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Dynamic Time Dependent Hexagonal Magnetoconvection*

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Abstract

The time dependence of the single mode hexagonal magnetoconvective system has been investigated numerically at high Rayleigh number. It is established that, in certain parameter ranges, the system has oscillatory solutions which not only have a periodic nature, but also develop into chaotic and intermittent solutions. Further, the system generates nonzero mean kinetic and magnetic helicity together with substantial magnetic field amplification. These features are shown to be maintained in time without any externally imposed rotation of the system.

1. Introduction

High resolution solar observations indicate that very strong, small scale magnetic fields exist in the photosphere and they are concentrated into ropes which emerge through it. Further, the scale of these ropes is estimated to be only a few hundred km across (Stenflo 1976) with their strength appraised to vary between 1000 and 2000 G (Harvey 1977). These features appear to have a well defined spatial relationship with the photospheric granular convection pattern.

In sunspots, where the fields are considered to be stronger, normal convection is seen to be inhibited, and only oscillatory motions prevail. These oscillations have been directly accredited to the influence of the magnetic force (Chandrasekhar 1961). Several attempts have been made to account for these oscillations by using the linear theory of hydromagnetic convection, which establishes that the diffusivities play a major role in determining the nature of the instability (Cowling 1976). If the magnetic diffusivity η is small then the magnetic Reynolds number is large and the field lines are swept around with the convecting fluid inducing a disturbance field $\boldsymbol{B}^*$ which exerts a stabilizing force and in turn tends to reverse the motion. If the magnetic diffusivity is smaller than the thermal diffusivity, linear theory predicts that the flow will be reversed and oscillatory motion with growing amplitude—overstability—sets in.

However, it could be unwise to account for oscillatory motions such as those seen in sunspots, which clearly are finite amplitude oscillations, by extrapolating from

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the linear theory. As pointed out by Weiss (1975), the quadratic restoring force is underestimated by the linear theory, since

$$|(\boldsymbol{B}.\nabla).\boldsymbol{B}| > |(\boldsymbol{B}_0.\nabla).\boldsymbol{B}|, \quad \text{where} \quad \boldsymbol{B} = \boldsymbol{B}_0+\boldsymbol{B}^*,$$

when the field is disturbed and enhanced. It is the quadratic nature of the magnetic force which encourages finite amplitude oscillations to develop. When the finite amplitude modal equations with hexagonal planform are considered in the absence of a magnetic field, we have established (Lopez and Murphy 1985) that the presence of a vertical component of vorticity also leads to finite amplitude oscillations which are not predicted from the linear theory. In the magnetoconvective Rayleigh–Benard model, which is considered in the present paper, both of these sources of oscillations are present and it is difficult to isolate and identify individual effects.

The kinematic studies of magnetoconvection represent the next stage on from the linear theory, where the magnetic induction equation is solved with a prescribed velocity field. Although these models adequately describe the early stages of field growth, the later stages of flow–field interaction are recognized to be beyond the scope of the kinematic theory. However, it is at this stage of the time evolution where the nonlinear oscillations, which are of prime interest (e.g. in the study of umbral dots), manifest themselves (Knobloch and Weiss 1984).

The two dimensional kinematic study by Weiss (1966) demonstrated the process of kinematic flux expulsion leading to the formation of flux sheets between the rolls. For the formation of flux tubes, however, a three-dimensional velocity pattern is clearly necessary. In this regard, Clark and Johnson (1967) pioneered the use of hexagonal velocity patterns in kinematic studies of the induction equation. The prescribed velocity is now only required to satisfy continuity, which in the Boussinesq approximation is $\nabla . \boldsymbol{u} = 0$. In the Clark and Johnson (1967) case, as well as in most other kinematic studies (e.g. Galloway and Proctor 1983), the structure of the velocity field satisfying continuity with a hexagonal planform has been taken to be as simple as possible. Consequently the forms taken for the velocity do not account for a vertical component of the vorticity, and hence have no mean kinetic helicity. In fact, the vertical structure of the imposed velocity is sinusoidal, a form which is representative of convective motions only very close to marginal stability. The value of this approach in modelling possible dynamos is limited, due to the restricted solutions for the velocity leading to zero mean kinetic helicity. However, this aspect could be compensated for by an externally imposed Coriolis force to generate asymmetries which may then lead to dynamo action.

Kinematic studies have also shown that the velocity field required for dynamo action is one which is characterized by a nonzero mean kinetic helicity (Childress 1976), and its importance is now well accepted (Moffatt 1977; Hide 1982). In Rayleigh–Benard type dynamo models, the helicity of the system has in the past been externally generated by the imposed rotation of the system or by imposing a shear flow on the mean flow. Kennett (1976), using the concept of 'minimal systems', severely truncated the modal equations to a point where the vertical structures of the velocity and magnetic fields retained only one sine or cosine mode, depending on the boundary conditions, and one horizontal mode for which the coupling moments vanished. Nevertheless, she was able to show that this minimal system led to finite amplitude oscillations, with nonzero magnetic field, which were claimed to be suggestive of dynamo action. However, with the high symmetry of the solutions, no nonzero mean helicity is possible. Baker's (1978) modal calculations also suggest the existence of a convective dynamo effect in a Benard layer without rotation. His truncation of the modal equations is much less severe than that employed by Kennett (1976), and as a consequence the vertical structures of the velocity and magnetic fields can be accurately solved implicitly in the system. Further, the expansions adopted for the velocity and magnetic field are general for solenoidal fields, with $\nabla \cdot \boldsymbol{u} = 0$ and $\nabla \cdot \boldsymbol{B} = 0$, giving zero mean horizontal values. The same formulation has been used throughout the present paper. However, Baker (1978) employed the 'mean field approximation' (see Van der Borght *et al.* 1972) in which a great deal of the nonlinearity of the modal system is lost. Possibly the most significant consequence of employing this approximation to a magnetoconvective system is that there can be no nonzero mean helicity. In the single mode mean field system there is not even any local helicity since the vertical vorticity vanishes. In Baker's (1978) calculations, a $1\frac{1}{2}$ mode and a 2 mode mean field approximation was used where the vertical components of vorticity are zero. In this case, a nonzero local helicity is now possible, depending on the choice of horizontal planform. However, the mean helicity still vanishes for these higher mode mean field solutions. Hence, at best, the Baker (1978) system may lead to a 'second-order dynamo', 'first-order dynamos' requiring a nonzero mean helicity (Roberts 1972).

A hexagonal horizontal modal expansion enhances the role of the nonlinear terms in the modal equations leading to the generation of the vertical components of vorticity and of current density, depending on the values of the parameters of the system. This produces solutions for which there is an associated nonzero mean helicity, which we call type II solutions and which are required for the alpha effect in dynamo action. Baker and Spiegel (1975) were the first to note the existence of type II solutions in rotating non-magnetic modal hexagonal equations. However, the existence of type II solutions was directly accredited to rotation (Baker 1978), and it was not until the work of Lopez and Murphy (1983) that the possibility of a nonzero vertical vorticity, and hence a nonzero mean helicity, without the presence of rotation or a magnetic field was shown to exist.

With the generation of the vertical component of current density, there is an associated mean twisting of the magnetic field, a mean magnetic helicity in essence. The observations of Babcock (1961) and Piddington (1983) suggest that the solar magnetic flux tubes could be helically twisted. Even though flux tubes are not a direct consequence of the single mode equations, due to the broad horizontal resolution, helically twisted fields do result from the type II solutions, and this is expected to be a feature of the full multimode equations.

So far, in order to solve the magnetoconvective Rayleigh–Benard problem most authors have had to make geometric simplifications, mainly resulting from the inadequate size and speed of available computing facilities. The geometry of the convective planform adopted is of crucial importance as ropes and sheets have different dynamical properties. Further, ropes cannot be formed by a two-dimensional roll pattern. Proctor and Galloway (1979) have stated that an axisymmetric geometry can model a hexagonal planform. While there may be some clear advantages in using a cylindrical representation of a hexagonal cell, such as being able to solve explicitly in the radial direction, a modal representation is preferable in that interactions between neighbouring cells are simulated and the possible generation of mean helicity is also included. The nonlinear diffusive terms in the momentum equation become more significant in a hexagonal representation of the planform than in the axisymmetric representation. This effect is measured in the modal representation of the momentum equation constant C, which is the third moment of the planform function. It takes the value $C = \sqrt{\frac{1}{6}} \sim 0 \cdot 408$ for hexagons, while for cylindrical cells $C \sim 0 \cdot 176$ and is zero in any mean field representation.

The oscillations reported from the axisymmetric model of Galloway and Moore (1979) would correspond to the curvature force $|(\boldsymbol{B}.\nabla).\boldsymbol{B}|$ as discussed by Weiss (1975), and do not include any of the finite amplitude oscillations which are associated with the generation of the vertical component of vorticity, and hence mean helicity, as described for ordinary Rayleigh–Benard convection by Busse (1972) and numerically investigated for the hexagonal situation by Lopez and Murphy (1985).

2. Equations and Method of Solution

The model considered in this investigation describes the interaction between an initially uniform vertical magnetic field and a horizontal layer of fluid which is heated from below and contained between two isothermal stress-free boundaries. The governing equations are given by the momentum equation (Chandrasekhar 1961)

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho \boldsymbol{u}.\nabla \boldsymbol{u} + \nabla P - \rho \boldsymbol{G} - \mu \nabla^2 \boldsymbol{u} + \frac{1}{4\pi} \mu^* \boldsymbol{H} \times (\nabla \times \boldsymbol{H}) = 0, \tag{1}$$

the continuity equation

$$\partial \rho / \partial t + \nabla.(\rho \boldsymbol{u}) = 0 \tag{2}$$

and the induction equation

$$\partial \boldsymbol{H}/\partial t + \eta \nabla \times (\nabla \times \boldsymbol{H}) - \nabla \times (\boldsymbol{u} \times \boldsymbol{H}) = 0. \tag{3}$$

When the Boussinesq approximation is considered equation (2) reduces to

$$\nabla.\boldsymbol{u} = 0, \tag{4}$$

and the heat transport equation takes the form

$$\rho C_v \partial T/\partial t + \rho C_v \boldsymbol{u}.\nabla T - K \nabla^2 T = 0. \tag{5}$$

Here the physical constants μ, K and C_v represent the viscosity, coefficient of thermal conductivity and specific heat at constant volume. The magnetic permeability is denoted by μ^* with $\boldsymbol{G}$ defined in component form by $(0, 0, g)$, g being the acceleration due to gravity. The modal equations which follow are derived from these equations in a manner described by Van der Borght and Murphy (1973) where the non-dimensional scalings are given; in this case only a single horizontal mode has been retained in the time dependent system, its horizontal extent measured by the non-dimensional horizontal wave number a:

$$\partial T_0/\partial t = \mathrm{D}^2 T_0 - \mathrm{D}(F W), \tag{6}$$

$$\partial F/\partial t = (\mathrm{D}^2 - a^2)F - W\mathrm{D}T_0 - C(2W\mathrm{D}F + F\mathrm{D}W), \tag{7}$$

$$\frac{1}{\sigma}\frac{\partial\psi}{\partial t} = -Ra^2F + (\mathrm{D}^2 - a^2)\psi + Q\tau\mathrm{D}\phi$$

$$-(C/\sigma)(W\mathrm{D}\psi + 2\psi\mathrm{D}W + 3Z\mathrm{D}Z) + Q\tau C(H\mathrm{D}\phi + 2\phi\mathrm{D}H + 3\chi\mathrm{D}\chi), \tag{8}$$

$$\psi = (\mathrm{D}^2 - a^2)W, \tag{9}$$

$$\partial\phi/\partial t = \tau(\mathrm{D}^2 - a^2)\phi + \mathrm{D}\psi + C(\mathrm{D}H\mathrm{D}^2W - \mathrm{D}W\mathrm{D}^2H + H\mathrm{D}\psi - W\mathrm{D}\phi), \tag{10}$$

$$\phi = (\mathrm{D}^2 - a^2)H, \tag{11}$$

$$\partial\chi/\partial t = \tau(\mathrm{D}^2 - a^2)\chi + \mathrm{D}Z - C(2\chi\mathrm{D}W - 2Z\mathrm{D}H - H\mathrm{D}Z + W\mathrm{D}\chi), \tag{12}$$

$$\frac{1}{\sigma}\frac{\partial Z}{\partial t} = (\mathrm{D}^2 - a^2)Z + Q\tau\mathrm{D}\chi - QC(\chi\mathrm{D}H - H\mathrm{D}\chi) - (C/\sigma)(W\mathrm{D}Z - Z\mathrm{D}W), \tag{13}$$

where $\mathrm{D} \equiv \partial/\partial z$.

In this sixteenth-order nonlinear system of partial differential equations the variables $Z(z, t)$ and $\chi(z, t)$ represent the scaled vertical components of vorticity and current density, while $W(z, t)$, $F(z, t)$ and $H(z, t)$ define the vertical convective velocity, the temperature fluctuation and induced magnetic field. The mean temperature across the layer follows $T_0(z, t)$ and H_0 designates the strength of the externally impressed magnetic field, taken as constant. Now the physical determination of the temperature, velocity and magnetic fields along with the vorticity and current density at any point within the convecting fluid layer are given at time t in terms of the quantities obtained from the solutions of these equations by the expressions:

$$T(x, y, z, t) = T_0(z, t) + F(z, t)f(x, y), \tag{14}$$

$$\begin{aligned} \boldsymbol{u}(x, y, z, t) = \Bigg\{ & \left(\frac{\mathrm{D}W(z, t)}{a^2}\frac{\partial f(x, y)}{\partial x} + \frac{Z(z, t)}{a^2}\frac{\partial f(x, y)}{\partial y}\right), \\ & \left(\frac{\mathrm{D}W(z, t)}{a^2}\frac{\partial f(x, y)}{\partial y} - \frac{Z(z, t)}{a^2}\frac{\partial f(x, y)}{\partial x}\right), \\ & \{W(z, t)f(x, y)\}\Bigg\}, \end{aligned} \tag{15}$$

$$\begin{aligned} \boldsymbol{H}(x, y, z, t) = H_0\Bigg\{ & \left(\frac{\mathrm{D}H(z, t)}{a^2}\frac{\partial f(x, y)}{\partial x} + \frac{\chi(z, t)}{a^2}\frac{\partial f(x, y)}{\partial y}\right), \\ & \left(\frac{\mathrm{D}H(z, t)}{a^2}\frac{\partial f(x, y)}{\partial y} - \frac{\chi(z, t)}{a^2}\frac{\partial f(x, y)}{\partial x}\right), \\ & \{1 + H(z, t)f(x, y)\}\Bigg\}, \end{aligned} \tag{16}$$

$$\omega(x, y, z, t) = \left\{\left(\frac{\mathrm{D}Z(z,t)}{a^2}\,\frac{\partial f(x,y)}{\partial x} - \frac{(\mathrm{D}^2-a^2)\,W(z,t)}{a^2}\,\frac{\partial f(x,y)}{\partial y}\right),\right.$$

$$\left(\frac{\mathrm{D}Z(z,t)}{a^2}\,\frac{\partial f(x,y)}{\partial y} + \frac{(\mathrm{D}^2-a^2)\,W(z,t)}{a^2}\,\frac{\partial f(x,y)}{\partial x}\right), \tag{17}$$

$$\left.\{Z(z,t)\,f(x,y)\}\right\},$$

$$\xi(x, y, z, t) = H_0\left\{\left(\frac{\mathrm{D}\chi(z,t)}{a^2}\,\frac{\partial f(x,y)}{\partial x} - \frac{(\mathrm{D}^2-a^2)H(z,t)}{a^2}\,\frac{\partial f(x,y)}{\partial y}\right),\right.$$

$$\left(\frac{\mathrm{D}\chi(z,t)}{a^2}\,\frac{\partial f(x,y)}{\partial y} + \frac{(\mathrm{D}^2-a^2)H(z,t)}{a^2}\,\frac{\partial f(x,y)}{\partial x}\right), \tag{18}$$

$$\left.\{\chi(z,t)\,f(x,y)\}\right\}.$$

In addition to a, which can only take values within a finite range, there are four further free parameters in the set of equations (6)–(13) which control the degree of instability and ultimate time evolution of the convective regime. They depend upon H_0, the depth of the fluid layer d, the temperature difference ΔT maintained across it and the physical properties of the electrically conducting fluid medium. Specifically, they are defined by

$R = gad^3\Delta T/\kappa\nu$	the Rayleigh number,
$Q = \mu^* d^2 H_0/4\pi\mu\eta$	the Chandrasekhar number,
$\sigma = \nu/\kappa$	the Prandtl number,
$\tau = \eta/\kappa$	the magnetic Prandtl number,

where a is the coefficient of volume expansion, κ the coefficient of thermal diffusivity, ν the viscous diffusivity and η the magnetic diffusivity.

The geometry of the convective planform is determined by the choice of $f(x, y)$, which satisfies the Helmholtz equation

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = -a^2 f(x,y), \tag{19}$$

and the cell aspect ratio is defined by $a = kd$, k being the horizontal wave number. For self-interacting cells of hexagonal type $f(x, y)$ is given as

$$f(x,y) = \surd\tfrac{2}{3}\{2\cos(\tfrac{1}{2}\surd 3ax)\cos(\tfrac{1}{2}ay) + \cos(ay)\} \tag{20}$$

and the planform constant C, defined by the ratio

$$C = \iint_{\text{cell}} f^3(x,y)\,\mathrm{d}x\,\mathrm{d}y \bigg/ \iint_{\text{cell}} f^2(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{21}$$

has the numerical value $\surd\frac{1}{6}$.

For steady convection the Nusselt number

$$N = FW - \mathrm{D}T_0, \tag{22}$$

which gives a non-dimensional measure of the heat flow, is constant at all points across the layer. However, in this case we are determining the time evolution of $N(z, t)$ and its numerical value is obtained at any particular time step by averaging over the grid in the z direction. It is felt that this manner of evaluation is consistent with following the time variation of the dependent variables.

Within the modal expansion framework and for a single mode, the mean kinetic helicity, which is defined as the volume integral over a cell of $\boldsymbol{u}.(\nabla \times \boldsymbol{u})$ is now given by

$$H_{\mathrm{v}} = \int_0^1 [WZ + (1/a^2)\{\mathrm{D}W\mathrm{D}Z - Z(\mathrm{D}^2 - a^2)W\}]\,\mathrm{d}z. \tag{23}$$

Analogously, a mean magnetic helicity is defined as the volume integral over a cell of $\boldsymbol{H}.(\nabla \times \boldsymbol{H})$, giving

$$H_{\mathrm{m}} = H_0 \int_0^1 [H\chi + (1/a^2)\{\mathrm{D}H\mathrm{D}\chi - \chi(\mathrm{D}^2 - a^2)H\}]\,\mathrm{d}z, \tag{24}$$

where H_0, as defined previously, is the strength of the initially uniform vertical magnetic field. From equation (24) H_{m} establishes a measure of the total twisting of the magnetic field in the same sense that H_{v} is a measure of the total twisting of the velocity field. It is important to note that H_{v} and H_{m} are nonzero only if nonzero vertical components of vorticity and current density respectively follow from the solution of the system of equations (6)–(13).

Specifically, the method of solution is based on an implicit second-order backward finite difference scheme with coordinate stretching being used for the spatial integrations together with forward differencing in time. The resulting band matrix is then solved iteratively for the nonlinear terms with an efficient algorithm specifically developed for systems of second-order equations (Van der Borght 1980). The initial conditions employed to start the time integrations mimic small perturbations on the conductive state.

3. Boundary Conditions

The system (6)–(13) has been solved utilizing the stress-free boundary conditions, which requires

$$W(0, t) = W(1, t) = \mathrm{D}^2 W(0, t) = \mathrm{D}^2 W(1, t) = \mathrm{D}Z(0, t) = \mathrm{D}Z(1, t) = 0. \tag{25}$$

Further, the boundaries are taken to be isothermal, and hence

$$F(0, t) = F(1, t) = 0, \quad T_0(0, t) = 0, \quad T_0(1, t) = -1. \tag{26}$$

The appropriate form of the electrical boundary conditions follows from the requirement that the boundaries should be current-free, so the current density satisfies (Chandrasekhar 1961)

$$\chi(0, t) = \chi(1, t) = 0. \tag{27}$$

In publications where the interaction of convection and a magnetic field is considered, differences have been found in the form of the boundary conditions adopted for the magnetic field disturbance $H(z, t)$ corresponding to current-free boundaries. Van der Borght *et al.* (1972) have shown that the appropriate conditions on H for these current-free boundaries can be written as

$$\mathrm{D}H - aH = 0 \quad \text{on} \quad z = 0, \tag{28a}$$

$$\mathrm{D}H + aH = 0 \quad \text{on} \quad z = 1, \tag{28b}$$

where a is the horizontal wave number for cellular convection.

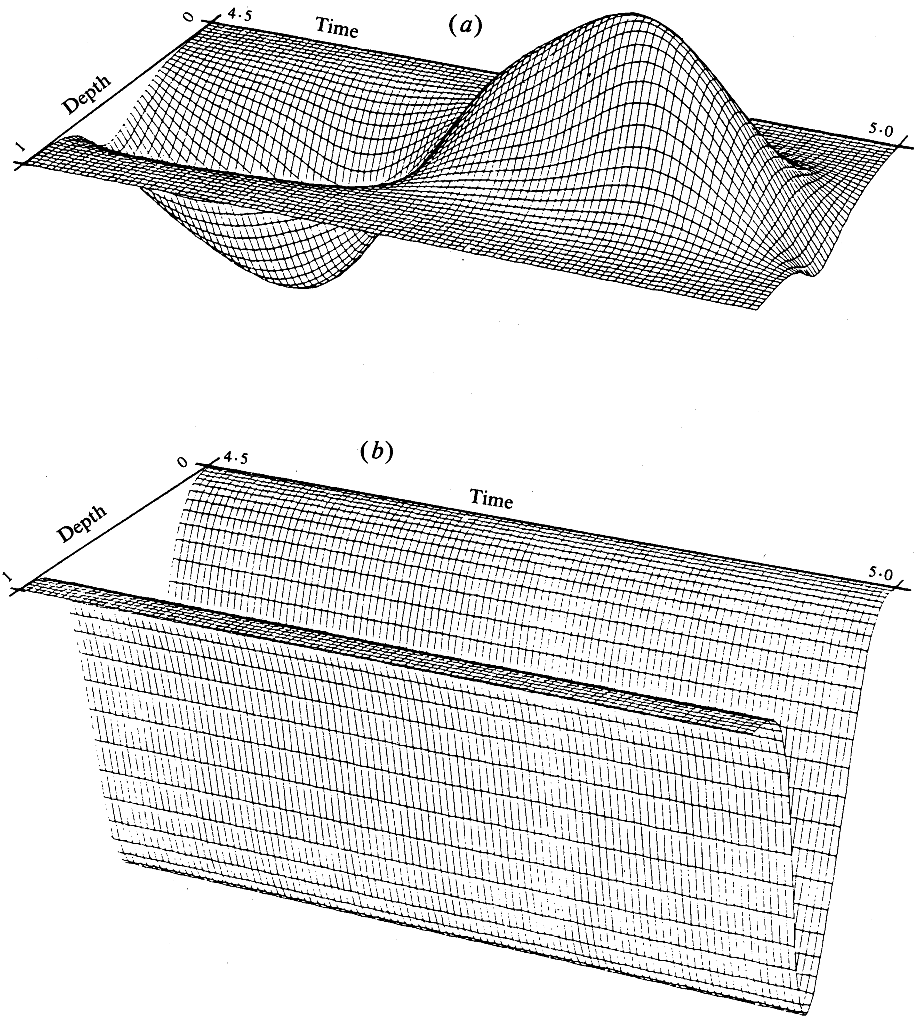


Fig. 1. Time evolution of vertical component of velocity from $t = 4 \cdot 5$–$5 \cdot 0$ for $R = 10^5$, $Q = 10^2$, $\sigma = 1$, $\tau = 1$ and $a^2 = 10$ with the boundary conditions (*a*) $\mathrm{D}H = 0$ and (*b*) $\mathrm{D}H \pm aH = 0$.

Recently, in their studies of magnetoconvection, a number of authors (e.g. Rudraiah 1981; Sharma and Sharma 1982) have employed an approximation to these boundary conditions on the magnetic field disturbance, using instead the conditions

$$\mathrm{D}H = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1. \tag{29}$$

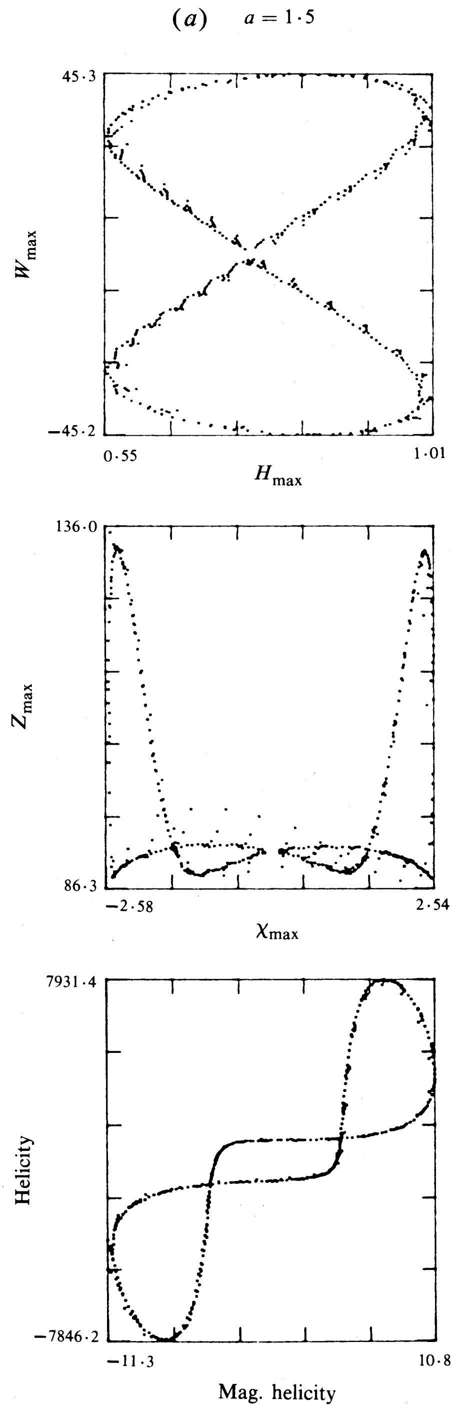
By making a direct comparison of the two sets of results obtained, the differences arising from employing these two forms of boundary conditions on the magnetoconvective system can be demonstrated. It is readily observed that the apparent slight difference in magnetic boundary conditions results in not only a quantitative but also a qualitative difference in both the evolution and the structure of the solutions.

For the particular case of $R = 10^5$, $Q = 10^2$, $\sigma = 1$, $\tau = 1$ and $a^2 = 10$ the most obvious difference is that the $\mathrm{D}H = 0$ solutions evolve into a periodically oscillating system, whereas the $\mathrm{D}H \pm aH = 0$ solutions evolve to a steady state, which is illustrated in the time evolutions of the vertical velocity shown in Figs 1*a* and 1*b* for the two different boundary conditions. However, of more physical importance is the difference in structure of the velocity and magnetic fields which evolve in the two cases. For $\mathrm{D}H = 0$, the vertical components of vorticity, as well as the vertical current density, decay very rapidly from their initial values. At $t = 0$ they were given values of $\sim 10^{-6}$ and at the point where the integrations were concluded they had values of $\sim 10^{-25}$ and were still decaying. This means that the flow structure is of type I, as described by Lopez and Murphy (1983), and the magnetic field inside the cell is predominantly vertical. Whereas, for the $\mathrm{D}H \pm aH = 0$ case, both the vertical components of vorticity and current density, after a time $\sim 0 \cdot 25$, have grown to dynamically significant values and the solutions have taken the type II form. In particular, these solutions have very desirable astrophysical features, especially the form of the magnetic field which now, due to the presence of the vertical components of vorticity and current density, possesses a mean helical nature.

Hence, the nature of the boundary conditions on H is crucial, and the approximation from $\mathrm{D}H \pm aH = 0$ to $\mathrm{D}H = 0$, which is usually made in order to facilitate the use of series expansion methods, should be avoided wherever possible.

4. Results

The exhibited time dependence of the solutions of equations (6)–(13), subject to the stated boundary conditions, may be due to a number of different effects. However, the presence of overstability is not necessarily one of them. Overstability is the term used to describe the state of the system at critical points of stability where the growth rate of the disturbance is complex. In nonlinear studies of magnetoconvection, overstability theory is no longer relevant, since the system is at a point in parameter space which is supercritical. Of course, the disturbances may still have real or complex growth rates, and initially will grow kinematically according to those growth rates. However, once they grow to a certain level, the nonlinear terms are no longer negligible, and the growth of the disturbances is halted. Up to this stage, the evolution has been controlled by the initial growth rate, but once the nonlinear terms become comparable in magnitude with the linear terms in the equations, the form of the evolution takes on a different character. It is this new character which is relevant to studies of nonlinear magnetoconvection.



Figs 2*a–g*. Evolutionary trajectories illustrating the phased coupling between the maxima of the vertical components of velocity and magnetic field disturbance (*top*), of the vorticity and current density (*middle*) and the mean helicity and the mean magnetic helicity (*bottom*) for $R = 10^5$, $Q = 10^3$, $\sigma = 1$, $\tau = 1$ and a as indicated.

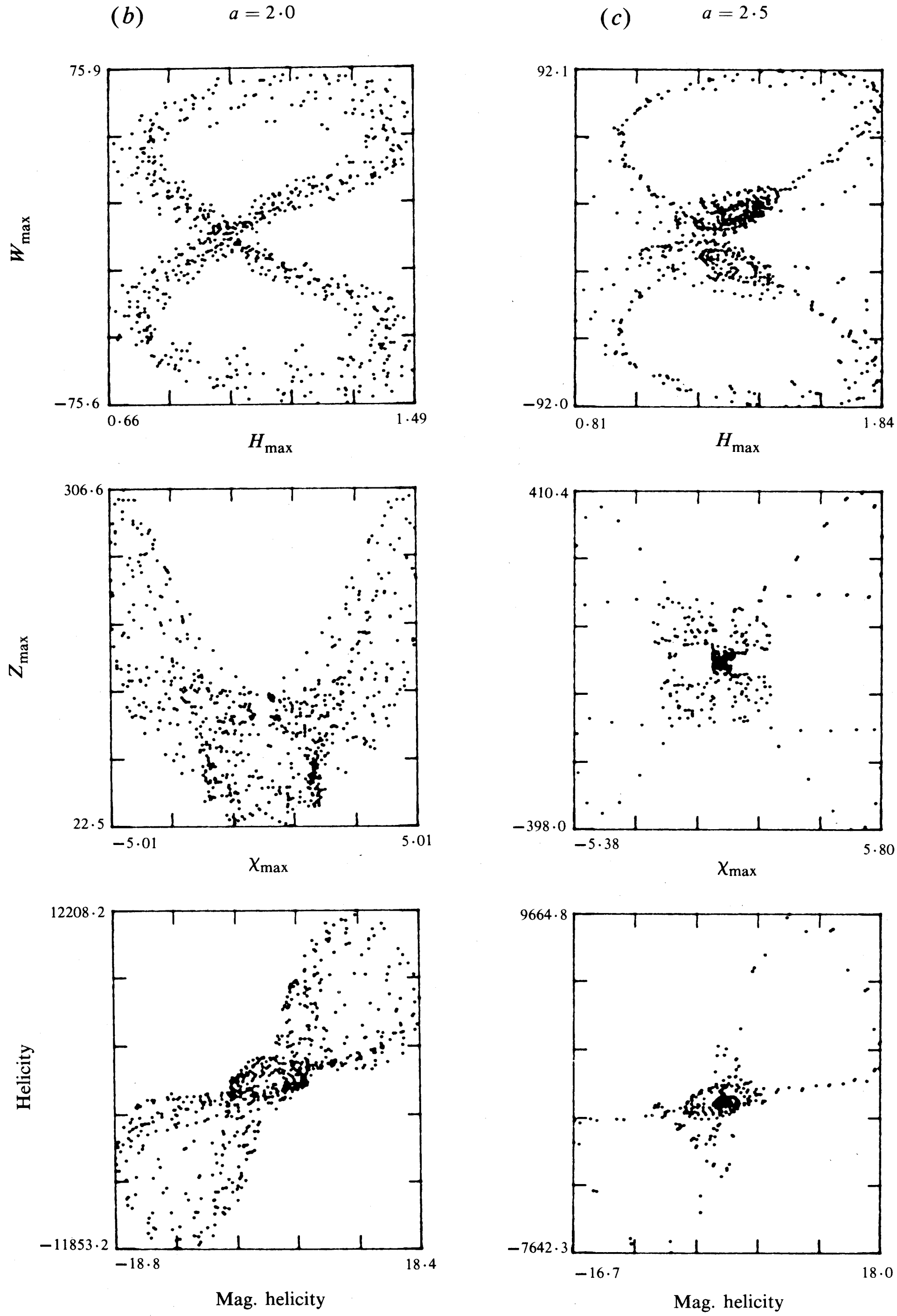
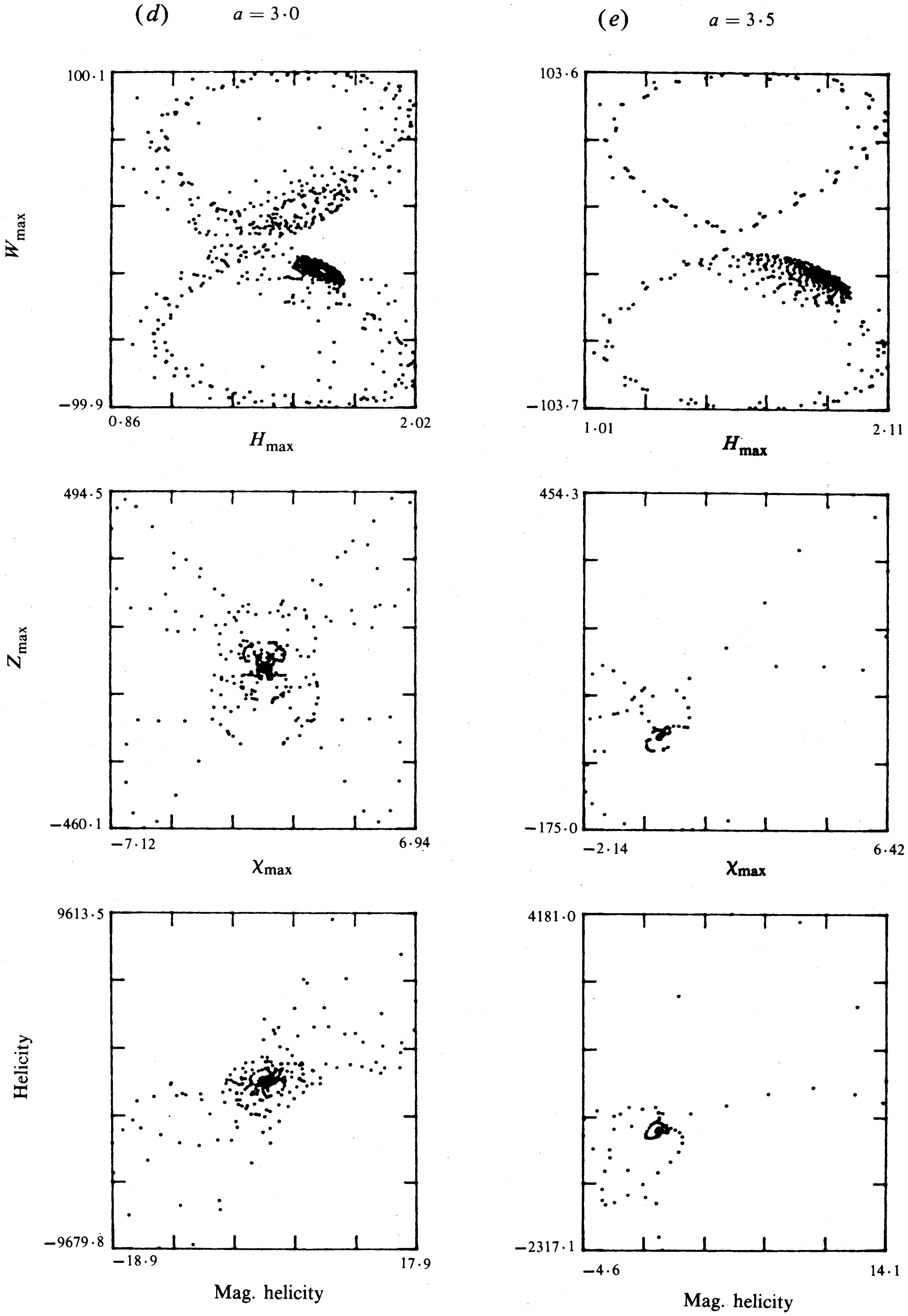


Fig. 2. (*Continued*)



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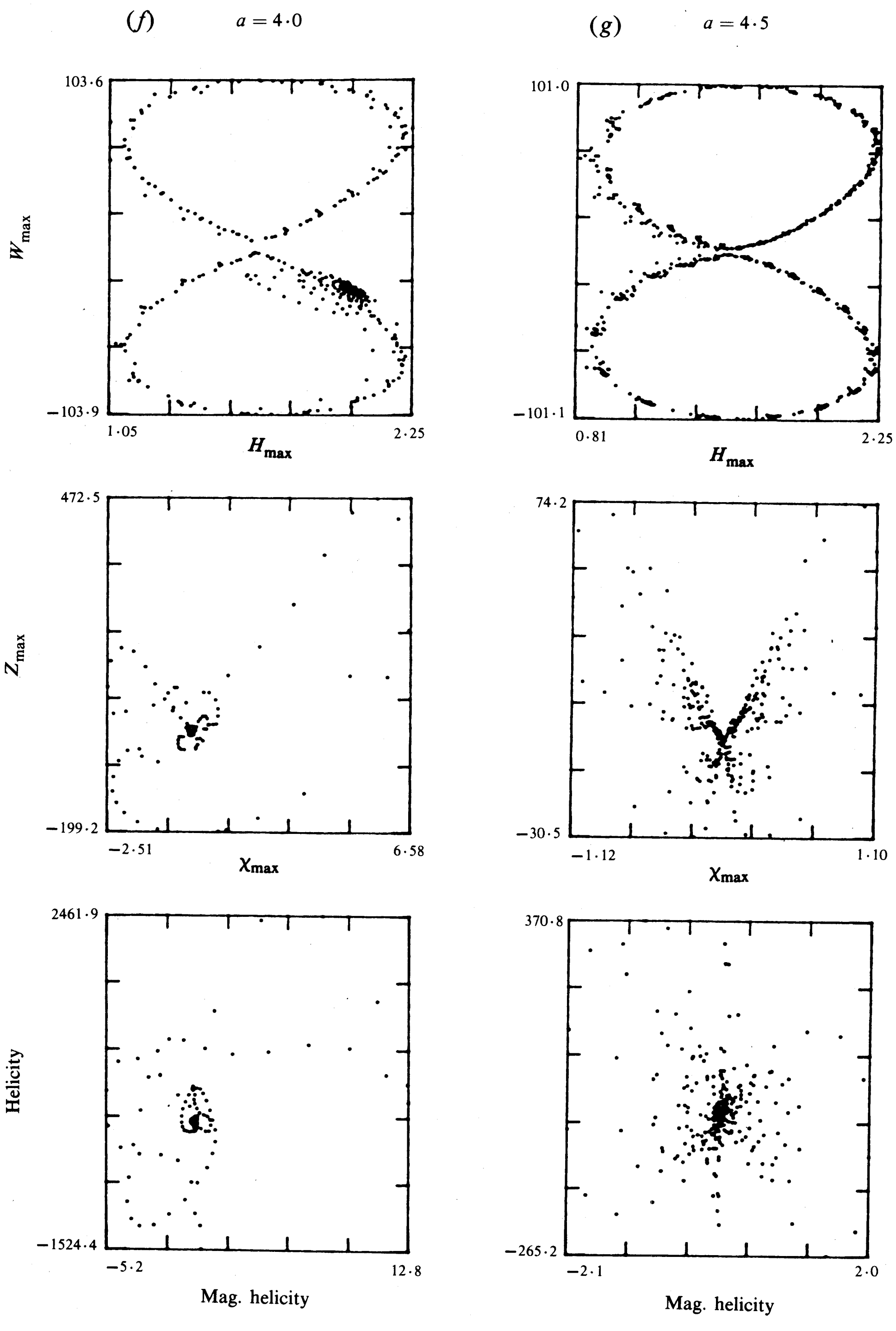
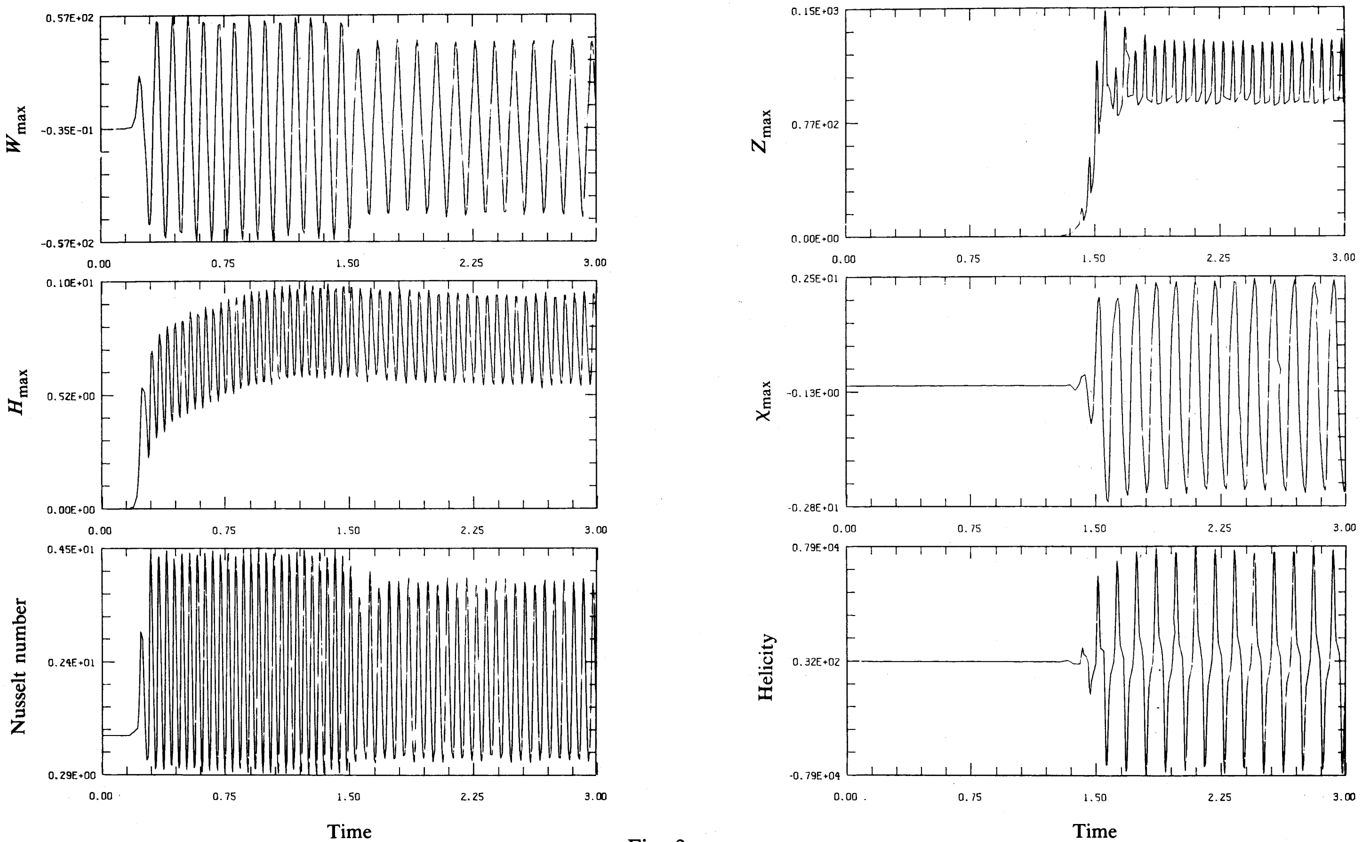


Fig. 2. (*Continued*)

(*a*) $a = 1\cdot5$



Figs 3*a–g*.

Figs 3*a–g*. Time series of maxima of the vertical components of velocity and vorticity (*top*), of magnetic field disturbance and current density (*middle*) and of the Nusselt number and the mean helicity (*bottom*) for $R = 10^5$, $Q = 10^3$, $\sigma = 1$, $\tau = 1$ and a as indicated.

(b) $a = 2{\cdot}0$

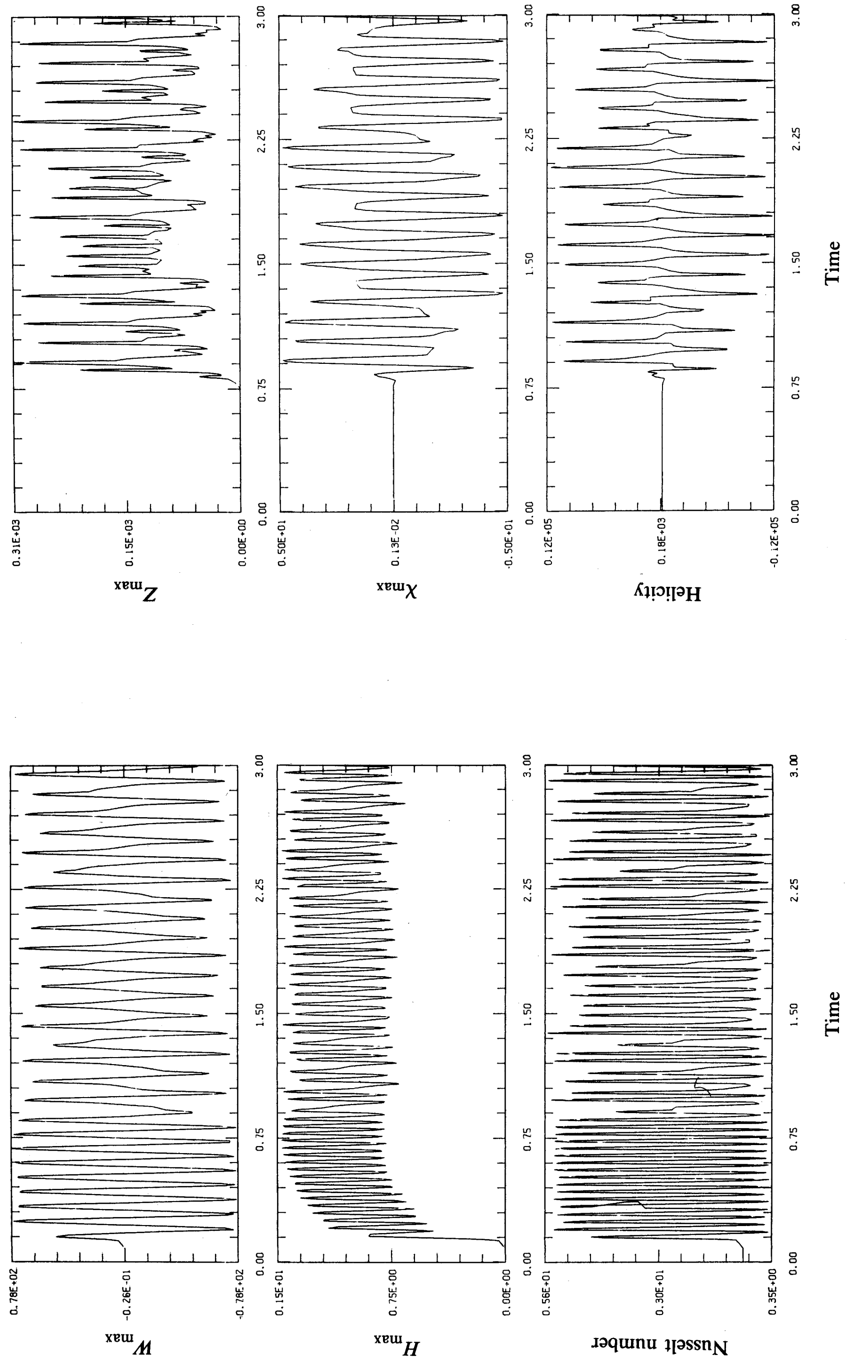


Fig. 3. (Continued)

(c) $a = 2 \cdot 5$

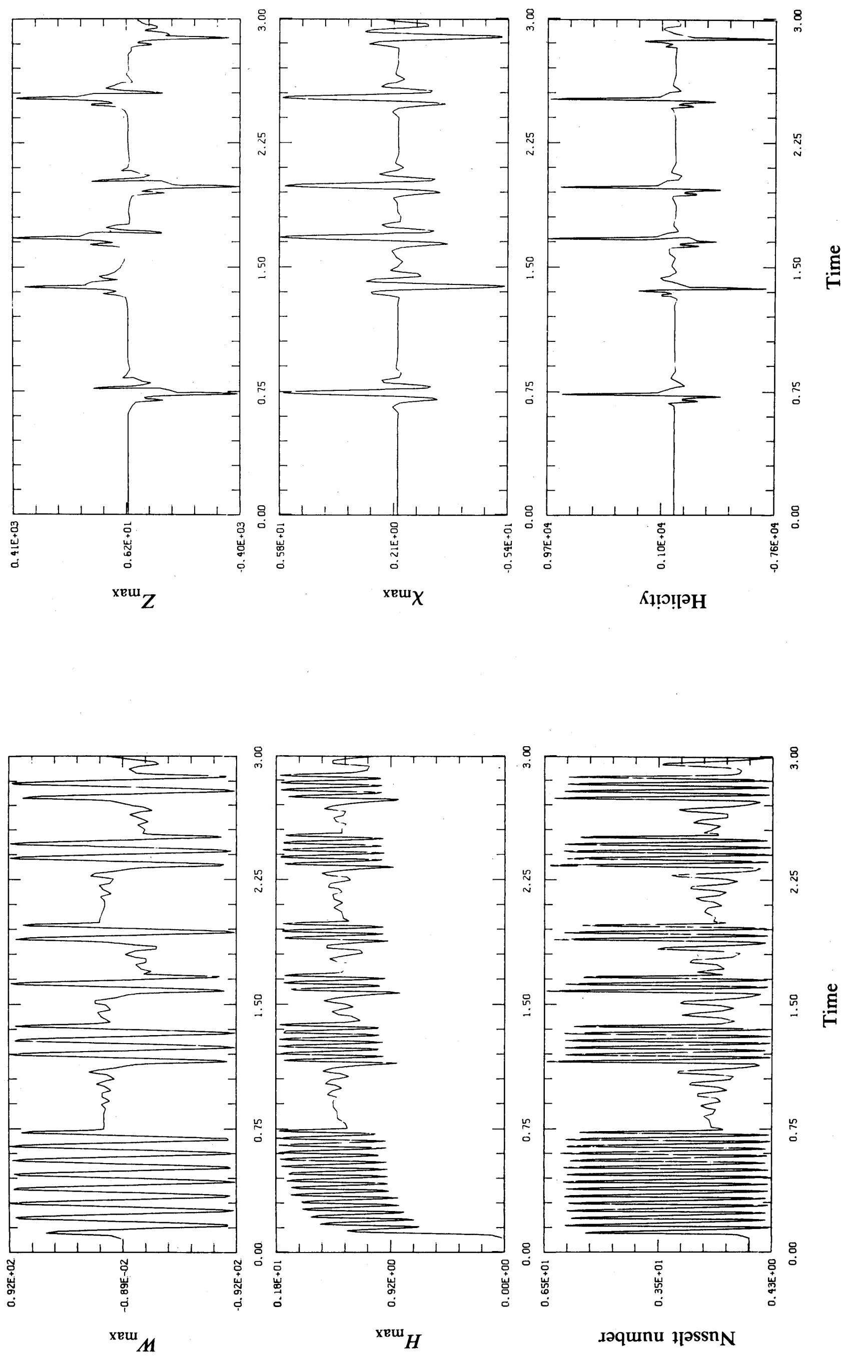
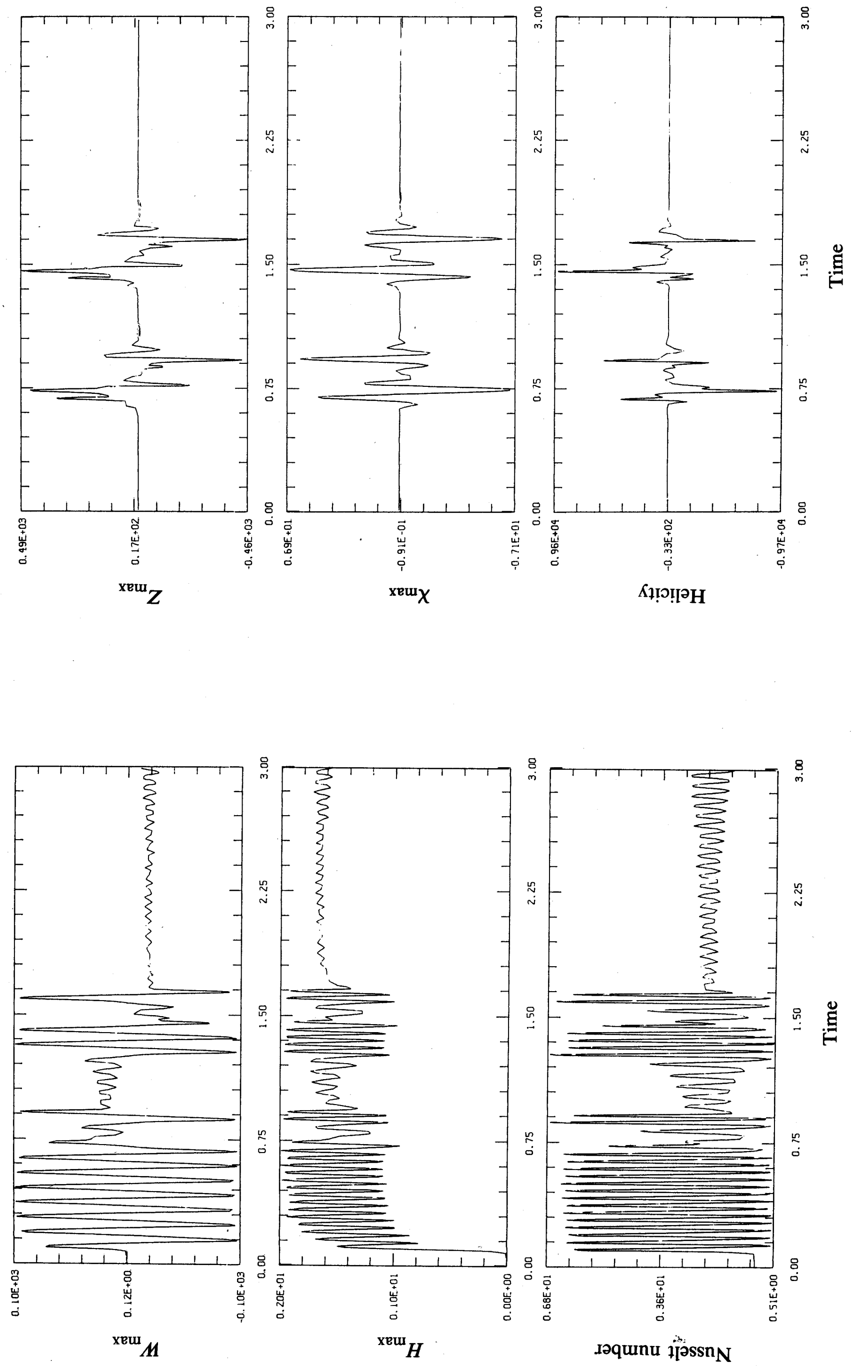


Fig. 3. *(Continued)*



(*d*) $a = 3{\cdot}0$

Fig. 3. (*Continued*)

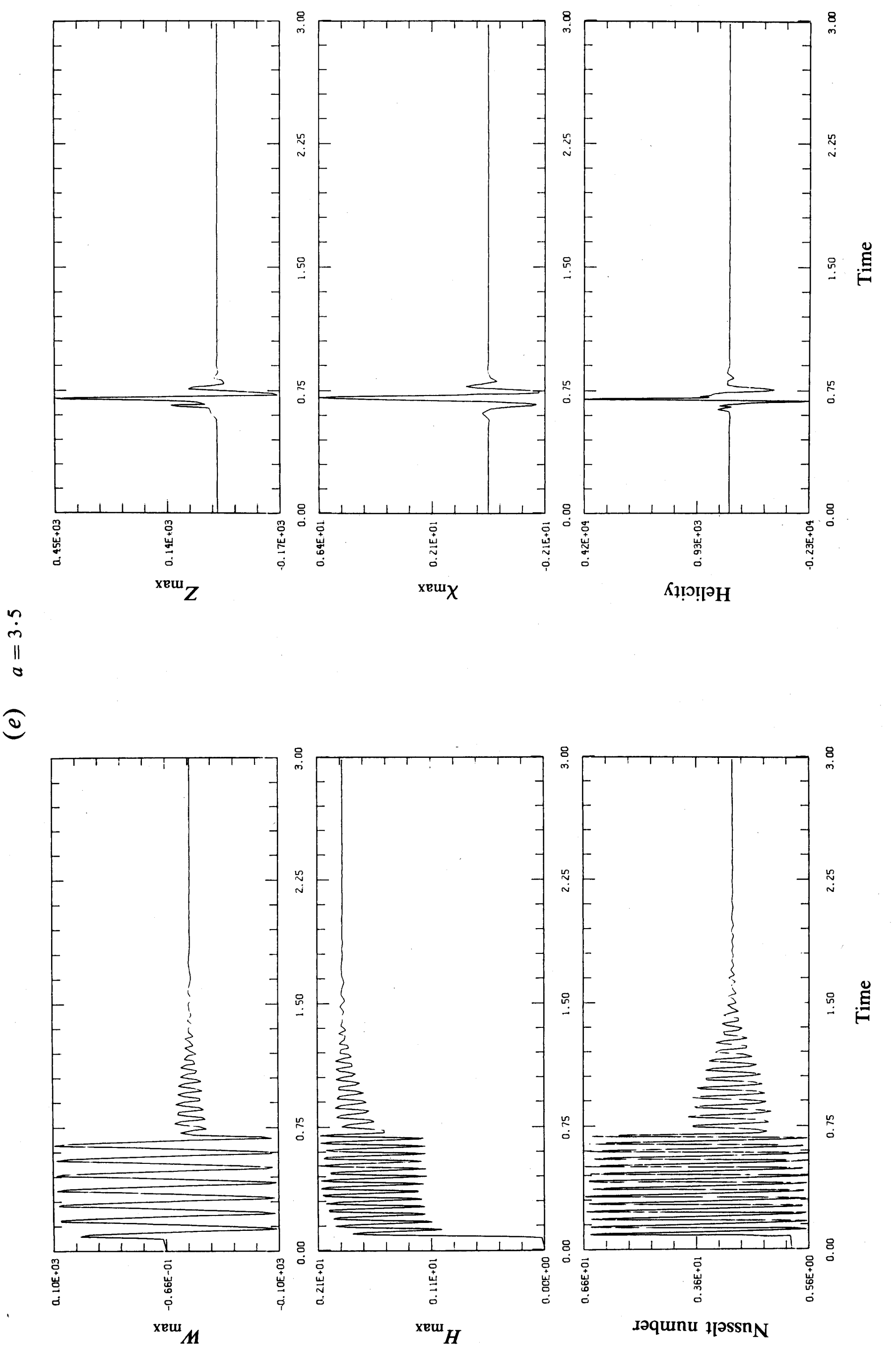


Fig. 3. *(Continued)*

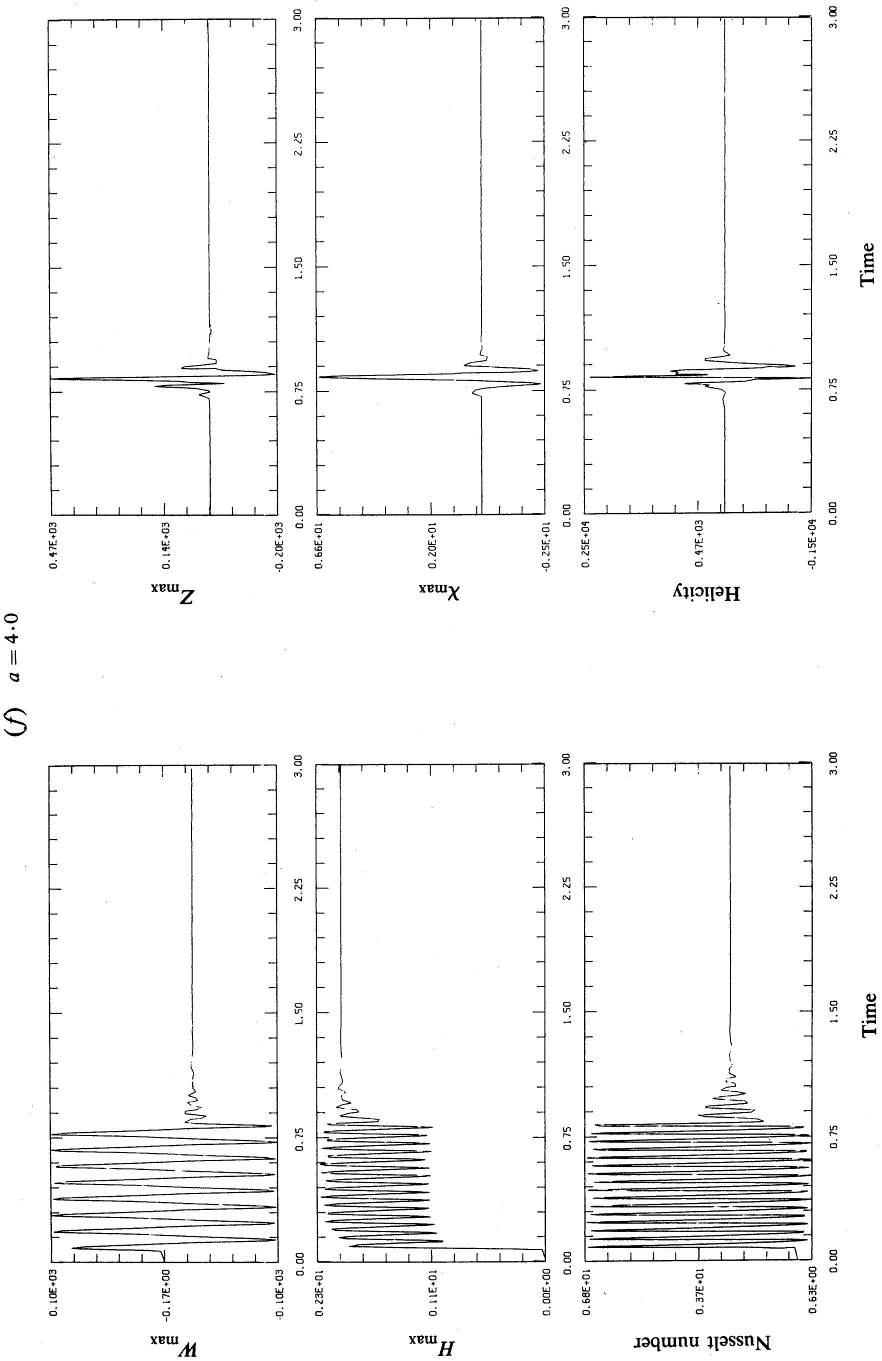


Fig. 3. (Continued)

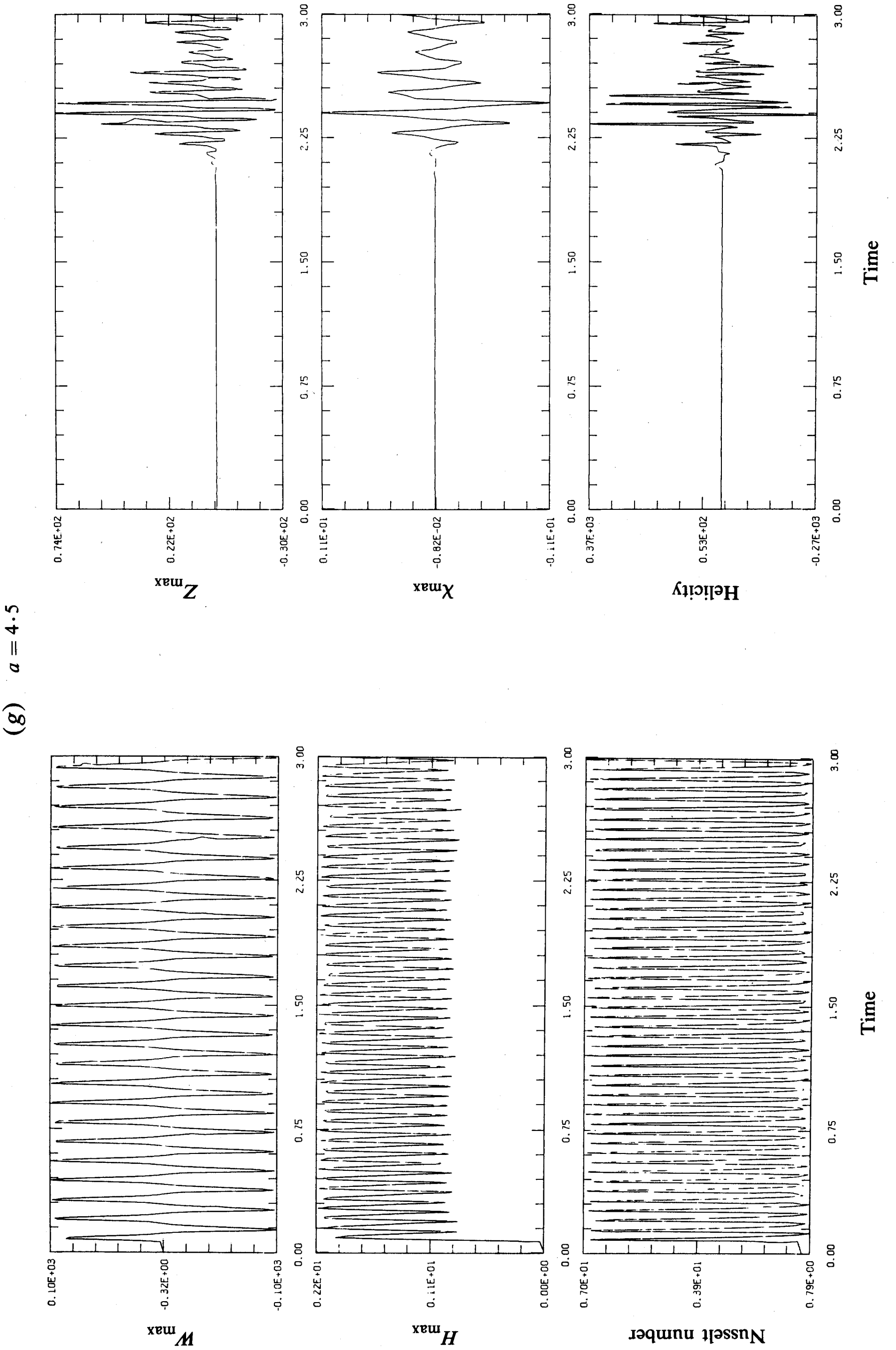


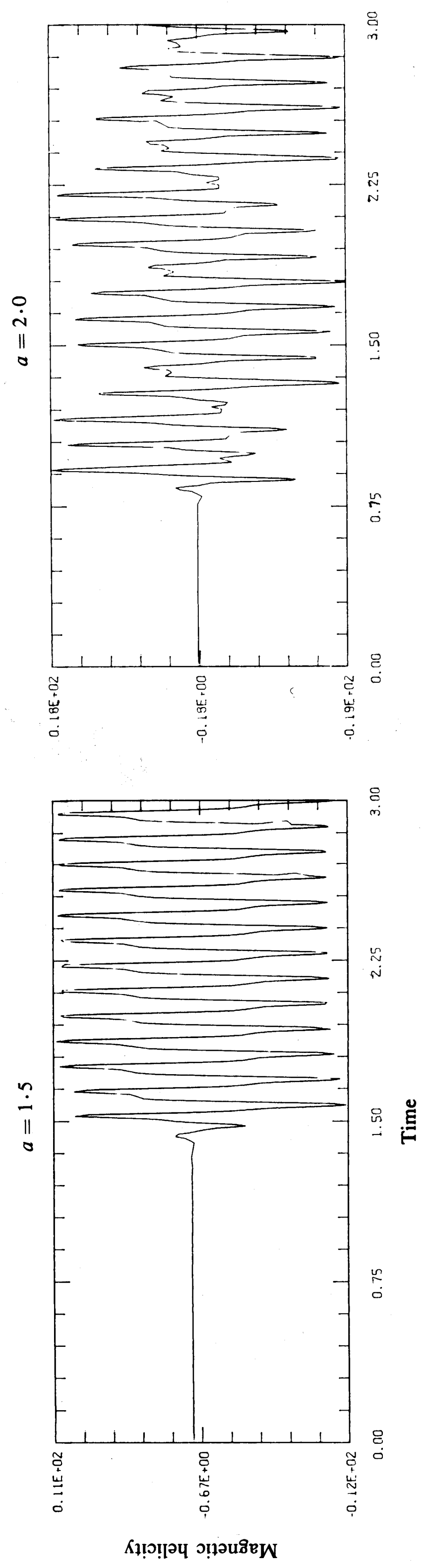
Fig. 3. (*Continued*)

The nonlinear behaviour of the evolutions depends critically on all the five parameters describing the system. Previously the non-magnetic system was described in a three-parameter space (Lopez and Murphy 1985) and a global classification of the system was found to be a formidable task. Here, the situation is further complicated with the addition of the Chandrasekhar number Q and the magnetic Prandtl number τ. A direct comparison with the results from the simplified truncated two-dimensional model of Knobloch *et al.* (1981) is not really feasible. The nonlinear terms in the equations which are responsible for the time dependence, and whose effect we are primarily concerned with, are neglected in their model, where they retain only a single Fourier sine or cosine mode in the x and z directions. Moreover, having been solved by using a perturbation method, their results would only be valid near the critical state. Clearly, from the large values of the Nusselt number N, being typically of order 10, the results presented here are from a system well beyond the critical state.

A small selection from the available fivefold parameter space has been made illustrating some of the typical nonlinear oscillations which are present at Rayleigh numbers larger than the critical Rayleigh number. In the sequences shown in Figs 2, 3 and 4, the Rayleigh number has been set at 10^5, the Chandrasekhar number at 10^3 and the Prandtl and magnetic Prandtl numbers at $1 \cdot 0$, while the aspect ratio a has been systematically varied from $1 \cdot 5$ through to $4 \cdot 5$ in steps of $0 \cdot 5$. Fig. 2 gives the evolutionary trajectories which illustrate the phasing between the maxima of the vertical components of velocity and magnetic field disturbance (*top*), the maxima of the vertical components of vorticity and current density (*middle*), and of the mean kinetic helicity and the mean magnetic helicity (*bottom*). Fig. 3 illustrates the corresponding time series for the maximum values of the vertical components of velocity, magnetic field disturbance, Nusselt number, vertical components of vorticity and current density, and the mean helicity. Finally, Fig. 4 gives the corresponding time series for the mean magnetic helicity.

At the smallest aspect ratio considered with $a = 1 \cdot 5$, which represents the widest cell size, the system is in a periodic oscillatory state which is evident from the limit cycles in Fig. 2*a*. The corresponding time series plots given in Fig. 3*a* demonstrate a phase transition from type I oscillatory to type II oscillatory behaviour, this being marked by the growth of the vertical components of vorticity and current density. The growth rates of the velocity and magnetic field disturbances are real while those of the vorticity and current density have an associated complex component which results in an oscillatory growth. The change in the amplitude of the velocity and magnetic field disturbance, as well as in the Nusselt number is obvious from Fig. 3*a*. As well, there is an increase in the period of the oscillations associated with the transition from type I to type II. A low frequency modulation is also observed, and is most pronounced in the Nusselt number. This modulation is apparent in the 'limit cycle' between $W_{\max}$, the maximum over z of $W(z, t)$, and $H_{\max}$, the maximum over z of $H(z, t)$, in Fig. 3*a* where it creates regular arcs making up the complete cycle.

The oscillatory nature of the mean kinetic helicity and the mean magnetic helicity is different from that exhibited by the other quantities. Whereas the others resemble a sinusoidal oscillation, both helicity oscillations are very 'spiked' at the extreme of their amplitudes, and broaden out near zero amplitude. This can be accounted for in that the helicities are nonlinear combinations of velocity and vorticity in the case



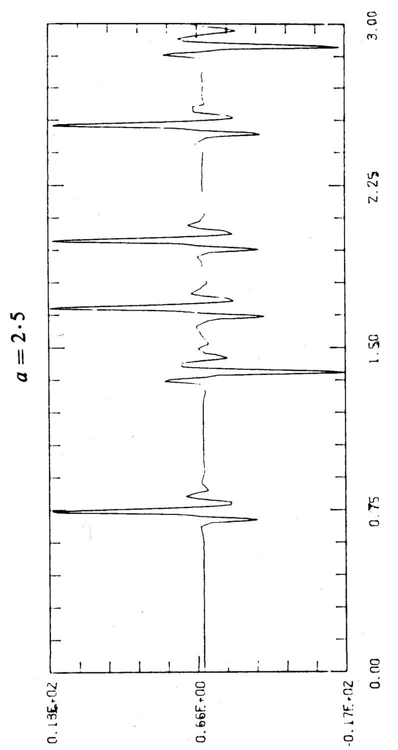


Fig. 4. Time series of the mean magnetic helicity for $R = 10^5$, $Q = 10^3$, $\sigma = 1$, $\tau = 1$ and a as indicated.

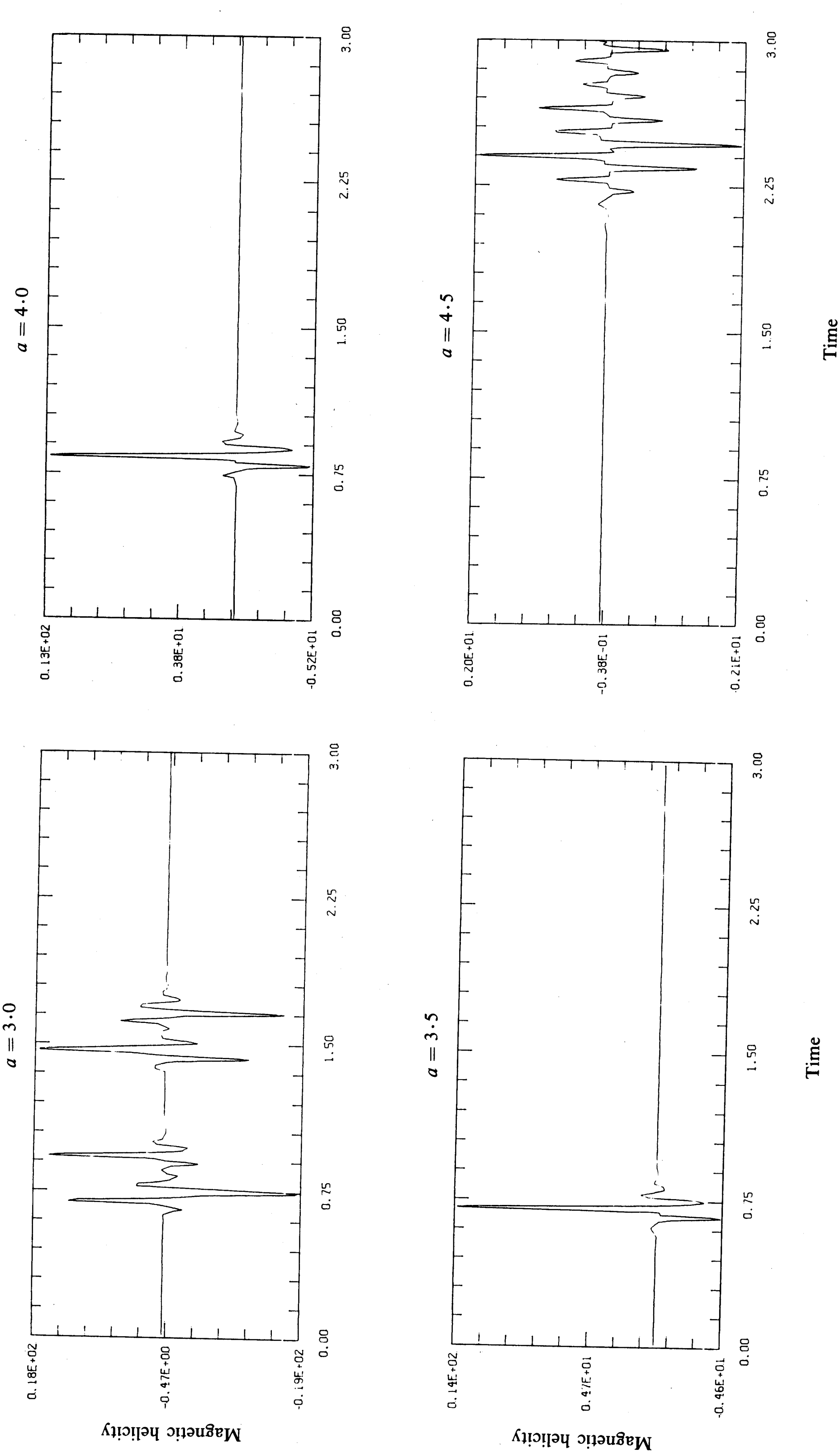


Fig. 4. *(Continued)*

of kinetic helicity, and of magnetic field and current density in the case of magnetic helicity. In both cases, one component has a period twice as large as that of the other; i.e., $W_{\max}$ has twice the period of $Z_{\max}$, and $\chi_{\max}$ has twice the period of $H_{\max}$. The corresponding helicities acquire the larger of the two periods, and their spiked nature near the maxima of the amplitudes is due to the nonlinear combination of the two different oscillators. The limit cycle between the two helicities demonstrates the complicated phasing between them, as well as their periodic nature. Further, a close examination of their time series displayed in Figs 3*a* and 4 clearly shows them to have the same period but with a slight phase shift.

Reducing the horizontal scale of the cell, by increasing a to $2\cdot0$, now reduces the time the system remains in the periodic type I state, before a transition to an aperiodic type II state occurs. The time series of the variables given by Fig. 3*b*, especially that of $Z_{\max}$, shows the chaotic nature of the system. However, Fig. 2*b* shows that there is still some coherency in the phasing, and portrays a perturbation of the limit cycles in Fig. 2*a* for the smaller value of a.

As a is further increased to $2\cdot5$ and $3\cdot0$, the transition to type II results in intermittent behaviour. The initial type I phase is still periodic as it was observed in the wider cells considered earlier. In the intermittent phase, the 'burst' segments are closely related to the type II oscillations found in the $a = 1\cdot5$ and $2\cdot0$ cases. This is observed by comparing Figs 2*b* and 2*c* where the large amplitude 'cycles' in Fig. 2*c* correspond to the 'burst' and have a similar 'structure' to those in Fig. 2*b*. The quieter sections have oscillations with much longer period than that of the 'burst' and have zero helicities associated with them.

Weiss *et al.* (1984) in their idealized dynamo model, which is essentially a complex generalization of the third-order 'minimal' systems of Lorenz (1963), have found aperiodic solutions where the magnetic field has bursts of cyclic activity separated by quiescent episodes during which the field is drastically reduced in amplitude and varies on a much slower timescale. Also, a limit cycle found by these authors (Fig. 3 of Weiss *et al.* 1984) between the magnetic field and the shear velocity is qualitatively very similar to that found in Fig. 2*a* for $a = 1\cdot5$ between $H_{\max}$ and $W_{\max}$.

As the cells become narrower, given by the values $a = 3\cdot5$ and $4\cdot0$, the intermittency of the second phase of the evolution has almost disappeared. The initial phase follows the same pattern as observed for the wider cells, periodic type I. Then, at about $t = 0\cdot75$, the vertical components of vorticity and current density, and consequently the mean kinetic and magnetic helicities give a high energy burst which lasts a relatively short period of time, and marks a transition from type I periodic to type I steady. These bursts of activity seen in the helicities are similar to those noted by Fautrelle and Childress (1982) in their Fig. 10 where the magnetic and kinetic helicities of their convective dynamo model have large bursts.

At even narrower cells, when $a = 4\cdot5$, the system appears to stay in the periodic type I state. At $t = 2\cdot25$, the vertical components of vorticity and current density grow considerably. However, judging from both the time series (Fig. 3*g*) and the 'limit cycle' in Fig. 2*g*, their growth does not have any noticeable effect on the nature of the evolution of $W(z, t)$ or $H(z, t)$, which continue in what is very nearly a periodic state with a low frequency modulation.

It is now evident that the influence of the vertical component of vorticity is reduced as the convective cells become narrower, and for large enough aspect ratio the system is no longer able to generate a vertical component of vorticity.

5. Conclusions

Overall, we have been able to produce periodic solutions with magnetic field amplifications, and further, obtain aperiodic solutions which Kennett (1976) was unable to detect but only suggest their possible existence. This was achieved without the introduction of rotation.

The nonlinearities introduced to the system by nonzero mean helicities appear necessary in order to generate aperiodic behaviour (Weiss *et al.* 1984), and their presence in kinematic dynamos is also required for dynamo action to be possible (Moffatt 1977). Here, a magnetoconvective model has been presented which allows for the generation of mean helicities and hence produces the aperiodic behaviour which is found in dynamo models and resembles the aperiodic nature of the solar magnetic cycle. Weiss *et al.* (1984), in their reduced dynamo system, were able to isolate a bifurcation structure which mimicked the magnetic cycles in the Sun and further, they hoped that the same features would be found from the relevant partial differential equations. Our modal system has a similar temporal character, and it is anticipated that a closer understanding of the Sun's magnetic cycles may emerge from further investigation.

Knobloch and Weiss (1984) on re-examining some aspects of linear and nonlinear magnetoconvection models concluded that umbral dots could originate through convective motions, their oscillations being qualitatively matched by the nonlinear oscillations from magnetoconvective models considered by Weiss (1981*a*, 1981*b*, 1981*c*). However, the nonlinear oscillations referred to result from a two-dimensional cellular structure. The oscillations found in our model may now provide a worthwhile extension to the understanding of the processes in umbral dots, although the effects of compressibility, boundary conditions and modal truncations are expected to have important consequences on any conclusions.

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