation (1). Firstly, in order to expand in terms of the small parameter λ we require $\lambda \ll 1$. For the cases of interest, $\omega \approx \Omega_{\mathrm{e}}$ and $\sin^2\theta \sim 1$ which imply that this requirement is satisfied provided $|n^2| \ll (c/v)^2$. In addition, the expansion of the real part of the plasma dispersion function as a power series in λ is only valid when $|z^2| \gg 1$. The power series is approximated by its first two terms which should be justified provided the magnitude of the ratio of the third-order term to the second-order term is $\lesssim \frac{1}{10}$. This requires $|3/2z^2| \lesssim \frac{1}{10}$ or

$$|\omega/\Omega_e - 1| \gtrsim 30^{\frac{1}{2}}|n\cos\theta|\,v/c\,. \tag{2}$$

The other approximation made in the derivation of equation (1) generally imposes a more stringent requirement on $|\omega/\Omega_e - 1|$ when $\theta \approx 90°$. Calculations of maser action for cold background plasmas indicate that for growth to occur θ must generally lie 'near' 90° (within $\sim 30°$). The approximation is to neglect all relativistic mass dependence—most importantly, that of the resonant denominator in the formulation given by Melrose (1980*a*; p. 41), namely $\omega - s\Omega_e/\gamma - k_{\|}\, v_{\|} + \mathrm{i}0$ where the subscript $\|$ denotes components parallel to the magnetic field. If the relativistic effects are small then $\gamma^{-1} \approx 1 - v^2/2c^2$ and the resonant denominator may be approximated by $\omega - s\Omega_e - k_{\|}\, v_{\|} + s\Omega_e\, v^2/2c^2 + \mathrm{i}0$. In the case of perpendicular propagation where $k_{\|}$ and hence the longitudinal Doppler term $k_{\|}\, v_{\|}$ are zero, the only velocity dependence of the denominator is that due to relativistic mass effects.

We assume that the relativistic effects are important over essentially the same frequency range for all the propagation angles of interest (i.e. $|\theta - 90°| \lesssim 30°$) to *estimate* the frequency regions for which the FLR approximation (1) is sufficient.

Calculations of the weakly relativistic dielectric tensor by Dnestrovskii *et al.* (1964) and Shkarofsky (1966) show that for perpendicular propagation and $\lambda \ll 1$, $\boldsymbol{\epsilon}$ can be expanded to first order in λ; it follows that $\boldsymbol{\epsilon}$ can be written in the form

$$\boldsymbol{\epsilon}(\theta = 90°) = \boldsymbol{\epsilon}^{(0)} + n^2 \boldsymbol{\epsilon}^{(1)}\,, \tag{3}$$

where $\boldsymbol{\epsilon}^{(0)}$ and $\boldsymbol{\epsilon}^{(1)}$ are independent of n (Robinson 1986). In contrast with the nonrelativistic (FLR) result (1), the hermitian parts of $\boldsymbol{\epsilon}^{(0)}$ and $\boldsymbol{\epsilon}^{(1)}$ both contain thermal contributions; $\boldsymbol{\epsilon}^{\mathrm{c}}$ is, by definition, independent of thermal effects. In addition, whilst $\boldsymbol{\epsilon}^{\mathrm{c}}$ and $\boldsymbol{\epsilon}^{\mathrm{T}}$ both exhibit a singularity at Ω_e—due to terms dependent on $(\omega^2 - \Omega_e^2)^{-j}$ for $j = 1, 2, 3$ arising from the nonrelativistic resonant denominator—neither $(\boldsymbol{\epsilon}^{(0)})^{\mathrm{h}}$ nor $(\boldsymbol{\epsilon}^{(1)})^{\mathrm{h}}$ exhibits such a singularity because the resonance is 'smeared out' by the relativistic Doppler effect.

For 'warm' plasmas, at frequencies outside a narrow band centred on the gyrofrequency, the functions occurring in $(\boldsymbol{\epsilon}^{(0)})^{\mathrm{h}}$ and $(\boldsymbol{\epsilon}^{(1)})^{\mathrm{h}}$ can be approximated by their asymptotic forms, yielding the FLR approximation $(\boldsymbol{\epsilon}^{(0)})^{\mathrm{h}} = \boldsymbol{\epsilon}^{\mathrm{c}}$ and $(\boldsymbol{\epsilon}^{(1)})^{\mathrm{h}} = \boldsymbol{\epsilon}^{\mathrm{T}}(\theta = 90°)$. We assume that the asymptotic approximation is valid provided the magnitude of the ratio of the second term in the asymptotic expansion to the first term is $\lesssim \frac{1}{10}$. Then at $\theta = 90°$, the FLR approximation to the dielectric tensor (equation 1) is valid if $\omega \approx \Omega_e$, $\lambda \ll 1$ and

$$|\omega/\Omega_e - 1| \gtrsim 25(v/c)^2\,. \tag{4}$$

In summary, at $T = 2\times10^6$ K corresponding to the temperature of the solar corona, $v/c = 0{\cdot}0184$ and we expect equation (1) to be valid provided $\omega \approx \Omega_e$, $|n| \ll 50$, $|\omega/\Omega_e - 1| \gtrsim \frac{1}{10}|n\cos\theta|$ and $|\omega/\Omega_e - 1| \gtrsim 0{\cdot}01$.

(b) Dispersion equation and dispersion relations

The general dispersion equation may be written as $An^4 - Bn^2 + C = 0$, where A, B and C are functions of the dielectric tensor elements (see for example Melrose

1980*a*; Ch. 2). The coefficients A, B and C may be expanded to first order in $n^2(v/c)^2$ using equation (1) and then written as

$$A = A_0 + n^2 A_1\,, \qquad B = B_0 + n^2 B_1\,, \qquad C = C_0 + n^2 C_1\,,$$

where the subscripted quantities are independent of n; A_0, B_0 and C_0 are independent of v/c whilst A_1, B_1 and C_1 are proportional to $(v/c)^2$; explicit expressions for these six quantities were derived by Sitenko and Stepanov (1957). The dispersion equation is then

$$A_1\, n^6 + (A_0 - B_1)\, n^4 + (C_1 - B_0)\, n^2 + C_0 = 0 \tag{5}$$

which is cubic in n^2. The important features of this equation are that the leading coefficient A_1 is a purely thermal term, being proportional to $(v/c)^2$, whilst the constant term C_0 is independent of the FLR corrections. In the cold plasma limit $v/c = 0$ and equation (5) reduces to a quadratic equation in n^2,

$$A_0\, n^4 - B_0\, n^2 + C_0 = 0\,, \tag{6}$$

the roots of which give the dispersion relations for the well-known magnetoionic modes.

The thermal dispersion equation (5) is of third order in n^2, in contrast with the cold plasma dispersion equation (6) which is quadratic. In principle this implies that thermal effects lead to an extra wave mode. However, at most frequencies the modifications may be regarded as changes to the topology of the cold plasma dispersion curves.

The topology of the solutions to dispersion equations such as (5) and (6) is determined by the cutoffs (where $n^2 = 0$) and resonances (where $|n^2|$ goes to infinity).

(i) *Cutoffs*

For a cold plasma, equation (6) implies that cutoffs occur where $C_0/A_0 = 0$; this only occurs where $C_0 = 0$. (We note that A_0 and C_0 both become infinite at the gyrofrequency in such a way that C_0/A_0 tends to a non-zero limit so no cutoff occurs at Ω_e.) Analytic expressions for the frequencies of the three cutoffs which occur in the magnetoionic modes may be obtained by solving the equation $C_0 = 0$ (see for example Melrose 1980*b*; p. 259).

When FLR effects are included, equation (5) implies that cutoffs occur where $C_0/A_1 = 0$. In general, A_1 is finite and non-zero at frequencies for which $C_0 = 0$ so the cold plasma cutoffs are unchanged by FLR corrections. This is consistent with the expansion in λ since, at a cutoff, $n = 0$ implies $\lambda = 0$ and ϵ^{h} reduces to ϵ^{c} (see equation 1). In addition, we find that A_1 becomes infinite at the gyrofrequency in such a way that

$$\lim_{\omega \to \Omega_e} (C_0/A_1) = 0\,,$$

which suggests that there is a cutoff at $\omega = \Omega_e$. However, this cutoff is unphysical since the necessary inclusion of relativistic effects near $\omega = \Omega_e$ removes the singularity at the gyrofrequency. [It should be noted that the cold plasma cutoffs are modified by relativistic thermal effects. This can be seen by setting $n = 0$ and taking the hermitian parts of the relativistic equation (3) giving $(\epsilon^{\rm h})_{n=0} = (\epsilon^{(0)})^{\rm h}$. Thermal contributions arising from the relativistic mass dependence of the electron gyrofrequency, $\Omega = \Omega_{\rm e}/\gamma$, are contained in $(\epsilon^{(0)})^{\rm h}$; these lead to a decrease in the cutoff frequencies (cf. for example, Winglee 1985; Robinson 1986).]

(ii) *Resonances*

Equation (6) implies that resonances in the magnetoionic modes occur where $A_0 = 0$. This equation may be solved analytically to give the frequencies of the two cold plasma resonances, namely

$$\omega_{\pm}^2(\theta) = \tfrac{1}{2}[(\Omega_{\rm e}^2+\omega_{\rm p}^2)\pm\{(\Omega_{\rm e}^2+\omega_{\rm p}^2)^2 - 4\omega_{\rm p}^2\,\Omega_{\rm e}^2\cos^2\theta\}^{\frac{1}{2}}]\,. \tag{7}$$

Provided $\omega_{\rm p} < \Omega_{\rm e}$ (which we assume for the rest of this discussion), $\omega_{\pm}(\theta)$ satisfy $\omega_{-}(\theta) < \Omega_{\rm e}$ and $\omega_{+}(\theta) > \Omega_{\rm e}$ for all θ.

The resonances in the FLR modes—as determined by equation (5)—occur at frequencies satisfying the equation $A_1 = 0$. This is entirely different from the condition for cold plasma resonances, $A_0 = 0$. It follows that in general $|\,n^2\,|$ does not become infinite at $\omega = \omega_{\pm}(\theta)$ when thermal effects are included; at these frequencies $|\,n^2\,|$ becomes very large but remains finite (Akhiezer *et al.* 1975; p. 232).

The equation, $A_1 = 0$, can be written as a quartic equation in $(\omega/\Omega_{\rm e})^2$ with coefficients dependent only on θ (i.e. independent of $\omega_{\rm p}/\Omega_{\rm e}$ and v/c). This quartic equation is found to have only two real solutions which we call $\omega_{\pm}^{\rm T}(\theta)$ and which satisfy $\omega_{-}^{\rm T}(\theta) < \Omega_{\rm e}$ and $\omega_{+}^{\rm T}(\theta) > \Omega_{\rm e}$ for all θ. The resonances at $\omega_{-}^{\rm T}(\theta)$ and $\omega_{+}^{\rm T}(\theta)$ occur in the modes which are modified versions of the cold plasma whistler (or slow ordinary) and z (or slow extraordinary) modes respectively.

We note that the approximations used to derive equation (5) break down at frequencies very close to a resonance because $|\,n^2\,| \ll (c/v)^2$ is no longer satisfied. However, for 'warm' plasmas with $(c/v)^2 \gg 1$, $\omega_{\pm}^{\rm T}(\theta)$ provide very good estimates of the true resonant frequencies and are useful in determining the topology of the dispersion curves.

(iii) *Thermal Modifications to the Plasma Modes*

At frequencies away from the cold plasma resonances and the gyrofrequency, two of the modes determined by equation (5) have almost exactly the same values of n^2 as the two modes determined by equation (6), whilst the third mode appears with very large values of $|\,n^2\,|$. At these frequencies then two thermal modes correspond to the cold plasma ordinary and extraordinary modes; the third mode does not satisfy the requirement $|\,n^2\,| \ll (c/v)^2$ and cannot be described by using this approximation. At frequencies very near $\Omega_{\rm e}$ we expect the behaviour of the cold plasma wave modes to be modified significantly by relativistic mass effects.

There is a small range of frequencies about the cold plasma resonances at $\omega_{\pm}(\theta)$ in which all three solutions of equation (5) satisfy the requirement $|\,n^2\,| \ll (c/v)^2$. Thus there are three physical wave modes in these frequency ranges.

The modifications to the cold plasma wave modes at frequencies near $\omega_{-}(\theta)$ may be briefly summarized as follows:

(i) The z mode, which is continuous at $\omega = \omega_{-}(\theta)$, is essentially unchanged by FLR effects.

(ii) The o mode (fast ordinary mode) no longer exhibits a resonance at $\omega = \omega_-(\theta)$. The dispersion curve for this mode is continuous at $\omega = \omega_-(\theta)$ with n^2 remaining large and negative for $\omega < \omega_-(\theta)$ and going to $-\infty$ as ω approaches zero.

(iii) The dispersion curve for the modified whistler mode is continuous at $\omega = \omega_-(\theta)$ since the thermal resonance frequency satisfies $\omega_-^{\mathrm{T}}(\theta) > \omega_-(\theta)$ for all θ ($\omega_{\mathrm{p}} < \Omega_{\mathrm{e}}$); n^2 remains large and positive before going to $+\infty$ at $\omega = \omega_-^{\mathrm{T}}(\theta)$.

The modifications to the cold plasma wave modes at frequencies near $\omega_+(\theta)$ are relevant to the discussion of growth in Section 3. The modified z and x (slow and fast extraordinary) modes show two distinct types of behaviour depending on the propagation angle θ (as suggested by Akhiezer *et al.* 1975). The two situations arise because while $\omega_+(\theta)$ increases monotonically from Ω_{e} at $\theta = 0°, 180°$ to $\omega_{\mathrm{UH}} = (\Omega_{\mathrm{e}}^2+\omega_{\mathrm{p}}^2)^{\frac{1}{2}}$ at $\theta = 90°$, $\omega_+^{\mathrm{T}}(\theta)$ decreases monotonically from $2\Omega_{\mathrm{e}}$ to Ω_{e}. So $\omega_+^{\mathrm{T}}(\theta) < \omega_+(\theta)$ for some small range of angles near $\theta = 90°$. Empirically we find $\omega_+^{\mathrm{T}}(\theta) \gtrless \omega_+(\theta)$ for $|\theta - 90°| \gtrless \delta\theta$, where

$$\delta\theta = m\{(1+\omega_{\mathrm{p}}^2/\Omega_{\mathrm{e}}^2)^{\frac{1}{2}} - 1\}$$

and $m \approx 55$ is independent of v/c.

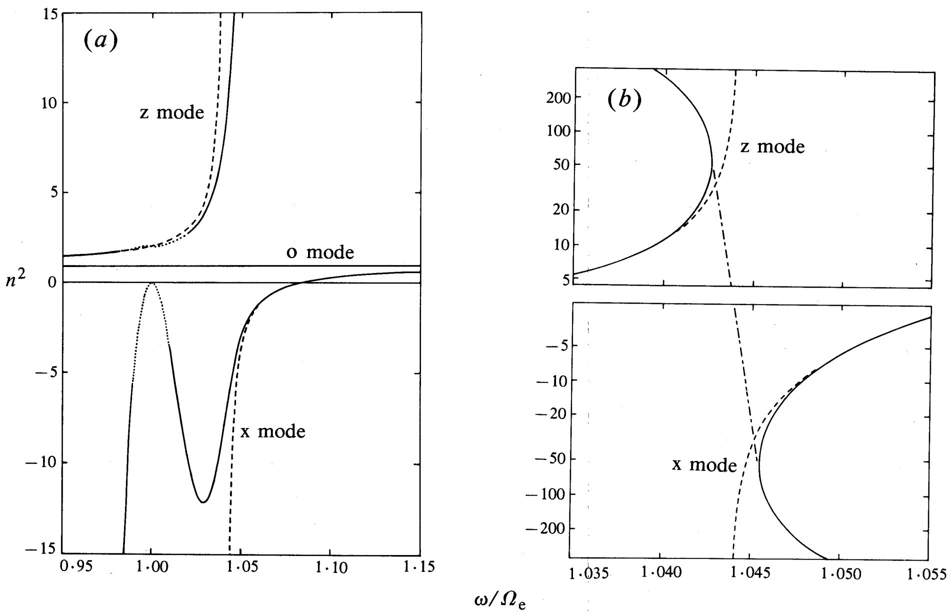

Fig. 1. Extraordinary mode dispersion relations including FLR effects (solid curves) for a background plasma temperature of 2×10^6 K with $\omega_{\mathrm{p}}/\Omega_{\mathrm{e}} = 0\cdot3$ and (*a*) $\theta = 105°$ and (*b*) $\theta = 91°$. In (*a*) the dotted portions near the gyrofrequency show the regions in which relativistic mass effects need to be considered. In (*b*) the alternating dashes indicate the region in which the modes are mutual complex conjugates (in n^2). The corresponding cold plasma modes (dashed curves) are shown for comparison.

The modifications to the extraordinary mode dispersion curves in the cases $\omega_+^{\mathrm{T}}(\theta) > \omega_+(\theta)$ and $\omega_+^{\mathrm{T}}(\theta) < \omega_+(\theta)$ are shown in Figs 1*a* and 1*b* respectively. The behaviour of the wave modes may be summarized as follows:

(i) The o mode, which is continuous at $\omega = \omega_+(\theta)$, is essentially unchanged by FLR effects.

(ii) The modified z mode exhibits a resonance at $\omega = \omega_+^{\mathrm{T}}(\theta)$. If $\omega_+^{\mathrm{T}}(\theta) > \omega_+(\theta)$ the dispersion curve for this mode is continuous at $\omega = \omega_+(\theta)$ with n^2 remaining large and positive for $\omega > \omega_+(\theta)$ and going to $+\infty$ at $\omega = \omega_+^{\mathrm{T}}(\theta)$—see Fig. 1*a*. If $\omega_+^{\mathrm{T}}(\theta) < \omega_+(\theta)$ the modified z-mode dispersion curve passes continuously through $\omega = \omega_+^{\mathrm{T}}(\theta)$ but 'bends back on itself' before reaching $\omega_+(\theta)$—see Fig. 1*b*. The dispersion relation is double-valued for a small range of frequencies above $\omega_+^{\mathrm{T}}(\theta)$.

(iii) If $\omega_+^{\mathrm{T}}(\theta) > \omega_+(\theta)$ the modified x-mode dispersion curve is continuous at $\omega = \omega_+(\theta)$ and displays an apparent (non-physical) cutoff at the gyrofrequency—see Fig. 1*a*. If $\omega_+^{\mathrm{T}}(\theta) < \omega_+(\theta)$ the dispersion curve 'bends back on itself' just above $\omega_+(\theta)$ and the dispersion relation is double-valued for some range of frequencies—see Fig. 1*b*.

Behaviour similar to that depicted in Fig. 1*b* also occurs (at perpendicular propagation) when relativistic effects are included—see Fig. 5 of Robinson (1986).

3. Growth

(a) Kinematics

The cyclotron resonance condition for the fundamental, $\omega - \Omega_{\mathrm{e}}/\gamma - k_{\|}\, v_{\|} = 0$, may be represented by an ellipse in velocity ($v_{\|}-v_{\perp}$) space (Hewitt *et al.* 1981). The condition for the resonant ellipse to have a non-vanishing semi-major axis implies the kinematic requirement $\omega^2 - \Omega_{\mathrm{e}}^2 - k_{\|}^2\, c^2 > 0$. The boundary of the region in ω–θ space for which resonance is possible may therefore be obtained by changing variables and replacing the inequality by an equality, giving

$$n^2 = (\omega - \Omega_{\mathrm{e}}^2)/\omega^2 \cos^2\theta\,, \tag{8}$$

where n^2 is determined by the appropriate dispersion relation (Hewitt and Melrose 1983).

Alternatively, if equation (8) is substituted into the dispersion equation (5) to eliminate n^2, the resulting equation may be written in the form

$$a \cos^6\theta + b\cos^4\theta + c\cos^2\theta + d = 0\,, \tag{9}$$

where the coefficients are functions of v/c, $\omega_{\mathrm{p}}/\Omega_{\mathrm{e}}$ and $\omega/\Omega_{\mathrm{e}}$. When v/c and $\omega_{\mathrm{p}}/\Omega_{\mathrm{e}}$ are treated as fixed parameters the solutions of equation (9) give the 'growth boundaries' in ω–θ space for the three wave modes determined by equation (5). The constant coefficient d is proportional to $(v/c)^2$ so in the cold plasma limit of $v/c = 0$, $d \equiv 0$ and equation (9) reduces to the quadratic equation

$$a_0 \cos^4\theta + b_0\cos^2\theta + c_0 = 0\,, \tag{10}$$

where

$$a_0 = P(S^2 - D^2) + \{S(S-P) - D^2\}(1 - Y^2),$$

$$b_0 = \{D^2 - S(S+P) + (1 - Y^2)(P - S)\}(1 - Y^2),$$

$$c_0 = S(1 - Y^2)^2;$$

$$P = 1 - (\omega_p/\Omega_e)^2\, Y^2, \qquad S = 1 - (\omega_p/\Omega_e)^2\, Y^2/(1 - Y^2),$$

$$D = -(\omega_p/\Omega_e)^2\, Y^3/(1 - Y^2), \qquad Y = \Omega_e/\omega.$$

(b) Numerical results

In this subsection we describe the modifications due to thermal (FLR) effects to the growth boundaries and growth rates due to maser emission at the fundamental. The cold plasma cases have been discussed in detail for the o, x and z modes by Hewitt *et al.* (1982; hereafter referred to as HMR), Hewitt and Melrose (1983; hereafter HM) and Hewitt *et al.* (1983; hereafter HMD). Our method of calculation is that described in HMR and our growth rates are for the distribution function defined in equations (11) and (12) of that paper. Specifically, we consider a hot Maxwellian distribution with a hole at pitch angles $\alpha > \alpha_0$ and with the distribution falling off as $[\sin\{\frac{1}{2}\pi(\pi-\alpha)/(\pi-\alpha_0)\}]^N$ for $\alpha > \alpha_0 > \frac{1}{2}\pi$. Here we consider only $N = 6$ and $\alpha_0 = 150°$ with the other parameters as chosen in HMR.

Since maser emission at the fundamental occurs at frequencies just above Ω_e (and above the cutoff frequency in the case of the x mode), the relevant modifications to the dispersion relations due to the thermal background plasma are those illustrated in Figs 1*a* and 1*b*.

(i) *The o mode*

We find that the o-mode growth boundaries and growth rates are essentially unchanged by the FLR effects of a thermal background plasma at a temperature of 2×10^6 K. This is consistent with the agreement between the dispersion relations for the o mode and the corresponding thermal mode in the frequency range shown in Fig. 1*a*.

(ii) *The x mode*

The growth properties of the x mode for a cold background plasma have been treated in some detail by HMR and by HM. The general shape of the x-mode growth boundary is shown in Fig. 14 of HMR and the fact that growth occurs in two bands is discussed in HM—see their Fig. 1. Our calculations show that when thermal effects are included the x-mode growth boundary is shifted to larger values of $|\cos\theta|$. The change is very small along the 'upper arm' of the boundary but increases near the 'nose' and along the 'lower arm', i.e. at frequencies approaching the cutoff. The shape of the boundary is such that if we consider the temporal growth rate maximized over frequency $\Gamma_{\max}/\Omega_e$, as a function of angle, the thermal effects appear quite large at angles near the nose. However, changes to $\Gamma_{\max}/\Omega_e$ maximized over θ, as a function of ω/Ω_e, are much smaller. An example of the x-mode growth rates, as a function of θ, calculated according to the cold plasma and FLR approximations, is presented

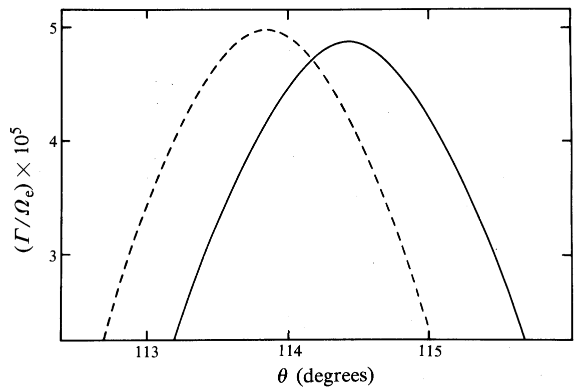

Fig. 2. Temporal growth rates, as a function of θ, for the x mode at the fundamental with $\omega_p/\Omega_e = 0\cdot15$ and $\omega/\Omega_e = 1\cdot0325$. The solid curve shows the growth rate including FLR effects for a background plasma temperature of 2×10^6 K and the dashed curve shows the cold plasma results for comparison.

in Fig. 2. In general, the growth rate maximized over both ω/Ω_e and θ decreases when thermal effects are included but the change is typically $\lesssim 10\%$. The shift of the growth boundary to larger values of $|\cos\theta|$ does lead to a shift in the values of ω/Ω_e and θ at which the maximum growth rate occurs (see Fig. 2). This shift is most significant in the lower band due to its extremely narrow bandwidth (see HM). The changes in the position of the growth boundary and of the maximum growth rate in θ space due to a thermal background plasma at 2×10^6 K, are presented in Fig. 3*a* for $\omega_p/\Omega_e = 0\cdot15$. Calculations for ω_p/Ω_e in the range $0\cdot1$–$0\cdot3$ have shown that the corrections due to thermal effects do not strongly depend on the plasma frequency.

(iii) *The z mode*

The growth properties of the z mode for a cold plasma background have been discussed in detail by HMD; the growth boundaries and the existence of two bands are shown in their Fig. 1. Our calculations show that when thermal effects are included the z-mode growth boundary is shifted to larger values of $|\cos\theta|$, as was the x-mode boundary. In contrast with the x mode, the z-mode growth boundary is not significantly changed along the lower arm, but thermal effects increase near the nose and along the upper arm, i.e. at frequencies approaching the upper hybrid frequency ω_{UH}. Once again, the change in the maximum growth rate is generally small with decreases of $\lesssim 10\%$ being typical. Fig. 3*b* gives the results for the z mode at $\omega_p/\Omega_e = 0\cdot35$. The major point to note here is that as ω approaches ω_{UH} the difference between the angles at which the growth boundary and maximum growth occur for a cold background decreases to zero. This is because the region between the growth boundary and the cold plasma resonance at $\omega_+(\theta)$ becomes very narrow as ω approaches ω_{UH} (see Fig. 1, HMD). This forces the maximum growth to occur at angles closer and closer to the growth boundary until in the limit of $\omega = \omega_{UH}$ the two are coincident. That this behaviour is modified by the FLR effects is obvious from the sharp 'turnover' of the two relevant curves in Fig. 3*b*. This 'turnover' is due to the qualitative change when FLR effects are included in the dispersion of the z mode at frequencies near $\omega_+(\theta)$ and for propagation angles close to 90°.

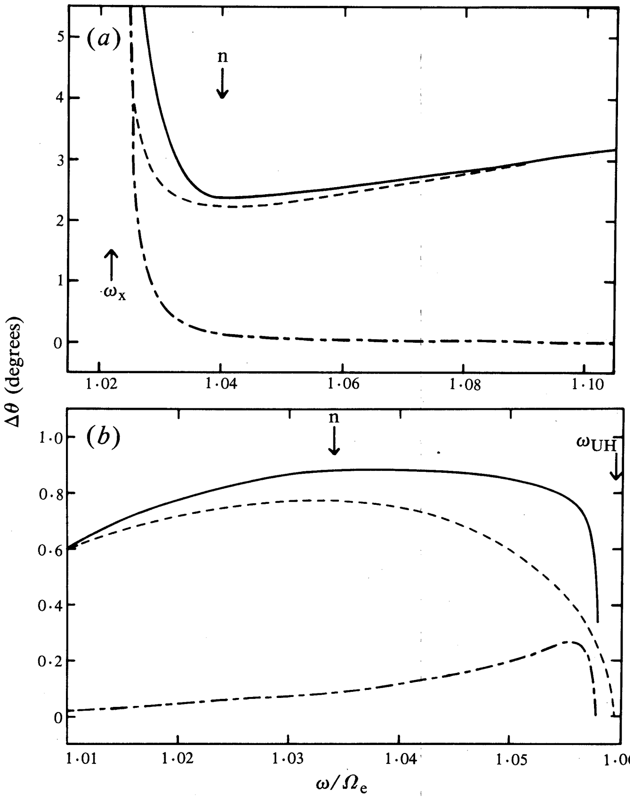

Fig. 3. Positions of the growth boundaries and maximum temporal growth rates for (*a*) the x mode at the fundamental with $\omega_p/\Omega_e = 0\cdot15$ and (*b*) the z mode at the fundamental with $\omega_p/\Omega_e = 0\cdot35$. The magnitude of an angular shift to larger values of $|\cos\theta|$ is denoted by $\Delta\theta$, measured relative to the cold plasma growth boundary. The curves of alternating dashes are the FLR growth boundaries and the solid curves are the positions of the maximum temporal growth rate Γ_{max}/Ω_e (maximized over θ, as a function of ω/Ω_e) for the FLR mode. The dashed curves show the position of maximum growth for the cold plasma mode for comparison. In (*a*) 'ω_x' denotes the x-mode cutoff and in (*b*) ω_{UH} denotes the upper hybrid frequency. The 'nose' of the growth boundary separating the upper and lower bands is denoted by 'n'.

4. Conclusions

In this paper we have investigated the modifications to the cold plasma dispersion relations, and to the growth rates for fundamental cyclotron maser emission, due to the finite Larmor radius effects of a 'warm' background Maxwellian electron plasma. Our analysis is valid at all frequencies below $2\Omega_e$ except in a narrow region centred on Ω_e where a full relativistic treatment is necessary in order to evaluate the thermal corrections. However, for most parameter regimes, there is a frequency band on either side of this central region in which thermal effects are still important but where the FLR approximation is sufficient. At frequencies sufficiently removed from Ω_e the wave modes of the thermal plasma may be approximated by the magnetoionic modes.

We have also shown that although, in principle, the inclusion of FLR effects leads to an extra wave mode, at most frequencies this extra mode appears with very large $|n^2|$ and so cannot be treated by using our method of approximation, nor is it important. At those frequencies where our treatment is valid for all three 'modes', the extra mode appears as a modification to the topology of the cold plasma modes and not as a separate branch.

The FLR modifications do not generally lead to large changes in the growth rates due to fundamental electron cyclotron maser emission because, in most parameter regimes where growth is important, the FLR wave modes are simple modifications of the magnetoionic modes. The results may be briefly summarized as follows:

(i) The FLR modifications to the o-mode dispersion relation and hence to the growth properties are entirely negligible.

(ii) The FLR modifications to the x-mode dispersion relation are very small at frequencies above the cutoff. As a result the maximum growth rates are only marginally affected but there is a shift in the position (in ω–θ space) of maximum growth.

(iii) The FLR modifications to the z-mode dispersion relation at frequencies near the upper cold plasma resonance are the most significant. However, the maximum growth rates are again only slightly affected although there is a small shift in the position of maximum growth.

Our results imply that FLR effects on fundamental electron cyclotron maser emission are likely to be important only in situations where z-mode emission occurs very close to the upper hybrid frequency. In most other cases it appears to be valid to assume that emission occurs in the magnetoionic wave modes.

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