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Finite Element Calculations for Non-radial Oscillations of Stars

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Abstract

Application of the finite element method to a variational principle of Chandrasekhar, although not suitable for computing stable g modes of non-radial oscillation of stars, is found to be satisfactory for unstable g modes as well as f and p modes.

1. Introduction

Much effort has been devoted to numerical computation of normal modes of non-radial adiabatic oscillations of stars, which are important in stellar stability theory (Cox 1980; Clement 1984). Andrew (1967) used a variational principle of Chandrasekhar (1964) for this computation and found that, when the continuous piecewise linear coordinate functions of the finite element method were used, the method was successful for the f and p modes but failed even to detect the g modes, yielding instead some completely spurious modes. However, the variational principle did yield g modes when certain polynomial coordinate functions were used. These polynomials span the same space as eigenfunctions of the homogeneous model and, except for models of unrealistically low central condensation, the eigenfunctions obtained for g modes were much less accurate than those obtained for f and p modes (Andrew 1968 and the references therein).

Clement (1984) also noted that spurious modes may be obtained when nonanalytic coordinate functions are used (as in the finite element method) and Sobouti (1977) confirmed that results depend crucially on the choice of coordinate functions. The existing theory of variational methods still does not appear to cover this problem (Andrew 1984), but theoretical studies of some related problems (Andrew 1969, 1970, 1971*a*, 1971*b*) suggest that computed g modes could be expected to be particularly sensitive to the choice of coordinate functions.

Although solutions of this problem can be computed satisfactorily by shooting methods, there are several reasons why further study of Chandrasekhar's variational approach is of much more than theoretical interest. First, variational methods, which are considerably faster than shooting methods, are quite often used for problems of non-radial stellar oscillations and, provided suitable coordinate functions are used, have been entirely successful (Clement 1984). Second, the finite element method is one of the commonest methods used for numerical solution of other eigenvalue problems in physics, and full understanding of the partial failure of the finite element coordinate

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functions for this problem may have application to other problems. In this connection it should be noted that an inappropriate choice of coordinate functions or elements in the finite element method has produced spurious modes in other problems in fluid mechanics (Walters 1983) and magnetohydrodynamics (Rappaz 1981). Finally, Sobouti's (1977) and Clement's (1984) search for coordinate functions suitable for computation of g modes has already highlighted important physical properties of these modes and additional study along these lines may well contribute further to physical understanding.

The computations of Andrew (1967) used a single stellar model—one of a massive star with a convectively neutral core surrounded by a convectively stable envelope (Van der Borght 1964). A single finite element calculation was made by using a grid with 25 equal subintervals. In the present paper similar calculations are made with various grids and different stellar models—polytropes with indices 1 and 3, and two composite models studied by Smeyers (1966) which have a homogeneous core and a polytropic envelope with index $n = 3$.

2. The Variational Principle

Following the method of Andrew (1967), we neglect the perturbation ϕ' of the gravitational potential, as has been done in several other studies, and we consider only the spherical harmonic $l = 2$. In his investigations with polynomial coordinate functions, Andrew (1968) found that neglect of ϕ' did not qualitatively affect the performance of the variational method. Unless ϕ' is neglected, Chandrasekhar's variational principle involves a repeated integral and the finite element matrices do not have the sparsity which is normally one of the main advantages of the finite element method over other variational methods.

With the above approximation, Chandrasekhar's variational principle for a polytrope of index n may be written as

$$\int_0^1 \{f_1(x)\,\xi^2 - 2x f_2(x)\,\xi\,\mathrm{d}\xi/\mathrm{d}x + x^2 f_3(x)(\mathrm{d}\xi/\mathrm{d}x)^2 + f_3(x)\,\eta^2 + 2f_2(x)\,\xi\eta - 2x f_3(x)\,\eta\,\mathrm{d}\xi/\mathrm{d}x\}\,\mathrm{d}x = \lambda \int_0^1 f_4(x)\{\xi^2 + \eta^2/l(l+1)\}\,\mathrm{d}x, \quad (1)$$

where

$$f_1(x) = 9x^2\Gamma_1\theta^{n+1}(x) + 6(n+1)x^3\theta^n(x)\theta'(x) + n(n+1)x^4\theta'^2(x)\theta^{n-1}(x),$$

$$f_2(x) = -3x^2\Gamma_1\theta^{n+1}(x) - (n+1)x^3\theta^n(x)\theta'(x),$$

$$f_3(x) = x^2\Gamma_1\theta^{n+1}(x), \qquad f_4(x) = x^4\theta^n(x).$$

Here θ satisfies the dimensionless Emden equation

$$\theta''(x) + (2/x)\theta'(x) + (R/\alpha)^2\theta^n(x) = 0$$

and the boundary conditions

$$\theta(0) = 1, \qquad \theta'(0) = \theta(1) = 0;$$

$\alpha = [(n+1)p(0)/\{4\pi G\rho^2(0)\}]^{\frac{1}{2}}$, R is the radius of the star, $\rho(0)$ and $p(0)$ are the central values of density and pressure and G is the gravitational constant. In (1) the components ξ and η of the eigenfunction (which correspond to radial and transverse components of the displacement) are as defined by Andrew (1967), and $\lambda = \sigma^2 R^2\rho(0)/p(0)$, where perturbations are proportional to $\exp(i\sigma t)$ and t is time. Table 1 gives the usual dimensionless eigenvalues

$$\omega^2 = \sigma^2 R^3/GM = -\lambda/(n+1)\theta'(1)$$

which are $\frac{3}{4}$ of the value in Robe's (1968) units. As is usual in work with polytropes, we took $\Gamma_1 = \frac{5}{3}$.

Table 1. Comparison of eigenvalues ω^2

Mode	Polytrope $n = 1$			Polytrope $n = 3$		
	Finite element method		Shooting	Finite element method		Shooting
	$N = 25$	$N = 35$	method[A]	$N = 25$	$N = 35$	method[A]
f	2·536	2·577	2·683	9·548	9·510	9·480
p_1	10·62	10·90	11·73	17·57	17·49	17·37
p_2	23·99	24·31	26·41	30·28	29·86	29·36
p_3	43·51	43·12	46·04	47·50	46·27	44·71
p_4	70·05	68·09	70·45	69·77	67·00	63·28
p_5	104·1	99·75	99·60	97·58	92·48	83·47
p_6	146·3	138·5	133·4	130·9	123·1	109·9
p_7	197·3	184·6	172·0	168·0	159·0	138·0
p_8	258·4	238·6	215·2	208·5	199·5	168·7
g_1	−0·291	−0·290	−0·289	—	—	5·326
g_2	−0·126	−0·130	−0·135	—	—	2·938
g_3	−0·049	−0·057	−0·079	—	—	1·864

[A] Values obtained by Robe (1968).

3. Results and Discussion

For the stable polytrope ($n = 3$), our results are very similar to those of Andrew (1967). With N equal subintervals, the finite element method (again with continuous piecewise linear coordinate functions) gave estimates of the f mode and the first N p modes but the remaining modes were spurious. Accuracy of both eigenvalues and eigenfunctions for p modes increased as N increased.

With the unstable polytrope ($n = 1$), estimates of the f mode and the first N p modes were again obtained, but this time good results were also obtained for g modes. However, for each N, only about half as many g modes were obtained as p modes, the remaining modes being spurious.

Table 1 compares some of the eigenvalues we obtained for the two polytropes with more accurate values obtained by Robe (1968) by means of a shooting method. With a few minor exceptions, all modes classified as genuine had the same number of zeros in both ξ and η as found by Robe. For more details of our results and further discussion see Fiedler and Andrew (1985).

For the composite models, which have both g^+ (stable) and g^- (unstable) modes, the finite element method yielded g^-, f and p modes as well as spurious modes but no g^+ modes. Relatively few g^- modes were obtained, especially for the model with a smaller (unstable) core. This is probably because the zeros of g^- modes are all in the unstable part of the star and the uniform grid used here is not suitable for locating close zeros.

Since the divergence of the displacement is known to be $O(r^{l-1})$ as $r \to 0$, the eigenfunctions satisfy $\eta(0) = 3\xi(0)$ (Andrew 1967). This requirement was not imposed in our calculations, but nearly all computed modes satisfied it approximately, and $\eta(0)/\xi(0)$ did not vary systematically between genuine and spurious modes. This suggests that difficulties with the finite element method may be more fundamental for this problem than for some problems in acoustic and electromagnetic wave theory, where a penalty technique by Winkler and Davies (1984) has achieved some success in eliminating spurious modes often obtained in finite element calculations.

All our results are consistent with the tentative explanations of Andrew (1969, 1970). The classical theory of variational methods for eigenvalue problems involves minimax principles. Only the unstable g^- modes, and no stable modes, can be characterized this way. We believe this may be why the method succeeded for the g^- modes but not for the g^+ modes.

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