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Is there a Connection between Local and Global (In-)Stability?

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Abstract

We review two criteria which have been used to predict the onset of large scale stochasticity in Hamiltonian systems. We show that one of them, due to Toda and based on a local stability analysis of the equations of motion, is inconclusive. An approach based on the local Riemannian curvature K of trajectories correctly predicts chaos if $K < 0$ everywhere, but no further conclusions can be drawn. New (counter-)examples are provided.

1. Introduction

In recent years a clear picture of the qualitative behaviour of Hamiltonian systems has emerged and a hierarchy of statistical properties has been found (Ford 1973; Berry 1978). The simplest property is ergodicity which applies, for example, to almost all initial conditions on the invariant manifold of integrable systems, so it does not imply chaos. Next, one has weak-mixing and mixing, the latter being required for statistical mechanics to apply. Kolmogorov systems (so-called K systems) and exponentially unstable systems (so-called C systems) differ only in their mathematical definitions (using measure theoretic and metric elements, respectively). The strongest chaotic property presently known is that of a Bernoulli shift which can be immersed in the flow (referred to as B systems). Further discussions can be found in Ford (1973) and Arnold and Avez (1968).

Except for the rare cases in which the Hamiltonian system is integrable and others in which it is chaotic (i.e. K, C or B systems), most systems show a divided phase space. There are regions (the so-called islands) inside of which the motion is quasi-periodic and others which are densely filled by a single trajectory (the so-called stochastic layers). The ratio of chaotic to quasi-periodic regions depends on the parameters of the system. If we take the energy E as the parameter [given by $E = H(p_{i0}, q_{i0})$, where p_{i0} and q_{i0} denote the initial values], we typically have for small energies a situation where the KAM theorem (Moser 1967) applies and the phase space is dominated by quasi-periodic regions (outside integer resonances). With increasing energy the chaotic regions grow in size and within a rather small energy range around a critical energy E_c the system turns to predominantly chaotic behaviour. In other words, the quantity E_c defines the energy onset of substantial irregularity (Hamilton

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and Brumer 1981). For simple but typical Hamiltonian systems the 'width' of the stochastic layers can be approximated by $\exp(-E_c/E)$ (Chirikov 1979; Escande 1982). This explains the dramatic change in the qualitative behaviour of the system.

At present, the only two methods known to yield reliable estimates of the critical energy E_c are the surface of section technique (Berry 1978) and the stability analysis of orbits (Benettin *et al.* 1977). The first method has the disadvantage of being numerically feasible in systems with two degrees of freedom only. In this method one follows the successive crossings of the trajectories through a surface intersecting the energy shell, for example, the (p_2, q_2) plane at the point $q_1 = 0$. If after a sufficient number of iterates, the resulting points form a closed curve, called an invariant curve, the trajectory corresponding to them lies on an invariant torus or KAM surface. If, instead, these points are dense in a two-dimensional area in the plane, then the trajectory corresponding to them is irregular. Thus for a fixed energy E we are normally forced to calculate these points for a sufficiently high number of different initial values. The second method requires time-consuming numerical calculations for many orbits. One has to find initial conditions in the chaotic region. Moreover, one has the difficulty of deciding between marginal stability (Casati *et al.* 1980) for integrable systems and exponential growth perhaps with a small exponent. Consequently, an analytical estimate for the critical energy E_c would be highly desirable.

Three different approaches have been proposed: one based on the local curvature (Arnold 1978; Van Velsen 1978), one based on a stability analysis of the equation of motion (Toda 1974; Brumer and Duff 1976; Benettin *et al.* 1977; Tabor 1981; Steeb and Kunick 1985; Steeb *et al.* 1985), and one based on the Mori projector formalism (Mo 1972). The last method, although analytical in its origin, requires extensive numerics to compute phase space averages and will not be considered further. In addition, the arguments given by Mo (1972) are incomplete (compare Tabor 1981; Marchesoni *et al.* 1982). The remaining two methods are local versions of valid global methods which study the stability of single orbits, i.e. they are evaluated along a reference orbit. This dependence on the reference orbit makes them intractable to analytical methods. Thus a natural approximation would be to disregard the reference orbit and study them for all points of the phase space. That this is more than just a 'dubious step', as noted by Tabor (1981), will become evident from discussions in Section 3.

2. Questions

From the discussion in the Introduction, one is led to investigate the following questions:

(i) Does local instability everywhere imply global instability?

(ii) Does local stability everywhere imply global stability?

If neither is the case, one would like to ask:

(iii) Does the existence of regions of local instability imply the occurrence of a divided phase space?

In the next section we discuss each of these questions for the two criteria mentioned in the Introduction, i.e. local curvature and stability analysis of the equation of motion.

3. Examples and Counter-examples

(a) Local Curvature

For the curvature criterion we write the Hamiltonian function as

$$H(q, \dot{q}) = \tfrac{1}{2} \sum_{i,j=1}^{n} a_{ij}(q)\, \dot{q}_i\, \dot{q}_j + U(q). \tag{1}$$

The solutions $q_i(t)$ are not only extremes of Hamilton's variational principle but also of the Maupertuis–Euler–Lagrange–Jacobi principle (Synge 1926)

$$\delta \int_{M_0}^{M_1} \left(2\{E - U(q)\} \sum_{i,j=1}^{n} a_{ij}(q)\, \mathrm{d}q_i\, \mathrm{d}q_j \right)^{\frac{1}{2}} = 0, \tag{2}$$

where M_0 and M_1 are the endpoints of the trajectory. This, however, may be interpreted as the variational equation for geodesics in a space with the metric element

$$\mathrm{d}s^2 = \sum_{i,j=1}^{n} g_{ij}\, \mathrm{d}q_i\, \mathrm{d}q_j \tag{3}$$

and the coefficients

$$g_{ij} = 2\{E - U(q)\}\, a_{ij}(q). \tag{4}$$

Now the behaviour of nearby geodesics is a question well studied in differential geometry (Synge 1926). The evolution of the difference η is given by the solutions to the Jacobi equation

$$\mathrm{D}^2\eta/\mathrm{d}t^2 = -K(q(t))\,\eta, \tag{5}$$

where $\mathrm{D}/\mathrm{d}t$ denotes the covariant derivative. Now the case of interest deals with initial conditions perpendicular to an orbit. Then the covariant derivative becomes an ordinary one and we can write

$$\mathrm{d}^2\eta/\mathrm{d}t^2 = -K(q(t))\,\eta, \tag{6}$$

where $K(q(t))$ is the Riemannian curvature calculated along the orbit. Assuming that K varies slowly with time t we have (at least locally and for short times) exponential instability for $K < 0$ and oscillatory behaviour for $K > 0$. The local approach now disregards the dependence on the orbits and defines stability (at a particular point) as $K > 0$ and instability as $K < 0$. With these preparations we can answer the questions in Section 2:

(i) It is known that a local curvature bounded from above by a negative number implies strong stochastic properties such as K, C, or even B systems (Anosov 1967). Thus the answer to question (i) is affirmative.

(ii) That local stability does not imply global stability is most easily seen by considering the problem of the billiard table. The discontinuities along the boundaries can be smoothed out by regarding it as the limit of motion on the surface of a three-dimensional body with one cross section equal to the billiard and a very small extension perpendicular to it (Arnold and Avez 1968). Bunimovich (1979) proved that two-dimensional billiards in domains bounded by straight lines and arcs of a circle are B systems if they satisfy certain geometrical conditions. A famous example of this class is the stadium, formed by two semicircles with equal radii, connected by straight line segments. The curvature of the semicircles is, of course, positive, that of the straight line segments being zero, i.e. $K \geqslant 0$, thus providing a counter-example to question (ii). An analytical counter-example is provided by the quartic Hamiltonian

$$H(p, q) = \tfrac{1}{2}(p_1^2 + p_2^2) + \tfrac{1}{2} q_1^2\, q_2^2 \,. \tag{7}$$

A simple computation yields

$$K(q) = (q_1^2 + q_2^2) \frac{E + U}{(E - U)^3}\,, \tag{8}$$

where $U(q) = \frac{1}{2} q_1^2\, q_2^2$ and consequently $K > 0$ except for $q_1 = q_2 = 0$. In a careful numerical analysis, Carnegie and Percival (1984) have shown that the system is not integrable; as a matter of fact no islands could be located. The equations of motion scale invariantly under $t \rightarrow \alpha^{-1} t$, $q_i \rightarrow \alpha q_i$, $p_i \rightarrow \alpha^2 p_i$. Corresponding to any motion of energy E, there is a similar motion for all other energies. The properties of the motion for any energy can be determined by simple scaling from the properties of the motion on the energy shell $E = 1$. A detailed analysis (singular point analysis, Toda–Brumer criterion, quantum chaos) of this system has been given by Steeb *et al.* (1985).

(iii) Counter-examples to the expectation expressed in question (iii) are easily found. One may take, for example, the billiard ball in an annulus (angular momentum is conserved) or radially symmetric potentials with a sufficiently strong inflection point.

In summary, only in the case $K \leqslant -a^2 < 0$ everywhere can we expect the local property (namely instability) to carry over into global behaviour.

(b) Stability Analysis of the Equation of Motion

We consider now the Toda (1974) criterion for a Hamiltonian with two degrees of freedom. More specifically, let us consider

$$H(p, q) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + U(q)\,, \tag{9}$$

with equations of motion

$$\dot{p}_i = -\partial U/\partial q_i\,, \qquad \dot{q}_i = p_i/m_i\,. \tag{10}$$

Linearizing around a solution $\{q(t), p(t)\}$ we obtain the variational equations

$$\begin{bmatrix} \delta \dot{q}_1 \\ \delta \dot{q}_2 \\ \delta \dot{p}_1 \\ \delta \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1/m_1 & 0 \\ 0 & 0 & 0 & 1/m_2 \\ -U_{q_1 q_1} & -U_{q_1 q_2} & 0 & 0 \\ -U_{q_2 q_1} & -U_{q_2 q_2} & 0 & 0 \end{bmatrix}_{\{p(t),\, q(t)\}} \begin{bmatrix} \delta q_1 \\ \delta q_2 \\ \delta p_1 \\ \delta p_2 \end{bmatrix}, \tag{11}$$

where indices on U indicate partial derivatives, e.g. $U_{q_1 q_2} = \partial^2 U/\partial q_1 \partial q_2$. We let $\mathbf{M}$ be the matrix on the right-hand side of (11). Again, dropping the dependence on the orbit, we compute the eigenvalues of the 4×4 matrix $\mathbf{M}$ and define local stability as all eigenvalues having a negative real part, and local instability as at least one eigenvalue having a positive real part. We obtain the secular equation

$$\lambda^4 + (U_{q_1 q_1}/m_1 + U_{q_2 q_2}/m_2)\lambda^2 + (1/m_1 m_2)(U_{q_1 q_1} U_{q_2 q_2} - U^2_{q_1 q_2}) = 0, \tag{12}$$

which is a quadratic equation in $\mu = \lambda^2$. The condition for local stability now becomes

$$U_{q_1 q_1}/m_1 + U_{q_2 q_2}/m_2 > 0, \tag{13}$$

$$U_{q_1 q_1} U_{q_2 q_2} - U^2_{q_1 q_2} > 0. \tag{14}$$

We note that the gaussian curvature of the potential U is given by

$$K_g(q) = \frac{U_{q_1 q_1} U_{q_2 q_2} - U^2_{q_1 q_2}}{\{1 + (U_{q_1})^2 + (U_{q_2})^2\}}, \tag{15}$$

thus relating local stability to the convexity of the potential surface. Toda (1974) applied this criterion successfully to the Hénon Heiles system. He found that at energy $E \leqslant E_c = 1/12$ no inflection point appears inside the region in configuration space accessible to a trajectory. However, it also hints that the argument is incorrect, because at energies below E_c stochastic layers already exist, although exponentially small. This is shown dramatically by the unequal mass Toda lattice

$$H(p, q) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \exp(q_1) + \exp\{-(q_1 - q_2)\} + \exp(-q_2). \tag{16}$$

Both conditions for local stability are satisfied, yet the system shows large stochastic regions. How this comes about is best illustrated by the example given in Appendix A in Tabor (1981). There the exponential instability is linked to resonance phenomena, where the discontinuities are not crucial, but make the calculation easier. A continuous model is given by

$$H(p, q) = \tfrac{1}{2}p^2 + \{\tfrac{1}{2}(\omega_1 + \omega_2) + \tfrac{1}{2}(\omega_2 - \omega_1)\cos t\} q^2, \tag{17}$$

where $(\omega_1 + \omega_2) > (\omega_2 - \omega_1) > 0$. This leads to Mathieu's equation, but the result remains the same. Whenever ω_1 and ω_2 are adjusted such that the coefficients in equation (17) lie in an unstable band of Mathieu's equation, the motion is unstable and exponentially separating. The basic ingredient in the discrete model is that the product of two matrices with purely imaginary eigenvalues may have real and positive eigenvalues.

This suggests that the converse might be possible too, i.e. a product of two matrices with real eigenvalues may yield a matrix with imaginary eigenvalues. Thus we are led to the following counter-example to question (ii). Let us consider the Hamiltonian

$$H(p, q) = \tfrac{1}{2}(p-aq)^2 - \tfrac{1}{2}\omega^2 q^2 . \tag{18}$$

The equations of motion are

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} -a & 1 \\ \omega^2 - a^2 & a \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \tag{19}$$

and the general solution is given by

$$\begin{bmatrix} q(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \cosh(\omega t) - (a/\omega)\sinh(\omega t) & (1/\omega)\sinh(\omega t) \\ (\omega^2 - a^2)(1/\omega)\sinh(\omega t) & \cosh(\omega t) + (a/\omega)\sinh(\omega t) \end{bmatrix} \begin{bmatrix} q_0 \\ p_0 \end{bmatrix}. \tag{20}$$

If we let $\mathbf{M}(a, t)$ be the matrix on the right-hand side of (20), then $\det \mathbf{M}(a, t) = 1$. Also, if we let $a = 0$ for a period T_1 and let $a > 0$ for a period T_2, then the time evolution of an initial point (p_0, q_0) after a time $T_3 = T_1 + T_2$ is given by the product $\mathbf{M}_3 = \mathbf{M}_2 \mathbf{M}_1$, where $\mathbf{M}_1 = \mathbf{M}(a{=}0, \ t{=}T_1)$ and $\mathbf{M}_2 = \mathbf{M}(a{>}0, \ t{=}T_2)$. Since $\det \mathbf{M}_1 = \det \mathbf{M}_2 = \det \mathbf{M}_3 = 1$, the eigenvalues of $\mathbf{M}_3$ are determined by $\operatorname{tr} \mathbf{M}_3$, namely real eigenvalues for $|\operatorname{tr} \mathbf{M}_3| > 2$ and complex eigenvalues for $|\operatorname{tr} \mathbf{M}_3| < 2$. From the condition $|\operatorname{tr} \mathbf{M}_3| < 2$ it follows that

$$|(1-A^2)\cosh\{\omega(T_1+T_2)\} + A^2\cosh\{\omega(T_1-T_2)\}| < 1, \tag{21}$$

where $A = a/2\omega$. If we let $T_1 = T_2 = T$ and $A^2 = 1+\delta$, then the condition for complex eigenvalues becomes

$$|-\delta\cosh(2\omega T) + 1 + \delta| < 1 \tag{22}$$

or

$$0 < \delta < 2/\{\cosh(2\omega T) - 1\} . \tag{23}$$

To obtain the behaviour for long times, we may write $t = nT_3 + \tau$ for some positive integer n and $0 \leqslant \tau < T_3$. The asymptotic behaviour is governed by the one-dimensional Lyapunov exponent (Eckmann and Ruelle 1985)

$$\lambda = \lim_{T\to\infty} \sup_{|\bar{x}|=1} \frac{1}{T} \ln \frac{|\mathbf{M}(T)\bar{x}|}{|\bar{x}|} . \tag{24}$$

Now $\mathbf{M}(T_3)$ acts like a rotation and $\mathbf{M}(\tau)$ is finite, so asymptotically $\lambda \to 0$ and the motion is stable. Consequently, for suitable a, ω and T the motion becomes stable although it is locally exponentially separating.

There are ample counter-examples to the question (iii), for example, all radially symmetric potentials have an inflection point, thus rendering the Toda (1974) criterion inconclusive in all three cases. We note that both criteria may lead to contradictory predictions. For the Hamiltonian system with the potential $U(q) = \frac{1}{2} q_1^2 q_2^2$ we found $K > 0$, i.e. local stability, whereas the Toda criterion yields local instability everywhere (Steeb and Kunick 1985; Steeb *et al.* 1985).

4. Conclusions

In conclusion we can say that, except in the case of negative Riemannian curvature, there is, in general, no connection between local and global properties. As noted in Section 1 the local criteria were derived from valid global methods so that the above counter-examples emphasize the importance of the reference orbit.

References

Anosov, D. V. (1967). *Proc. Steklov. Math. Inst.* **90**, 1.

Arnold. V. I. (1978). 'Mathematical Methods in Classical Mechanics', Appendix 1 (Springer: New York).

Arnold, V. I., and Avez, A. (1968). 'Ergodic Problems in Classical Mechanics' (Benjamin: New York).

Benettin, G., Brambilla, R., and Galgani, L. (1977). *Physica* A **87**, 381.

Berry, M. V. (1978). *In* 'Topics in Nonlinear Dynamics' (Ed. S. Journa), AIP Conf. Proc., Vol. 46, p. 16.

Brumer, P., and Duff, J. W. (1976). *J. Chem. Phys.* **65**, 3566.

Bunimovich, L. A. (1979). *Commun. Math. Phys.* **65**, 295.

Carnegie, A., and Percival, I. C. (1984). *J. Phys.* A **17**, 801.

Casati, G., Chirikov, B. V., and Ford, J. (1980). *Phys. Lett.* A **77**, 91.

Chirikov, B. V. (1979). *Phys. Rep.* **52**, 263.

Eckmann, J. P., and Ruelle, D. (1985). *Rev. Mod. Phys.* **57**, 617.

Escande, D. F. (1982). *Physica* D **6**, 119.

Ford, J. (1973). *Adv. Chem. Phys.* **24**, 155.

Hamilton, J., and Brumer, P. (1981). *Phys. Rev.* A **23**, 1941.

Marchesoni, F., Sparpaglione, R. S., and Grigolini, P. (1982). *Phys. Lett.* A **88**, 113.

Mo, K. C. (1972). *Physica* **57**, 445.

Moser, J. (1967). *Math. Ann.* **169**, 163.

Steeb, W.-H., and Kunick, A. (1985). *Lett. Nuovo Cimento* **42**, 89.

Steeb, W.-H., Villet, C. M., and Kunick, A. (1985). *J. Phys.* A **18**, 3267.

Synge, A. (1926). *Phil. Trans. R. Soc. London* A **226**, 31.

Tabor, M. (1981). *Adv. Chem. Phys.* **46**, 73.

Toda, M. (1974). *Phys. Lett.* A **48**, 335.

Van Velsen, J. F. C. (1978). *Phys. Lett.* A **67**, 325.

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