Aust. J. Phys., 1985, **38**, 97–112

Poloidal Flow in Axisymmetric Pulsar Magnetospheres

R. R. Burman

Department of Physics, University of Western Australia,
Nedlands, W.A. 6009.

Abstract

This paper deals with dissipation-free flow in steadily rotating axisymmetric pulsar magnetospheres; for each species, relativistic inertia is balanced by the Lorentz force. Knowledge of integrals of the motion, including a complete integral for the limiting case of purely toroidal flow, is used to manipulate the fundamental electromagnetic and hydrodynamic equations into convenient forms. Consideration of the flow dynamics, incorporating plasma drift across the magnetic field and injection along it, provides the physical basis of a description of flow in which the poloidal motion is closely tied to the poloidal magnetic field lines (L. Mestel *et al.* to be published 1985). Particular attention is paid to flows whose toroidal part tends towards corotation as the symmetry axis is approached, and implications of the results for model building are discussed.

1. Introduction

In the 17 years since Jocelyn Bell discovered the first pulsar, no solution for their magnetospheric structure has been found, in spite of many attempts. The approaches made can be classified into two overlapping, but nonetheless different, classes which can perhaps be characterized as that of the theoretical astronomer and that of the mathematical physicist. The former approach, and much the more popular, is to make guesses and assumptions in proposing a model, and to see where they lead. The latter approach eschews postulating models; rather it aims ultimately to yield a model soundly based on carefully developed and reasonably rigorous mathematical physics.

Perhaps of most permanent value among the efforts of theorists so far has been the uncovering of integrals of the coupled electromagnetic and hydrodynamic equations for these steadily rotating relativistic systems, particularly, but not only, for the cases of axisymmetry and cylindrical symmetry. These results have recently been expanded and set out in an elegant and systematic form in a paper by Westfold (1981). A complete integral for purely toroidal flow in axisymmetric magnetospheres can be added to the list (Burman 1984*a*).

These integrals will undoubtedly feature prominantly in the process of obtaining a self-consistent pulsar magnetosphere model. Now that quite a few integrals are known, it is time to take the next step in the spirit of the mathematical physicist's path to understanding the pulsar magnetosphere. This is, I suggest, to use the integrals to re-organize and simplify the fundamental electromagnetic and hydrodynamic equations—to try, in fact, to transform the equations into the optimum form for

0004-9506/85/010097$02.00

physical interpretation and for further study, both analytical and numerical. I shall attempt to do this, in Sections 2 and 3*a*, for axisymmetric systems.

L. Mestel, Y.-M. Wang and K. C. Westfold (to be published 1985; hereafter referred to as MWW) have introduced a pulsar magnetosphere model in which electrons leave the star with non-negligible, but not highly relativistic, speeds; the poloidal motion of the electrons is along the poloidal magnetic field lines and is only moderately accelerated until they reach a limiting surface near which rapid acceleration occurs. The basic formalism MWW developed predicts, in addition, a second class of flows, which do not encounter a region of rapid acceleration (Burman 1984*b*). In Section 3, I shall study the physical basis of the MWW formalism by examining the driving forces and the resulting plasma drift.

In Section 4, I shall treat flows whose toroidal part tends towards corotation as the symmetry axis is approached, and go on in Section 5 to discuss implications of those results for model building.

2. The Integrals

Let ϖ, ϕ and z be cylindrical polar coordinates, with the z axis as the rotation axis of the star. The system under consideration is steady in the rotating frame: the changes in time at points fixed in the inertial frame of the star result from the steady rotation of a structure at angular frequency Ω. The dimensionless coordinates $\Omega\varpi/c$ and $\Omega z/c$ will be denoted by X and Z, and the unit toroidal vector by $\boldsymbol{t}$; the vacuum speed of light is c.

It follows from Faraday's law and $\nabla . \boldsymbol{B} = 0$ that the electric and magnetic fields are connected by $\boldsymbol{E} + X\boldsymbol{t} \times \boldsymbol{B} = -\nabla\Phi$ (Mestel 1971; Westfold 1981) where the gauge-invariant potential Φ is defined in terms of the familiar scalar and vector potentials ϕ and $\boldsymbol{A}$ as $\phi - XA_\phi$ (Endean 1972*a*). Thus $\boldsymbol{E}$ is expressed as the sum of a part $X\boldsymbol{B} \times \boldsymbol{t}$, generated by the rotation of the magnetic field structure, and a non-corotational part $-\nabla\Phi$.

Endean (1972*a*, 1972*b*) pointed out that, under the steady-rotation constraint, there exists a constant of the motion Ψ_k for particles of species k:

$$\Psi_k \equiv \Phi + (\gamma_k m_k c^2/e_k)(1 - Xv_{k\phi}/c), \tag{1}$$

where γ_k, m_k, e_k and $v_{k\phi}$ denote the Lorentz factor, rest mass, charge and ϕ-component of velocity of the particles of this species. If all particles of species k are nonrelativistic in an arbitrarily thin neighbourhood of the stellar surface, taken to be a perfect conductor, then Ψ_k is constant throughout all the space connected to the surface by flow lines of that species (Burman and Mestel 1978).

It is convenient to let $\boldsymbol{u}_k$ denote $\boldsymbol{v}_k - \Omega\varpi\boldsymbol{t}$, the flow velocity reduced by the local velocity of corotation with the star, and $\boldsymbol{B}_k^*$ denote $\boldsymbol{B} + (m_k c/e_k)\nabla \times \gamma_k \boldsymbol{v}_k$, the magnetoidal or magneto-inertial field of species k; we note that $\boldsymbol{B}_k^* = \nabla \times \boldsymbol{A}_k^*$ where $\boldsymbol{A}_k^*$ denotes $\boldsymbol{A} + (m_k c/e_k)\gamma_k \boldsymbol{v}_k$, the magnetic vector potential augmented by an inertial term (see e.g. Wright 1978).

For a species represented as a cold dissipation-free fluid, the equations of motion, expressing the balance of the Lorentz force by relativistic inertia, can be written as (Burman and Mestel 1978)

$$c^{-1}\boldsymbol{u}_k \times \boldsymbol{B}_k^* = \nabla\Psi_k . \tag{2}$$

This shows that Ψ_k is constant both on lines of $\boldsymbol{u}_k$ and on lines of $\boldsymbol{B}_k^*$.

Attention will now be restricted to the axisymmetric case, in which the magnetic and rotation axes of the star are either parallel or antiparallel. There is now no distinction between constancy on lines of $\boldsymbol{v}_k$ and on lines of $\boldsymbol{u}_k$. The poloidal magnetic field $\boldsymbol{B}_\mathrm{p}$ is expressible as $\varpi^{-1}\boldsymbol{t} \times \nabla P$ where $P \equiv -\varpi A_\phi$. The definition of Φ shows that $\Phi = \phi + \Omega P/c$.

The variable defined by

$$P_k^* \equiv P - (m_k c/e_k)\varpi\gamma_k v_{k\phi} \tag{3}$$

is the Stokes stream function of $\boldsymbol{B}_\mathrm{p}$ augmented by an inertial term: the poloidal part of $\boldsymbol{B}_k^*$ is $\varpi^{-1}\boldsymbol{t} \times \nabla P_k^*$. The toroidal component of the equation of motion (2) takes the form $\boldsymbol{v}_k . \nabla P_k^* = 0$, which is just the statement that P_k^* is a constant of the motion of species k (Mestel *et al.* 1979). The definitions of Ψ_k, Φ and P_k^* combine to give $\Psi_k - \Omega P_k^*/c = \gamma_k m_k c^2/e_k + \phi$: the two integrals Ψ_k and P_k^* contain the energy integral, stating that $\gamma_k m_k c^2 + e_k \phi$ is a constant of the motion (Mestel *et al.* 1979).

The subscript k labelling the species will now be dropped; electromagnetic quantities augmented by inertial terms will be distinguished by asterisks. The following dimensionless variables will be used: $\tilde{\nabla} \equiv (c/\Omega)\nabla$, $\boldsymbol{V} \equiv \boldsymbol{v}/c$ (with toroidal and poloidal parts V_ϕ and V_p), $\tilde{\Phi} \equiv e\Phi/mc^2$, $\tilde{\Psi} \equiv e\Psi/mc^2$, $\tilde{P} \equiv (e/mc^2)\Omega P/c$, $\tilde{P}^* \equiv \tilde{P} - X\gamma V_\phi$, $\tilde{\boldsymbol{A}} \equiv e\boldsymbol{A}/mc^2$ and $\tilde{\boldsymbol{A}}^* \equiv \tilde{\boldsymbol{A}} + \gamma \boldsymbol{V}$.

Not only are $\tilde{\Psi}$ and $\tilde{P}^*$ constants of the motion for each species, but each $\tilde{\Psi}$ is a function of the corresponding $\tilde{P}^*$ only: this follows from the poloidal part of the equation of motion (2) on eliminating u_ϕ between its ϖ and z parts and using the fact that $\tilde{P}^*$ is a constant of the motion. Consequently, the poloidal equation of motion reduces to

$$\tilde{B}_\phi^* \boldsymbol{V}_\mathrm{p} = \{U_\phi + X\tilde{\Psi}'(\tilde{P}^*)\}X^{-1}\boldsymbol{t} \times \tilde{\nabla}\tilde{P}^*, \tag{4}$$

where $\tilde{\boldsymbol{B}}^*$ denotes $e\boldsymbol{B}^*/mc\Omega$, $U_\phi \equiv V_\phi - X$ and the prime denotes differentiation with respect to the argument. Equation (4) states that the poloidal flow is everywhere parallel to the poloidal part of the magnetoidal field of the species, and that V_p and V_ϕ are related by

$$\tilde{B}_\phi^* V_\mathrm{p}/\tilde{B}_\mathrm{p}^* = U_\phi + X\tilde{\Psi}'. \tag{4'}$$

Equation (4) can be written in the form

$$\boldsymbol{V}_\mathrm{p} = \kappa \tilde{\boldsymbol{B}}_\mathrm{p}^* \tag{5a}$$

by introducing κ through the relation

$$V_\phi = \kappa \tilde{B}_\phi^* + (1 - \tilde{\Psi}')X. \tag{5b}$$

Combining these into a single vector equation gives [Mestel *et al.* 1979, equation (2.27)]

$$\boldsymbol{V} = \kappa \tilde{\boldsymbol{B}}^* + (1 - \tilde{\Psi}')X\boldsymbol{t}. \tag{6}$$

Although equation (5a) expresses coincidence of the lines of the poloidal flow of a species with those of the poloidal part of its magnetoidal field, it is not an algebraic relation but a differential equation, because of the vorticity contribution to $\tilde{\boldsymbol{B}}_{\mathrm{p}}^{*}$. It is the vorticity contribution that introduces an 'inertial drift' or departure of the poloidal flow from the poloidal magnetic field lines (Burman and Mestel 1979; Mestel *et al.* 1979). Equation (4′) is a first-order partial differential equation relating V_{p} and V_{ϕ}, generalizing the equation $U_{\phi}+X\Psi' = 0$, which describes purely toroidal flow, to incorporate general poloidal flow.

Equation (4) shows that

$$X\tilde{\nabla}.(\tilde{B}_{\phi}^{*}V_{\mathrm{p}}/X) = \boldsymbol{t}\times\tilde{\nabla}\tilde{P}^{*}.\nabla\omega \tag{7a}$$

$$= \frac{\partial\omega}{\partial X}\frac{\partial\tilde{P}^{*}}{\partial Z}-\frac{\partial\omega}{\partial Z}\frac{\partial\tilde{P}^{*}}{\partial X}, \tag{7b}$$

where ω denotes V_{ϕ}/X, equal to the local azimuthal angular speed of the species normalized to the angular speed of the star. In the special case of purely toroidal flow, this result shows that $\omega = \omega(\tilde{P}^{*})$, a function of $\tilde{P}^{*}$ only; a complete integral and singular solution for purely toroidal flow can be deduced by equating the right-hand side of equation (7b) to zero (Burman 1984*a*).

Under the steady-rotation constraint, the hydrodynamic equation of continuity for a species of particle number density n takes the form $\nabla.(n\boldsymbol{u}) = 0$, and this reduces further to $\nabla.(n\boldsymbol{v}_{\mathrm{p}}) = 0$ under axisymmetry. This becomes, on invoking equation (4) for $\boldsymbol{V}_{\mathrm{p}}$, the vanishing of a scalar triple product:

$$\boldsymbol{t}\times\tilde{\nabla}\tilde{P}^{*}.\tilde{\nabla}\{(n/\tilde{B}_{\phi}^{*})(U_{\phi}+X\tilde{\Psi}')\} = 0.$$

This means that the equations of motion and continuity together provide a further integral:

$$n/n_0 = \tilde{B}_{\phi}^{*}\chi'/(U_{\phi}+X\tilde{\Psi}'), \tag{8}$$

where χ' is a function of $\tilde{P}^{*}$ only; a constant reference particle number density n_0 has been inserted to make χ' dimensionless. Equations (4) and (8) for $\boldsymbol{V}_{\mathrm{p}}$ and n show that

$$(n/n_0)\boldsymbol{V}_{\mathrm{p}} = X^{-1}\boldsymbol{t}\times\tilde{\nabla}\chi. \tag{9}$$

So the integral $\chi(\tilde{P}^{*})$ of $\chi'(\tilde{P}^{*})$, a constant of the motion, is a dimensionless Stokes stream function for the poloidal particle flux of a species (Westfold 1981, Section 4). Equation (9) shows that the poloidal part of the electric current density can be expressed by the equation

$$\boldsymbol{j}_{\mathrm{p}}/n_0 e_0 c = X^{-1}\boldsymbol{t}\times\tilde{\nabla}\bar{S}, \tag{10}$$

where e_0 is a reference particle charge and $\bar{S}$ denotes $e\chi/e_0$ summed over the species; n_0 and e_0 have been taken to be fixed constants, not changing from species to species. The poloidal part of Ampère's law is satisfied by (Mestel *et al.* 1979)

$$\tilde{B}_{\phi} = -\tilde{S}/X, \tag{11}$$

where $\tilde{S}$ denotes $-(\omega_0/\Omega)^2\bar{S}$ with $\omega_0^2 \equiv 4\pi n_0 e_0 e/m$; both $\bar{S}$ and $\tilde{S}$ are dimensionless; ω_0 is a reference (species-dependent) angular plasma frequency.

The integrals (1) and (3) can be written in dimensionless form as

$$\tilde{\Psi}(\tilde{P}^*) \equiv \tilde{\Phi} + \gamma(1 - XV_\phi), \tag{1'}$$

$$\tilde{P}^* \equiv \tilde{P} - X\gamma V_\phi. \tag{3'}$$

Taking the gradient of (1′) and using (3′) gives

$$\tilde{\nabla}\tilde{\Phi} = (1 - \tilde{\Psi}')\gamma V_\phi \boldsymbol{i} - \boldsymbol{\mu} + \tilde{\Psi}'\tilde{\nabla}\tilde{P}, \tag{12}$$

where $\boldsymbol{i}$ is the unit cylindrical radial vector and

$$\boldsymbol{\mu} \equiv \tilde{\nabla}\gamma - (1 - \tilde{\Psi}')X\tilde{\nabla}(\gamma V_\phi), \tag{13a}$$

giving

$$\begin{aligned}\boldsymbol{\mu}/\gamma^3 = &\{1 - (1 - \tilde{\Psi}')(1 - V_p^2)X/V_\phi\}V_\phi \tilde{\nabla} V_\phi \\ &+ \{1 - (1 - \tilde{\Psi}')XV_\phi\}V_p \tilde{\nabla} V_p;\end{aligned} \tag{13b}$$

the vector $\boldsymbol{\mu}$ vanishes in the case of purely toroidal flow. Equation (12) shows that

$$\tilde{\nabla}^2\tilde{\Phi} = X^{-1}(\partial/\partial X)\{(1 - \tilde{\Psi}')\gamma X V_\phi\} - \tilde{\nabla}.\boldsymbol{\mu} + \tilde{\Psi}'\tilde{\nabla}^2\tilde{P} + \tilde{\nabla}\tilde{P}.\tilde{\nabla}\tilde{\Psi}', \tag{14}$$

which will be required soon.

Forms taken by the Gauss and Ampère laws under the constraints of steady rotation and axisymmetry have been given by Mestel *et al.* (1979). The former is

$$\tilde{\nabla}^2(\tilde{\Phi} - \tilde{P}) = -(\omega_0/\Omega)^2 \rho^e / n_0 e_0, \tag{15}$$

where ρ^e denotes the net electric charge density. The azimuthal component of the Ampère law is

$$\tilde{\nabla}^2\tilde{P} + 2\tilde{B}_z = (\omega_0/\Omega)^2 X j_\phi / n_0 e_0 c, \tag{16}$$

where $\tilde{\boldsymbol{B}} \equiv e\boldsymbol{B}/mc\Omega$ and j_ϕ is the azimuthal component of the electric current density.

Equation (14) relates $\tilde{\nabla}^2\tilde{\Phi}$ to $\tilde{\nabla}^2\tilde{P}$ and $\tilde{\nabla}\tilde{P}$, together with the flow variables. Eliminating $\tilde{\Phi}$ and $\tilde{\nabla}^2\tilde{P}$ among equations (14)–(16) results in

$$\begin{aligned}(\omega_0/\Omega)^2\{\rho^e - c^{-1}(1 - \tilde{\Psi}')Xj_\phi\}/n_0 e_0 = &-2(1 - \tilde{\Psi}')\tilde{B}_z - \tilde{\nabla}\tilde{P}.\tilde{\nabla}\tilde{\Psi}' \\ &+ \tilde{\nabla}.\boldsymbol{\mu} - X^{-1}(\partial/\partial X)\{(1 - \tilde{\Psi}')\gamma X V_\phi\}.\end{aligned} \tag{17}$$

If all species present in a region have the same azimuthal velocity component, or if a single species predominates, then j_ϕ reduces to $\rho^e v_\phi$ and equation (17) expresses ρ^e in terms of the magnetic field and the flow variables. I have previously given the resulting ρ^e for purely toroidal flows described by the complete integral (18) below (Burman 1984*a*). More generally, when j_ϕ cannot be equated to $\rho^e v_\phi$, an expression for ρ^e obtained by using equation (14) to eliminate $\tilde{\Phi}$ from the Gauss law (15) will contain a $\tilde{\nabla}^2\tilde{P}$ contribution.

Purely toroidal flows are described by a complete integral, together with a singular solution corresponding to precise corotation with the star; these follow from a quasilinear first-order partial differential equation which toroidal flows must satisfy, corresponding to the vanishing of the right-hand side of (7b), thus demonstrating the existence of an underlying quasilinear structure to that highly nonlinear problem. Complete integrals are not unique; one form is

$$\omega = b/(a+\tilde{P}^*), \tag{18}$$

where a and b are constants for each species. The equation of motion for purely toroidal flow is $(1-\omega)\tilde{\nabla}\tilde{P}^* = \tilde{\nabla}\tilde{\Psi}$, which can be integrated, on using the complete integral, to give

$$\tilde{\Psi} = 1 + b(1/\omega + \ln\omega - C), \tag{19}$$

where C is a constant; comparison with the Endean integral (1′) shows that (Burman 1984*a*)

$$\tilde{\Phi} = 1 - \gamma(1-XV_\phi) + b(1/\omega + \ln\omega - C). \tag{20}$$

Flows that approach corotation on the axis of symmetry are described by equations (18)–(20) with $b = a$ and $C = 1$.

It is feasible that there could exist a set of solutions of the electro-hydrodynamic equations that does not contain a purely toroidal flow as a limiting case. If, however, equation (4), relating V_p to $\tilde{\boldsymbol{B}}^*_\mathrm{p}$, is to remain valid in the limiting case of purely toroidal flow, the factor in braces must vanish in that limit. Comparison with the complete integral (18) for toroidal flow determines the corresponding functional form of $\tilde{\Psi}(\tilde{P}^*)$:

$$\tilde{\Psi}'(\tilde{P}^*) = 1 - b/(a+\tilde{P}^*), \tag{21}$$

which can be integrated to give

$$\tilde{\Psi}(\tilde{P}^*) = 1 + b[(a+\tilde{P}^*)/b - \ln\{(a+\tilde{P}^*)/b\} - C], \tag{22}$$

where C is a constant. Comparison with the Endean integral (1′) shows that

$$\tilde{\Phi} = 1 - \gamma(1-XV_\phi) + b[(a+\tilde{P}^*)/b - \ln\{(a+\tilde{P}^*)/b\} - C]. \tag{23}$$

For flows that have pure corotation, rather than toroidal flow described by the complete integral, as a limiting case, equation (4) shows that $\tilde{\Psi}$ is constant.

For flows which are connected to a perfectly conducting star by dissipation-free flow lines, and which are nonrelativistic near the star, $\tilde{\Psi}$ is not merely constant on each flow line of the species concerned, but has the same value, namely one, throughout the flow (Burman and Mestel 1978). Motion of pure corotation with the star also has constant $\tilde{\Psi}$, the value being one if the flow extends continuously outward from the star. Flows having constant $\tilde{\Psi}$ may be characterized as being flows that are 'strongly linked' to the star: in general they are not disconnected from the star by any region devoid of the species concerned or by the intervention of any region in which dissipation is significant, but corotational flow is always a member of this set. Flows which satisfy these conditions, but not that of being nonrelativistic near the star, will be referred to as being 'weakly linked' to the star.

Equation (5a) has the form of the poloidal part of the generalized isorotation law stating that the reduced flow velocity $\boldsymbol{u}$ of a species is parallel to its magnetoidal field $\boldsymbol{B}^*$ (Burman and Mestel 1978, 1979). But that law, which is independent of any spatial symmetry, is restricted to flows that are strongly linked to the star, meaning that $\tilde{\Psi}$ is constant. The validity of equation (5a) does not require the flows to be linked to the star, but the equation has been derived for the axisymmetric case and is a poloidal relation only: equation (5b) shows that $U_\phi = \kappa\tilde{B}_\phi^*$ for flows strongly linked to the star, but not otherwise.

3. Flow Dynamics

(*a*) *MWW Formalism*

Equation (6) relating the flow velocity to the magnetoidal field, equivalent to equation (2.27) of Mestel *et al.* (1979), was formed by combining its poloidal and toroidal parts. An alternative decomposition, into its parts perpendicular and parallel to the magnetoidal field, will now be considered; the respective contributions to the velocity will be denoted by $\boldsymbol{V}_\perp$ and $\boldsymbol{V}_\parallel$.

An appropriate choice of κ in equation (6) provides the 'perpendicular' part of that equation:

$$\boldsymbol{V}_\perp = (1-\tilde{\Psi}')(S^*\tilde{\boldsymbol{B}}^*/\tilde{B}^{*2} + X\boldsymbol{t}), \tag{24}$$

where S^* stands for $-X\tilde{B}_\phi^*$. It follows that $V_\perp = |1-\tilde{\Psi}'|X\tilde{B}_\mathrm{p}^*/\tilde{B}^*$ and, with γ_m denoting the Lorentz factor corresponding to this speed, that

$$\gamma_\mathrm{m}^{-2} = 1-(1-\tilde{\Psi}')^2(X\tilde{B}_\mathrm{p}^*/\tilde{B}^*)^2 \tag{25}$$

$$= 1+(1-\tilde{\Psi}')^2(S^{*2}/\tilde{B}^{*2}-X^2). \tag{25'}$$

On expressing $V_\parallel^2$ as $\gamma_\mathrm{m}^{-2}-\gamma^{-2}$, the 'parallel' part of the equation takes the form

$$\boldsymbol{V}_\parallel = \Delta S^*\tilde{\boldsymbol{B}}^*/\tilde{B}^{*2}, \tag{26}$$

where

$$\Delta \equiv (\gamma_\mathrm{m}^{-2}-\gamma^{-2})^{\frac{1}{2}}\tilde{B}^*/S^* \tag{27}$$

$$= [(1-\tilde{\Psi}')^2+\{1-(1-\tilde{\Psi}')^2X^2-\gamma^{-2}\}(\tilde{B}^*/S^*)^2]^{\frac{1}{2}}. \tag{27'}$$

Recombining (24) and (26) into an equation for $\boldsymbol{V}$ yields equation (6) with

$$\kappa = (1-\tilde{\Psi}'+\Delta)S^*/\tilde{B}^{*2}. \tag{28}$$

This procedure of splitting equation (6) for $\boldsymbol{V}$ into parts perpendicular and parallel to $\tilde{\boldsymbol{B}}^*$, expressing $\boldsymbol{V}_\parallel$ in terms of γ and γ_m, and recombining the parts into equation (6) again, has provided a representation of the flow velocity formally equivalent to that introduced by MWW; they neglected inertial drift and considered flows strongly linked to the star. In the next two subsections, I shall look into the physical interpretation of their formalism.

(*b*) *Driving Forces*

Here I shall develop the equation of motion of a species into a form in which each term has a clear physical interpretation. By approximating this equation, and adapting plasma drift theory, I shall use it in Section 3c to elucidate the physical basis of the MWW formalism. Because of the physical nature of the argument, I have found it convenient to return to using quantities with dimensions.

By splitting the flow velocity into its toroidal and poloidal parts, and using the vector identity $\boldsymbol{\alpha} \times (\nabla \times \boldsymbol{\alpha}) \equiv \frac{1}{2}\nabla(\alpha^2) - \boldsymbol{\alpha} \cdot \nabla\boldsymbol{\alpha}$ with $\gamma \boldsymbol{v}_p$ for $\boldsymbol{\alpha}$, it is found that, under axisymmetry,

$$\boldsymbol{u} \times (\nabla \times \gamma \boldsymbol{v}) = (u_\phi/\varpi)\nabla(\varpi\gamma v_\phi) - \boldsymbol{t}\varpi^{-1}\boldsymbol{v}_p \cdot \nabla(\varpi\gamma v_\phi) + (1/2\gamma)\nabla(\gamma^2 v_p^2) - \boldsymbol{v}_p \cdot \nabla(\gamma \boldsymbol{v}_p).$$

Use of the expression $\varpi^{-1}\boldsymbol{t} \times \nabla P$ for $\boldsymbol{B}_p$ shows that

$$\boldsymbol{u} \times \boldsymbol{B} = \boldsymbol{t}\varpi^{-1}\boldsymbol{v}_p \cdot \nabla P - (u_\phi/\varpi)\nabla P + B_\phi \boldsymbol{v}_p \times \boldsymbol{t}.$$

These expressions are to be substituted into the equation of motion (2). On recalling that P^* is a constant of the motion of the species concerned, and invoking the definition (1) of Ψ to eliminate its gradient, equation (2) is found to take the form

$$-\nabla\Phi = (m/e)\{-\gamma v_\phi^2 \boldsymbol{i}/\varpi + \gamma v_p^2(\boldsymbol{p} \cdot \nabla)\boldsymbol{p} + \boldsymbol{v}_p(\boldsymbol{p} \cdot \nabla)(\gamma v_p)\} + (u_\phi/c\varpi)\nabla P + B_\phi \boldsymbol{t} \times \boldsymbol{v}_p/c, \tag{29}$$

where $\boldsymbol{p}$ denotes the unit tangent vector to the poloidal flow lines.

Under the steady-rotation constraint, the Lorentz force $\boldsymbol{F}/e$ per unit charge acting on a species can be expressed as $-\nabla\Phi + c^{-1}\boldsymbol{u} \times \boldsymbol{B}$ (Burman and Mestel 1978), becoming

$$\boldsymbol{F}/e = -\nabla\Phi - (u_\phi/c\varpi)\nabla P + c^{-1}\boldsymbol{v}_p \times \boldsymbol{B} \tag{30}$$

under axisymmetry. So the equation of motion (29) can be written

$$\boldsymbol{F}/m = -\gamma v_\phi^2 \boldsymbol{i}/\varpi + \gamma v_p^2(\boldsymbol{p} \cdot \nabla)\boldsymbol{p} + \boldsymbol{v}_p(\boldsymbol{p} \cdot \nabla)(\gamma v_p) + (e/mc)\boldsymbol{v}_p \times \boldsymbol{B}_p. \tag{29$'$}$$

The physical significance of the various terms on the right-hand side of this equation is clear: the first term represents the centripetal acceleration, directed toward the axis of symmetry, of the toroidal part of the motion; the second represents the centripetal acceleration associated with the poloidal part of the motion, and is directed toward the local centre of curvature of the poloidal flow line; the third represents the linear acceleration of the poloidal flow, and so is parallel to the tangent vector to the flow lines; the fourth term is a magnetic acceleration arising through departure of the poloidal flow from the poloidal magnetic field lines.

The equation of motion (29) can be regarded as giving the non-corotational electric field $-\nabla\Phi$ required to support the motion against inertia and against magnetic forces arising from drift across the rotating magnetic field structure. Inertia has been decomposed into three effects, namely the rotational inertia of the toroidal and poloidal parts of the flow and the linear acceleration of the poloidal flow; the magnetic forces, corresponding to the last two terms in equation (29), arise from toroidal motion relative to the rotating poloidal magnetic field structure and from poloidal motion relative to the toroidal magnetic field.

The differential equations (5) relating V_p and V_ϕ can be written in terms of dimensioned quantities as

$$\boldsymbol{v}_p = \lambda \boldsymbol{B}_p^* = \lambda \varpi^{-1} \boldsymbol{t} \times \nabla P^*, \tag{31a, b}$$

with

$$u_\phi = \lambda B_\phi^* - \Omega \varpi \tilde{\Psi}', \qquad \lambda \equiv e\kappa/m\Omega. \tag{31c, d}$$

The centripetal acceleration of the poloidal flow has magnitude $\gamma v_p^2/\rho$, where ρ is the local radius of curvature of the flow line, and is directed along the inward normal to the line. In mathematical terms, the unit tangent vector to the flow line rotates at rate $1/\rho$ around the instantaneous binormal to the line, which here is $\boldsymbol{t}$; that is, we have $(\boldsymbol{p}.\nabla)\boldsymbol{p} = \boldsymbol{t} \times \boldsymbol{p}/\rho = \boldsymbol{n}/\rho$ where $\boldsymbol{n}$ is the principal normal to the line, namely $-\nabla P^*/\varpi B_p^*$. Thus, the centripetal acceleration of the poloidal flow is given by

$$\gamma v_p^2 (\boldsymbol{p}.\nabla)\boldsymbol{p} = -(\gamma v_p^2/\rho)(\nabla P^*)/\varpi B_p^* = -(\gamma v_p/\rho)(\lambda/\varpi)\nabla P^*, \tag{32a, b}$$

where equation (31a) has been used to get equation (32b).

Equation (31b) shows that

$$B_\phi \boldsymbol{t} \times \boldsymbol{v}_p/c = -B_\phi(\lambda/c\varpi)\nabla P^*. \tag{33}$$

Because of equations (32) and (33), the terms in equation (29) associated with the centripetal acceleration of the poloidal flow and with the poloidal motion relative to the toroidal magnetic field combine into

$$-(\lambda/c\varpi)\{B_\phi + (mc/e)\gamma v_p/\rho\}\nabla P^*. \tag{34}$$

The equation of motion becomes

$$\begin{aligned} -\nabla\Phi = {} & (u_\phi/c\varpi)\nabla P - (\lambda/c\varpi)\{B_\phi + (mc/e)\gamma v_p/\rho\}\nabla P^* \\ & - (m/e)\gamma v_\phi^2 \boldsymbol{i}/\varpi + (m/e)\boldsymbol{v}_p(\partial/\partial s)(\gamma v_p), \end{aligned} \tag{35}$$

where $\partial/\partial s$ denotes differentiation along a poloidal flow line.

I shall make the approximation of regarding the quantity $\gamma v_p/\rho$ as an estimate of the relativistic vorticity $\nabla \times \gamma \boldsymbol{v}_p$ of the poloidal flow; it has the right sign and order of magnitude. Thus, the quantity in braces in equation (35) is approximately B_ϕ^*: the equation of motion becomes

$$\begin{aligned} -\nabla\Phi \approx {} & (u_\phi/c\varpi)\nabla P - (\lambda B_\phi^*/c\varpi)\nabla P^* - (m/e)\gamma v_\phi^2 \boldsymbol{i}/\varpi \\ & + (m/e)\boldsymbol{v}_p(\partial/\partial s)(\gamma v_p). \end{aligned} \tag{36}$$

I shall now consider flows in which the poloidal motion is closely tied to the poloidal magnetic field lines: the exact equations (31a, b) are replaced by

$$\boldsymbol{v}_p \approx \lambda \boldsymbol{B}_p = \lambda \varpi^{-1} \boldsymbol{t} \times \nabla P, \tag{37a, b}$$

but with u_ϕ still given by (31c). In other words, inertial drift of the poloidal flow across the poloidal magnetic field lines will be neglected, but the inertial contribution to B_ϕ^* will be retained.

In this approximation, the centripetal acceleration of the poloidal flow is orthogonal to $\boldsymbol{B}_p$, so ∇P^* in equations (32) is to be replaced by ∇P. Also, use of $\boldsymbol{v}_p \approx \lambda \boldsymbol{B}_p$ instead of $\boldsymbol{v}_p = \lambda \boldsymbol{B}_p^*$ shows that ∇P^* in equation (33), describing the effect of poloidal motion relative to the toroidal magnetic field, is to be replaced by ∇P. So ∇P replaces ∇P^* in the expression (34) combining these effects, and hence in the equation of motion (36), which becomes

$$-\nabla \Phi \approx -(\Omega/c)\tilde{\Psi}'\nabla P - (m/e)\gamma v_\phi^2 \boldsymbol{i}/\varpi + (m/e)\boldsymbol{v}_p(\partial/\partial s)(\gamma v_p); \tag{38}$$

equation (31c) has been used to substitute for $u_\phi - \lambda B_\phi^*$ in the ∇P term. Equation (38) will be used in the next subsection to study the motion across the magnetic field.

Since $\boldsymbol{B}_p$, which is the external magnetic field of the star as modified by the magnetospheric toroidal currents, is so large, neglect of the inertial part of $\boldsymbol{B}_p^*$ is likely to be a very accurate approximation throughout much of the magnetosphere; however, the approximation is subject to failure in inertial boundary layers, separating electron-dominated and ion-dominated zones, in which the number densities of the species drop rapidly and a process of inertial development of vorticity occurs. On the other hand, it is desirable to make allowance for the inertial part of B_ϕ^*: it cannot yet be said whether or under what circumstances B_ϕ, which is generated by the poloidal magnetospheric currents, is larger or smaller than $(mc/e)\nabla \times (\gamma \boldsymbol{v}_p)$.

(c) *Drift across* $\boldsymbol{B}$

Equation (38) expresses the non-corotational electric field $-\nabla\Phi$ required to support the motion against inertial and drift effects as the sum of three parts. The first part, which is absent for flows that are strongly linked to the star, supports a 'residual drift' at rate $u_\phi - \lambda B_\phi^*$, which equals $-\Omega\varpi\tilde{\Psi}'$, of the toroidal motion relative to the poloidal magnetic field structure; the second part is just the centripetal force per unit charge associated with the toroidal flow; the remaining part of $-\nabla\Phi$, after these two contributions have been subtracted, provides the linear acceleration of the poloidal flow along the poloidal magnetic field lines. The three contributions to Φ may be referred to as the 'residual drift', 'centrifugal' and 'linear acceleration' parts; i.e. $\Phi = \Phi^r + \Phi^c + \Phi^a$.

The usual plasma physics formula for the drift velocity in crossed electric and magnetic fields, namely $\boldsymbol{V} = \boldsymbol{E} \times \boldsymbol{B}/B^2$, can be adapted as follows to this problem involving steady relativistic rotation. Under steady rotation, we have $\boldsymbol{E} = X\boldsymbol{B} \times \boldsymbol{t} - \nabla\Phi$, which reduces to $\nabla(\Omega P/c - \Phi)$ under axisymmetry. But only the $-\nabla\Phi^r$ part of the non-corotational electric field is to be inserted into the drift velocity formula: it is that part which is relevant to maintaining the residual drift. The $-\nabla\Phi^c$ part, which is balanced by the centrifugal effect of the steady rotation, and the $-\nabla\Phi^a$ part, which produces the linear acceleration of the poloidal flow, are to be omitted. The corotational electric field $X\boldsymbol{B} \times \boldsymbol{t}$, generated by rotation of the magnetic field structure, provides a contribution to $\boldsymbol{V}$ which is the value it would take if inertia were neglected. The approximate equation of motion (38) shows that $\nabla\Phi^r = (\Omega/c)\tilde{\Psi}'\nabla P$. Hence the effective electric field, in so far as the usual drift velocity formula is concerned, is $(\Omega/c)(1-\tilde{\Psi}')\nabla P$. The resulting velocity, combining the steady rotation and the residual drift, is $\boldsymbol{V}_\perp$, where

$$\boldsymbol{V}_\perp = (1-\tilde{\Psi}')(\tilde{S}\tilde{\boldsymbol{B}}_p + X\tilde{B}_p^2\boldsymbol{t})/\tilde{B}^2. \tag{39}$$

The corresponding Lorentz factor is given by

$$\gamma_m^{-2} = 1 - (1 - \tilde{\Psi}')^2 (X\tilde{B}_p/\tilde{B})^2 \tag{40}$$

$$= 1 + (1 - \tilde{\Psi}')^2 (\tilde{\boldsymbol{S}}^2/\tilde{B}^2 - X^2). \tag{40'}$$

Equations (39) and (40) are just (24) and (25) with the inertial contribution to the magnetoidal field neglected.

Let us consider for a moment flows that are strongly linked to the star. Near the star, we have $B_\phi \ll B_p$ and the toroidal component of $\boldsymbol{V}_\perp$ is close to the corotational value X. The poloidal part $\tilde{S}\tilde{\boldsymbol{B}}_p/\tilde{B}^2$ of $\boldsymbol{V}_\perp$ is vanishingly small at the star: the drift motion can describe outflow from the star, but only with insignificant emission speed. The Lorentz factor γ_m is given by $\gamma_m^{-2} = 1 - X^2 + \tilde{S}^2/\tilde{B}^2$.

(*d*) *Injection along* $\boldsymbol{B}$

Outflow with significant emission speed requires the inclusion of a contribution $\boldsymbol{V}_\parallel$ to the velocity, as in Section 3*a*:

$$\boldsymbol{V}_\parallel = (\gamma_m^{-2} - \gamma^{-2})^{\frac{1}{2}} \boldsymbol{B}/B. \tag{41}$$

Combining (39) and (41) gives

$$\boldsymbol{V} = (1 - \tilde{\Psi}' + \Delta)\tilde{S}\tilde{\boldsymbol{B}}_p/\tilde{B}^2 + \{(1 - \tilde{\Psi}')\tilde{B}_p^2 - \Delta \tilde{B}_\phi^2\} X\boldsymbol{t}/\tilde{B}^2, \tag{42}$$

with

$$\Delta \equiv (\gamma_m^{-2} - \gamma^{-2})^{\frac{1}{2}} \tilde{B}/\tilde{S}; \tag{43}$$

these equations are equivalent to equation (6) with the inertial contribution to the magnetoidal field neglected and with

$$\kappa = (1 - \tilde{\Psi}' + \Delta)\tilde{S}/\tilde{B}^2. \tag{44}$$

This reproduces some of the basic flow equations introduced by MWW. The treatment shows that flow with $\gamma = \gamma_m$ corresponds to a plasma drift, perpendicular to $\boldsymbol{B}$, driven by the part $(\Omega/c)(1 - \tilde{\Psi}')\nabla P$ of the electric field in the presence of the magnetic field. The equations show that γ_m (when it is real) is a lower bound on the Lorentz factor (see MWW): pure drift flow is the least relativistic of flows incorporating poloidal motion. The present approach shows that flow described by the MWW formalism with $\gamma > \gamma_m$ may be interpreted as a plasma drift following injection: a significant injection velocity leads to the presence of the contribution $\boldsymbol{V}_\parallel$ to the flow velocity.

The linear poloidal acceleration can be found by comparing the approximate equation of motion (38) with the exact equation (12) for $\tilde{\nabla}\tilde{\Phi}$, resulting in

$$V_p(\partial/\partial\tilde{s})(\gamma V_p) \approx \boldsymbol{\mu} + \{1 - (1 - \tilde{\Psi}')X/V_\phi\}\gamma V_\phi^2 \boldsymbol{i}/X, \tag{45}$$

where $\tilde{s}$ denotes $\Omega s/c$, a dimensionless measure of distance along a poloidal flow line, and $\boldsymbol{\mu}$ is defined by (13a).

We consider flows that are not linked to the star, but have toroidal flow described by the complete integral (18) as a limiting case, so that $\tilde{\Psi}'$ and $\tilde{\Psi}$ are given by (21) and (22). Substituting (21) for $\tilde{\Psi}'$ into (39) for $V_\perp$ provides the flow velocity:

$$V_\phi = \{b/(a+\tilde{P}^*)\}X\tilde{B}_p^2/\tilde{B}^2\,, \tag{46a}$$

$$V_p = \{b/(a+\tilde{P}^*)\}\bar{S}\tilde{\boldsymbol{B}}_p/\tilde{B}^2\,. \tag{46b}$$

Since $\tilde{P}^* \equiv \tilde{P} - X\gamma V_\phi$, equation (46a) relating V_ϕ and γ can be rearranged into

$$\gamma V_\phi + (B_p^2/B^2)b/V_\phi \approx (a+\tilde{P})/X\,. \tag{47}$$

On using equation (46a), equations (21) and (22) for $\tilde{\Psi}'$ and $\tilde{\Psi}$ become

$$\tilde{\Psi}' \approx 1 - B^2 V_\phi/B_p^2 X\,, \tag{48}$$

$$\tilde{\Psi} \approx 1 + b\{B_p^2 X/B^2 V_\phi - \ln(B_p^2 X/B^2 V_\phi) - C\}\,. \tag{49}$$

Equations (46) describe flows that are not connected to the star by lines of dissipation-free flow. They extend the complete integral for purely toroidal flow to incorporate poloidal flow, provided it is almost along the poloidal magnetic field lines. Equation (39) for $V_\perp$ with $\tilde{\Psi}'$ put equal to zero describes flows that are strongly linked to the star, extending the singular integral describing pure corotation to incorporate poloidal flow, again provided it is closely tied to the poloidal magnetic field lines.

4. Inside the Light Cylinder

In a paper on purely toroidal flow (Burman 1984*a*), I paid particular attention to flows which approach corotation on the symmetry axis. These are described by the complete integral (18), together with equations (21) and (22) for $\tilde{\Psi}'$ and $\tilde{\Psi}$, on putting $b = a$ and $C = 1$. For the flow to remain close to corotation for a sensible distance from the star, the constant a needs to be huge, at least of order ω_{B_s}/Ω; here $\omega_{B_s} \equiv eB_s/mc$ with B_s denoting the polar magnetic field strength on the stellar surface. I shall now study the subclass of flows specified by taking $b = a$ and $C = 1$ in equations (46)–(49) for V_ϕ, V_p, $\tilde{\Psi}'$ and $\tilde{\Psi}$. Again, a will be taken to be at least of order ω_{B_s}/Ω. This will extend my previous calculations on purely toroidal flows inside the light cylinder to incorporate poloidal flow that is tied to the poloidal magnetic field lines.

With $b = a$, equation (47) relating V_ϕ to γ can be written in the form

$$\frac{V_\phi}{X} \approx \left(\frac{B_p^2}{B^2} + \frac{\gamma V_\phi^2}{a}\right)\Big/\left(1 + \frac{\tilde{P}}{a}\right). \tag{50}$$

The condition $\gamma \ll |a|$ will be very comfortably satisfied, implying that equation (50) can be approximated further to

$$\frac{V_\phi}{X} \approx \left(1 - \frac{B_\phi^2}{B^2}\right)\Big/\left(1 + \frac{\tilde{P}}{a}\right). \tag{51}$$

If a is written as $d\omega_{B_s}/\Omega$, where d is a constant, then, in the dipole approximation,

$$\tilde{P}/a \approx -(\Omega R/c)^2(R/2dr)\sin^2\theta; \tag{52}$$

here r and θ are the spherical polar coordinates in meridional planes and R is the radius of the star. Equation (52) gives a maximum value $(\Omega R/c)^2(2d)^{-1}$ for $|\tilde{P}/a|$, occurring at the equator on the stellar surface. Since the dipole approximation provides some idea of the magnetospheric magnetic field strength and since d is at least of the order of one, it is clear that $|\tilde{P}/a|$ is very small everywhere. So equation (51) can be yet further approximated to

$$V_\phi/X \approx (1 - B_\phi^2/B^2)(1 - \tilde{P}/a). \tag{53}$$

Since $B_\phi \to 0$ as $X \to 0$, equation (51) shows that $V_\phi \to X$ as $X \to 0$: flows of this subclass can be characterized as those for which the toroidal component of the flow velocity approaches the corotational value $\Omega\varpi$ on the symmetry axis. Equation (46b) shows that V_p varies as $-(B_\phi/B_p)X$ as $X \to 0$, and so vanishes faster than V_ϕ as the axis is approached.

Equation (51) shows that the flow is sub-rotating so long as

$$B_\phi^2/B^2 + \tilde{P}/a > 0. \tag{54}$$

Incorporation of poloidal flow has introduced the positive contribution B_ϕ^2/B^2 into the left-hand side of this inequality, and so has greatly increased the likelihood of sub-rotation.

The condition $\gamma \ll |a|$ cannot be a sufficient one for equation (51) to be valid: the relativity restriction $V_\phi < 1$ implies that the condition

$$X \lesssim \left(1 + \frac{\tilde{P}}{a}\right) \Big/ \left(1 - \frac{B_\phi^2}{B^2}\right) \tag{55}$$

must be satisfied. So, if $\tilde{P}/a$ is positive, equations (51) and (53) will be applicable from the axis to at least slightly beyond the light cylinder, further if B_ϕ is a substantial fraction of B in that vicinity. If $\tilde{P}/a$ is negative, then equations (51) and (53) will be applicable from the axis at least out to slightly inside the light cylinder, but very likely until well beyond it, depending on the relative size of $|\tilde{P}/a|$ and B_ϕ^2/B^2 in that vicinity.

Use of equation (51) for V_ϕ in equations (48) and (49) for $\tilde{\Psi}'$ and $\tilde{\Psi}$, with $b = a$ and $C = 1$, shows that

$$\tilde{\Psi}' \approx 1 - (1 + \tilde{P}/a)^{-1} \approx \tilde{P}/a, \tag{56a, b}$$

$$\tilde{\Psi} \approx 1 + a\{\tilde{P}/a - \ln(1 + \tilde{P}/a)\} \approx 1 + \tilde{P}^2/2a; \tag{57a, b}$$

the small size of $\tilde{P}/a$ has been used in obtaining the approximations (56b) and (57b).

Substitution of V_ϕ from equation (51) into the Endean integral (1) gives

$$\tilde{\Phi} \approx \tilde{\Psi} - \frac{1 + \tilde{P}/a - (XB_p/B)^2}{\{(1 + \tilde{P}/a)^2(1 - V_p^2) - X^2(B_p/B)^4\}^{\frac{1}{2}}}. \tag{58}$$

Similarly, the approximate equation of motion (38) becomes

$$\nabla\Phi \approx \frac{\tilde{P}/a}{1+\tilde{P}/a}\tilde{\nabla}\tilde{P} + \left(\frac{B_\mathrm{p}}{B}\right)^4 \frac{\gamma X\boldsymbol{i}}{(1+\tilde{P}/a)^2} - V_\mathrm{p}\frac{\partial}{\partial\tilde{s}}(\gamma V_\mathrm{p}) \tag{59a}$$

$$\approx \{(1+\tilde{P}/a)^2(1-V_\mathrm{p}^2) - X^2(B_\mathrm{p}/B)^4\}^{-\frac{1}{2}}(B_\mathrm{p}/B)^4 X\boldsymbol{i} + (\tilde{P}/a)X\Omega^{-1}\boldsymbol{\omega}_B \times \boldsymbol{t} - V_\mathrm{p}(\partial/\partial\tilde{s})(\gamma V_\mathrm{p}), \tag{59b}$$

where $\boldsymbol{\omega}_B \equiv e\boldsymbol{B}/mc$ and the small size of $\tilde{P}/a$ has been used in obtaining (59b).

So long as $X^2(B_\mathrm{p}/B)^4$ is not close to $1-V_\mathrm{p}^2$ and $(XB_\mathrm{p}/B)^2$ is not close to one, equation (58) for $\tilde{\Phi}$ can be replaced by

$$\tilde{\Phi} \approx 1 - \frac{1-(XB_\mathrm{p}/B)^2}{(1-X^2B_\mathrm{p}^4/B^4 - V_\mathrm{p}^2)^{\frac{1}{2}}} + \frac{\tilde{P}^2}{2a}. \tag{60}$$

So long as $X^2(B_\mathrm{p}/B)^4$ is not close to $1-V_\mathrm{p}^2$, equation (59b) for $\tilde{\nabla}\tilde{\Phi}$ can be replaced by

$$\tilde{\nabla}\tilde{\Phi} \approx (1-X^2B_\mathrm{p}^4/B^4 - V_\mathrm{p}^2)^{-\frac{1}{2}}(B_\mathrm{p}/B)^4 X\boldsymbol{i} - V_\mathrm{p}(\partial/\partial\tilde{s})(\gamma V_\mathrm{p}) + (\tilde{P}/a)X\Omega^{-1}\boldsymbol{\omega}_B \times \boldsymbol{t}. \tag{61}$$

In equation (58) for $\tilde{\Phi}$, expanding in powers of X^2, V_p^2 and B_ϕ^2/B^2 to second order and in $\tilde{P}/a$ to first order, yields

$$\tilde{\Phi} \approx \tilde{P}^2/2a + \tfrac{1}{2}(X^2 - V_\mathrm{p}^2) + \tfrac{1}{8}(X^2 - 3V_\mathrm{p}^2)(X^2 + V_\mathrm{p}^2); \tag{62}$$

to this order, $\tilde{P}/a$ and B_ϕ^2/B^2 do not contribute.

5. Implications for Model Building

The standard Goldreich–Julian (1969) model of the pulsar magnetosphere features zones of corotating electrons and positive ions, in which particle inertia and all forces other than the Lorentz force are neglected. The electron and ion zones are separated by the $B_z = 0$ surfaces. But this configuration is highly unstable to charge mixing across those surfaces (Jackson 1978; Burman 1981*b*, 1981*c*).

When particle inertia is allowed for, the non-corotational part of the electric field, with potential Φ, must be introduced in order to support the motion. If the assumption of corotation is retained, then the requisite Φ is proportional to the mass-to-charge ratio of the species: it is negative and comparatively small in the electron zones and positive and very much larger in the ion zone. So there is a big jump in Φ between the zones, and the resulting electric field is directed so as to accelerate electrons into the ion zone and ions into the electron zones; it is, beyond $X \approx 1/30$, sufficient to accelerate electrons to relativistic speeds (Burman 1981*b*, 1981*c*).

In a previous paper (Burman 1984*a*), I studied the problem of relaxing the constraint of corotation to one of toroidal flow. I found that the fundamental problem of mismatch of Φ and its derivatives between electron and ion zones remains. With the constant a expressed as $d\omega_{B_\mathrm{s}}/\Omega$, if d is independent of the species, then the jump in Φ, and the corresponding accelerating electric field between the zones, are much the same as with precisely corotating zones: relaxing the assumption of corotation to one of toroidal flow does not alleviate the mismatch problem at all.

An outcome of these analyses is that there can be no zones occupied solely by toroidal flowing particles: although the toroidal flow condition can be satisfied locally, global considerations imply that poloidal flow is endemic. There may well be zones that are dominated by particles whose motion is close to being toroidal, but those particles will have a poloidal part to their motion and, furthermore, those zones will be penetrated by fast poloidally flowing particles of the species of opposite sign.

Thus, poloidal flow must be incorporated into the analysis, and this paper is directed to that end. What, then, is the effect of removing the toroidal flow constraint? The equations of the last section provide a partial answer: the mismatch is at least ameliorated by allowing for the poloidal flow that must occur.

With the constant a written as $d\omega_{B_s}/\Omega$, equation (60) for Φ, and its expanded form (62), can be written as, respectively,

$$\Phi \approx \frac{mc^2}{e}\left(1 - \frac{1-(XB_p/B)^2}{(1-X^2B_p^4/B^4 - V_p^2)^{\frac{1}{2}}}\right) + \frac{\Omega}{2cdB_s}\left(\frac{\Omega P}{c}\right)^2, \tag{63}$$

$$\Phi \approx (mc^2/e)\{\tfrac{1}{2}(X^2 - V_p^2) + \tfrac{1}{8}(X^2 - 3V_p^2)(X^2 + V_p^2)\} + (\Omega/2cdB_s)(\Omega P/c)^2\,. \tag{64}$$

With d independent of the species, any jump in Φ between electron and ion zones will arise from the part of Φ that involves the mass-to-charge ratio of the species. Equation (63) shows that this contribution is reduced in size by the presence of V_p^2. To the extent that equation (64) is applicable, B_ϕ does not enter. Hence, at least for much of the region inside the light cylinder, allowance for the poloidal flow decreases the size of the potentially discontinuous part of Φ. (I am making the reasonable assumption that the effect is not so large as to invert the sign of the discontinuity.)

The above equations show that the poloidal speeds in the boundary layers will, for $X^2 \ll 1$, need to be a significant fraction of $\Omega\varpi$ if inclusion of poloidal flow is to have a substantial effect on the mismatch problem. In fact, for X^2 not too large, the value $X(1-X^2)^{\frac{1}{2}}$ for V_p in the boundary layer would reduce the potentially discontinuous part of Φ to zero. But X is the value taken by V_ϕ near the axis of symmetry, and the discussion following equation (53) shows that $V_p \ll X$ there.

Qualitatively, then, allowance for the poloidal flow at least helps to overcome the mismatch problem. Little more can be said until the analysis has been extended to allow for the departure of the poloidal flow from the poloidal magnetic field lines.

6. Concluding Remarks

In a recent paper on toroidal flow in axisymmetric pulsar magnetospheres (Burman 1984*a*), I showed that the qualitative implications of my earlier work based on corotation remain unaffected when the constraint of corotation is relaxed to one of purely toroidal flow. The mismatch of the non-corotational potential between electron and ion zones implies that there can be no zones occupied solely by toroidally flowing particles. There may well be zones dominated by particles whose motion is largely toroidal, but their motion will have at least a small poloidal part, and the zones will be penetrated by fast poloidally flowing particles of the opposite sign. The real problem in understanding the pulsar magnetosphere is not the Goldreich–Julian one of satisfying the boundary conditions on the star: it is that of matching regions of different species.

In the present paper I have used the integrals of the motion, including a complete integral for toroidal flow, to reorganize and simplify the fundamental electrodynamic and hydrodynamic equations for axisymmetric magnetospheres. The extension of the complete integral for purely toroidal flow to incorporate poloidal motion that is closely tied to the poloidal magnetic field lines has shown that the inclusion of the poloidal flow in the analysis at least reduces the mismatch problem. But further progress along these lines will require the theory to be extended to allow for the effect of inertial development of vorticity, whereby the poloidal flow departs from the poloidal magnetic field in the boundary layers separating the electron and ion zones.

Acknowledgments

I thank Professor L. Mestel, FRS, for access to the MWW work, and an anonymous referee for a helpful review; these factors led to an extensive revision of this paper.

References

Burman, R. R. (1981*a*). *Aust. J. Phys.* **34**, 91.
Burman, R. R. (1981*b*). *Aust. J. Phys.* **34**, 303.
Burman, R. R. (1981*c*). *Aust. J. Phys.* **34**, 317.
Burman, R. R. (1984*a*). *Aust. J. Phys.* **37**, 105.
Burman, R. R. (1984*b*). *Proc. Astron. Soc. Aust.* **5**, 467.
Burman, R. R., and Mestel, L. (1978). *Aust. J. Phys.* **31**, 455.
Burman, R. R., and Mestel, L. (1979). *Aust. J. Phys.* **32**, 681.
Endean, V. G. (1972*a*). *Nature Phys. Sci.* **237**, 72.
Endean, V. G. (1972*b*). *Mon. Not. R. Astron. Soc.* **158**, 13.
Goldreich, P., and Julian, W. H. (1969). *Astrophys. J.* **157**, 869.
Jackson, E. A. (1978). *Mon. Not. R. Astron. Soc.* **183**, 445.
Mestel, L. (1971). *Nature Phys. Sci.* **233**, 149.
Mestel, L., Phillips, P., and Wang, Y.-M. (1979). *Mon. Not. R. Astron. Soc.* **188**, 385.
Westfold, K. C. (1981). *Aust. J. Phys.* **34**, 595.
Wright, G. A. E. (1978). *Mon. Not. R. Astron. Soc.* **182**, 735.

Manuscript received 28 September, accepted 11 December 1984