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Universal Features of Tangent Bifurcation

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Abstract

We exhibit certain universal characteristics of limit cycles pertaining to one-dimensional maps in the 'chaotic' region beyond the point of accumulation connected with period doubling. Universal, Feigenbaum-type numbers emerge for different sequences, such as triplication. More significantly we have established the existence of *different* classes of universal functions which satisfy the *same* renormalization group equations, with the *same* parameters, as the appropriate accumulation point is reached.

1. Introduction

Considerable progress has been achieved during the last few years in our understanding of turbulent or chaotic behaviour in natural processes (Ruelle and Takens 1971; Ott 1981; Eckmann 1981; Hu 1982). Much of the insight has come from a study of one-dimensional nonlinear mappings, both in qualitative and quantitative terms (May 1976; Collet and Eckmann 1980). Experimental evidence from diverse scientific fields ranging from physics through chemistry to biology has accumulated, which provides substantial support to the scenario based upon the period-doubling route to chaos, not only in the regime before the onset of chaos, but in the regime beyond where turbulence has developed. However, in that chaotic domain there exist certain windows of stability connected with low period cycle structures and in their vicinity one may observe the phenomenon of intermittent periodicity (Manneville and Pomeau 1980; Hirsch *et al.* 1982; Hu and Rudnick 1982).

Most of the theoretical studies (Feigenbaum 1983) have been focussed on the neighbourhood of the accumulation point of the first pitchfork bifurcation sequence and many of the characteristic universal properties have been thoroughly investigated there. In this paper we wish to highlight a number of universal properties that lie beyond this region and pertain to tangent bifurcations. These properties are partly implied in the paper by Derrida *et al.* (1979) which described the self-similarity of chaotic bands and cycles in that regime, but they are not widely known. We will exhibit what we believe are several new features associated with windows of stability to the right of the onset of chaos. Apart from demonstrating the occurrence of universal numbers connected with period multiplications in a 'forward' and reverse' sense (see Sections 2–4), we also show that the solution of the corresponding renormalization group equations near the accumulation points is by no means unique,

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despite the scaling parameters being the same. This provides the necessary graphic support to McCarthy's (1983) mathematical analysis which also proposed a multiplicity of such solutions.

We have attempted to make this paper self-contained by providing all the numerical and other evidence needed. Sometimes we have not been able to avoid covering familiar ground; still, we believe that the various tables and figures will be of real value to the expert and nonexpert alike by exposing, at a glance, all the numerical details* about the attainment of the various limits for two typical mappings. In Section 2 we summarize the well-known properties of pitchfork sequences, and in Section 3 we show that another 'reverse' period-doubling sequence in the chaotic region is governed by the same universal constants, subject to one important proviso: namely, the occurrence of families of solutions of the (duplication) functional equation. Sections 3 and 4 generalize the work to period triplings; again we demonstrate the existence of many solutions to the (triplication) functional equation by examining the reverse sequences of functions as one approaches the period-tripling accumulation point. We conclude in Section 5 with a number of comments about fractional universal functions by tracking other function sequences.

2. Windows of Stability

It has long been established (Metropolis *et al.* 1973) that for smooth maps of the real axis onto itself of the type

$$x \to F(\lambda, x); \qquad a \leqslant x \leqslant b,$$

where F has a unique maximum $x = X$ in the interval $[a, b]$ and λ is constrained to lie in some specified range, there is a universal sequence of limit cycles. This sequence is independent of the detailed form of the mapping F beyond the conditions stated. For example, it applies to the typical maps

$$\text{(A)} \qquad x \to \lambda x(1-x), \quad 0 < x < 1, \ 1 < \lambda < 4 \qquad \text{with } X = \tfrac{1}{2}; \tag{1a}$$

$$\text{(B)} \qquad x \to x e^{\lambda(1-x)}, \quad 0 < x < \infty, \ 0 < \lambda < \infty \qquad \text{with } X = 1/\lambda. \tag{1b}$$

(In fact, for these two examples F possesses a quadratic maximum and we will largely be restricting our attention to this class of functions.) In the chaotic region, beyond some critical value of λ (see equations 2 below), there exist infinitely many parameter values characterized by stable limit cycles of finite order; 'windows of stability', as May (1976) has phrased the regions in their vicinity. The order in which these windows succeed one another is independent of the map (even the character of the F maximum) within the constraints. This is the content of *structural universality*.

As they appear in order, the low period cycles up to 8 are listed in Table 1 for the reader's convenience. There we specify the 'superstable' λ-parameter values for which $x = X$ is one of the fixed points of the cycle for maps A and B. In what follows we shall denote by λ_{2^n} that value of λ for which the 2^n cycle is superstable. As the cycle period increases, so does the multiplicity of cycle *structures* as has been fully documented by Metropolis *et al.* (1973). We have followed May (1976) in Table 1 by

* All our computations were carried out on TRS-80 microcomputers to double precision.

appending a lower case letter to distinguish between different cycles of the same order, although this labelling is not of much value except for the low order cycles: already at period 9 there occur 28 different cycles and not enough letters in the alphabet to accommodate them.

Table 1. Superstable parameter values (up to 8 cycles) for mappings A and B

Cycle	Map A	Map B	Cycle	Map A	Map B
2	3·23606798	2·25643121	8h	3·94421350	3·48286345
4a	3·49856170	2·59351893	7e	3·95103216	3·50943386
8a	3·55464086	2·67100426	4b	3·96027013	3·59011302
6a	3·62755753	2·77263994	8i	3·96093370	3·60907717
8b	3·66219250	2·81656251	7f	3·96897686	3·70138725
7a	3·70176915	2·85991838	8j	3·97372426	3·73428947
5a	3·73891491	2·91759985	6d	3·97776642	3·77387587
7b	3·77421419	2·98514113	8k	3·98140895	3·80592983
8c	3·80077094	3·03277660	7g	3·98474762	3·82392739
3	3·83187406	3·11670045	8l	3·98774550	3·85101848
6b	3·84456879	3·17360416	5c	3·99026705	3·92280940
8d	3·87054098	3·25777911	8m	3·99251952	4·02352830
7c	3·88604588	3·29362781	7h	3·99453781	4·07007407
8e	3·89946895	3·33449413	8n	3·99621960	4·10314846
5b	3·90570647	3·36398510	6e	3·99758312	4·18096812
8f	3·91204662	3·39276769	8o	3·99864115	4·30421131
7d	3·92219340	3·41870460	7i	3·99939706	4·39226269
8g	3·93047300	3·43427458	8p	3·99984936	4·57119266
6c	3·93753644	3·45595376			

The first truly chaotic place is the accumulation point of the 2^n cycles

$$\lambda_{2^\infty} = \lim_{n\to\infty} \lambda_{2^n} = 3\cdot569945671, \qquad \text{map A}; \tag{2a}$$

$$= 2\cdot692368853, \qquad \text{map B}; \tag{2b}$$

associated with 'pitchfork' or 'forward' bifurcations, and the passage to it from smaller parameter values λ is known as the 'period-doubling route to chaos'. Feigenbaum (1978, 1979) noticed that when the chaotic point was approached from below, the ratios of the relative differences between successive λ_{2^n}

$$\lim_{n\to\infty} \frac{D\lambda_{2^n}}{D\lambda_{2^{n+1}}} \left(\equiv \frac{\lambda_{2^n}-\lambda_{2^{n-1}}}{\lambda_{2^{n+1}}-\lambda_{2^n}}\right) = \delta, \tag{3a}$$

as well as the ratios of the relative spacings,

$$\lim_{n\to\infty} \frac{Dx_{2^n}}{Dx_{2^{n+1}}} \left(\equiv \frac{x^*_{2^n}-X}{x^*_{2^{n+1}}-X}\right) = -\alpha, \tag{3b}$$

between the central fixed point X and the nearest fixed point

$$x^*_{2^n} = [F]^{2^{n-1}}(\lambda_{2^n}, X), \tag{4}$$

tended geometrically to two universal constants δ and α respectively, independently of mapping details (see Table 2).† This is termed *metric universality*. When the functions possess a quadratic maximum as in (1a) or (1b) the universal constants turn out to be

$$\delta = 4\cdot 6692\ldots, \qquad \alpha = 2\cdot 5029\ldots. \tag{5}$$

To simplify the notation it proves useful to shift origin and rescale x by a factor a,

$$x \to X + ax,$$

whereupon

$$F(x) \to f(x),$$

with an a-dependent normalization $f(0)$. Often a is chosen so that $f(0) = 1$. For instance, with our two mappings, we have

$$\text{(A)} \qquad \begin{aligned} f(x) &= -1/2a + \lambda(1/4a - ax^2); \\ a &= \tfrac{1}{4}\lambda - \tfrac{1}{2} \quad \text{ensures} \quad f(0) = 1; \end{aligned} \tag{6a}$$

$$\text{(B)} \qquad \begin{aligned} f(x) &= -1/\lambda a + (1/\lambda a + x)e^{\lambda - 1 - a\lambda x}; \\ a &= (e^{\lambda-1} - 1)/\lambda \quad \text{fixes} \quad f(0) = 1. \end{aligned} \tag{6b}$$

In this way the spacing between the centremost fixed points may be reinterpreted as

$$Dx_{2^n} = [f]^{2^{n-1}}(\lambda_{2^n}, 0).$$

As noted, there exist windows of stability to the right of λ_{2^∞} and it is on these windows that we wish to exclusively focus attention. In any given window of period k, it is well known (Feigenbaum 1978, 1979) that if one studies harmonics of period $k\cdot 2^n$ which arise by pitchfork bifurcation, the sequences of

$$Dx_{k\cdot 2^n} = x^*_{k\cdot 2^n} - X = [f]^{c\cdot 2^{n-1}}(\lambda_{k\cdot 2^n}, 0), \tag{7a}$$

$$D\lambda_{k\cdot 2^n} = \lambda_{k\cdot 2^n} - \lambda_{k\cdot 2^{n-1}}, \tag{7b}$$

are again characterized by the same universal constants α and δ (see Table 3):

$$\lim_{n\to\infty} \frac{Dx_{k\cdot 2^n}}{Dx_{k\cdot 2^{n+1}}} = -\alpha, \qquad \lim_{n\to\infty} \frac{D\lambda_{k\cdot 2^n}}{D\lambda_{k\cdot 2^{n+1}}} = \delta. \tag{8a, b}$$

In (7a) the integer c is determined by the precise details of the k-cycle structure. For the 3-cycle we have $c = 2$.

The most noticeable window in the chaotic region, because it is the widest, is connected with the 3-cycle; that cycle is born (May 1976; Collet and Eckmann (1980)

† A few notational points: Feigenbaum (1978, 1979) used d_n in place of our Dx_{2^n}, and λ_n instead of our λ_{2^n}. Our more explicit formulae are necessary later. Also, N-iterates of F, as in (4), will be written as $[F]^N$ rather than as F^N to avoid subsequent confusion when a host of universal functions are introduced.

Table 2*a*. Forward bifurcations $1 \cdot 2^n$ for mappings A and B

Cycle	Map A		Map B	
	λ	Dx	λ	Dx
2	3·236067978	−0·310016994	2·256431209	−1·113644594
4a	3·498561699	0·116401770	2·593518933	0·200505017
8a	3·554640863	−0·045975211	2·671004264	−0·107148265
16a	3·566667380	0·018326176	2·687782643	0·037739867
32a	3·569243532	−0·007318431	2·691386189	−0·015822652
64a	3·569795294	0·002923675	2·692158376	0·006197856
128a	3·569913465	−0·001168087	2·692323776	−0·002495613
256a	3·569938774	0·000466690	2·692359200	0·000993953
512a	3·569944195	−0·000186459	2·692366787	−0·000397622

Table 2*b*. Ratios of successive $D\lambda$ and Dx for mappings A and B

Cycle	Map A		Map B	
	$R\lambda$	Rx	$R\lambda$	Rx
4a	4·681	−2·663	4·350	−5·554
8a	4·663	−2·532	4·618	−1·871
16a	4·668	−2·509	4·656	−2·839
32a	4·669	−2·504	4·667	−2·385
64a	4·669	−2·503	4·669	−2·553
128a	4·669	−2·503	4·669	−2·484
256a	4·669	−2·503	4·669	−2·511
512a	—	−2·503	—	−2·500

Table 3*a*. Forward bifurcations $3 \cdot 2^n$ for mappings A and B

Cycle	Map A		Map B	
	λ	Dx	λ	Dx
3	3·831874055	−0·457968514	3·116700451	−2·343057612
6b	3·844568792	0·027235706	3·173604163	0·081983089
12b	3·848344657	−0·011051342	3·190739426	−0·037885476
24b	3·849198054	0·004430880	3·194602580	0·014375982
48b	3·849383110	−0·001771810	3·195440367	−0·005874092
96b	3·849422845	0·000708039	3·195620245	0·002326936
192b	3·849431360	−0·000282899	3·195658791	−0·000932966
384b	3·849433184	0·000113019	3·195667047	0·000372242
768b	3·849433575	−0·000045159	3·195668815	−0·000148815

Table 3*b*. Ratios of successive $D\lambda$ and Dx for mappings A and B

Cycle	Map A		Map B	
	$R\lambda$	Rx	$R\lambda$	Rx
6b	3·362	−16·815	3·321	−28·580
12b	4·419	−2·464	4·436	−2·164
24b	4·617	−2·494	4·611	−2·635
48b	4·657	−2·501	4·658	−2·447
96b	4·667	−2·502	4·667	−2·524
192b	4·669	−2·503	4·669	−2·494
384b	4·669	−2·503	4·669	−2·506
768b	—	−2·503	—	−2·501

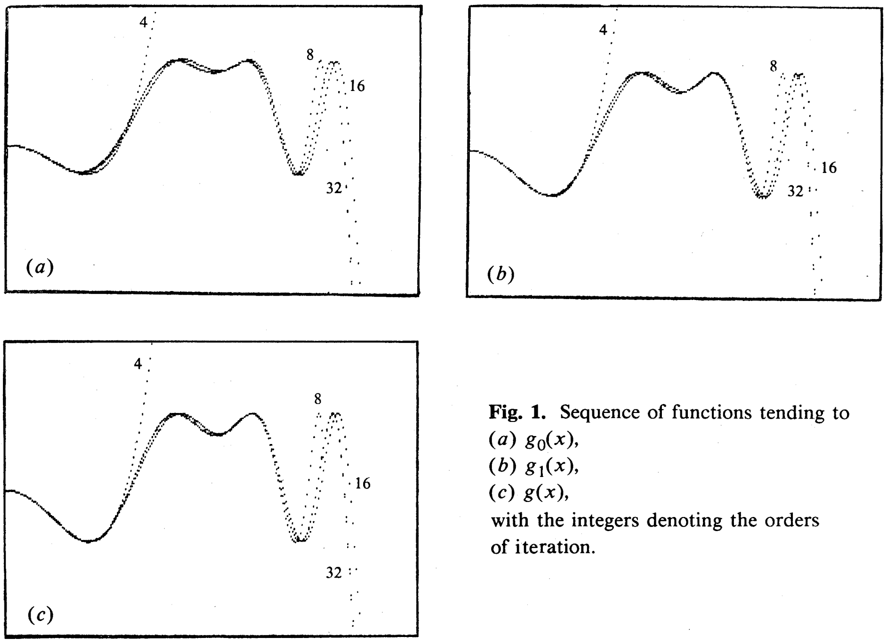


Fig. 1. Sequence of functions tending to
(*a*) $g_0(x)$,
(*b*) $g_1(x)$,
(*c*) $g(x)$,
with the integers denoting the orders of iteration.

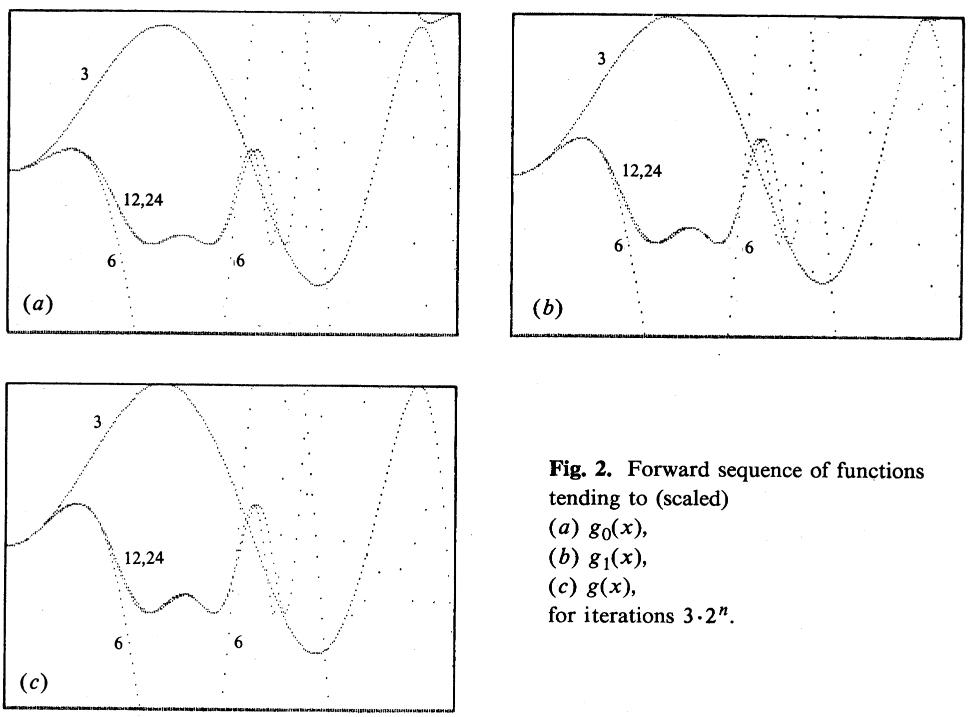
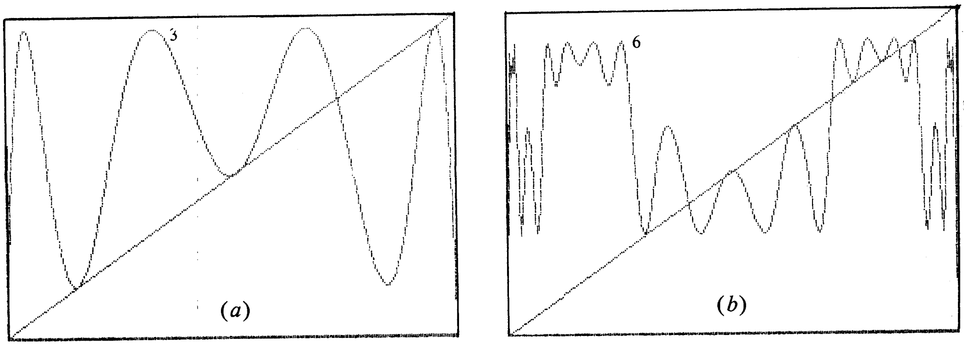


Fig. 2. Forward sequence of functions tending to (scaled)
(*a*) $g_0(x)$,
(*b*) $g_1(x)$,
(*c*) $g(x)$,
for iterations $3 \cdot 2^n$.



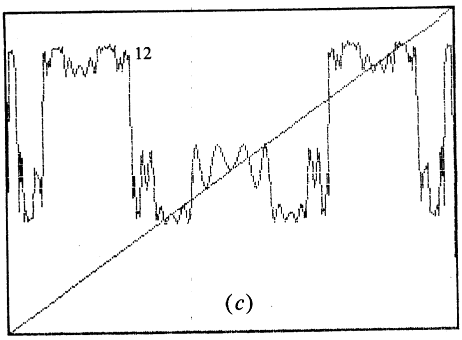


Fig. 3. The iterate
(a) $[F]^3$,
(b) $[F]^6$,
(c) $[F]^{12}$,
evaluated at λ_3, $\lambda_{2\cdot3}$ and $\lambda_{2^2\cdot3}$ respectively.

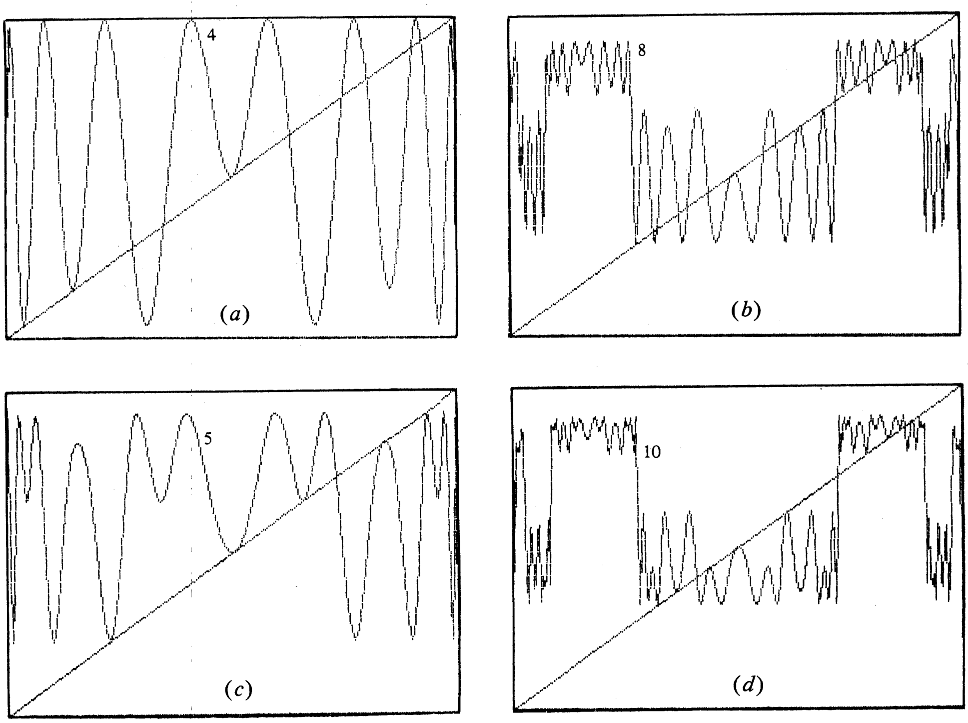


Fig. 4. Iterate (a) $[F]^4$, (b) $[F]^8$, (c) $[F]^5$ and (d) $[F]^{10}$ evaluated at λ_4, $\lambda_{2\cdot4}$, λ_{5a} and $\lambda_{2\cdot5a}$ respectively.

by 'tangent bifurcation' in the region to the right of λ_{2^∞}. [Just below this window there is an almost stable triplication pattern giving rise to the 'intermittency' phenomenon (see Manneville and Pomeau 1980; Hirsch *et al.* 1982; Hu and Rudnick 1982).] For this particular case we shall term the tangent bifurcation a 'trifurcation', recognizing it as a mathematical misdemeanour—thus period tripling $N \to 3N$ does not happen as λ varies continuously. Likewise, there are possibilities of fourfold, fivefold period multiplications as we keep on increasing λ.

Our investigations are primarily concerned with the sequences $k\cdot 2^n$ and $k\cdot 3^n$ in the forward and backward sense (see Section 3), and we have discovered that for such period doublings and triplings properties analogous to Feigenbaum's metric universality prevail. Specifically we have studied these sequences for the two popular maps A and B. It should be clear that any conclusions we draw from both mappings almost certainly apply to other maps with the same general characteristics; namely, one-dimensional non-invertible maps with a unique quadratic maximum.

By comparing the shapes of iterated maps $[f]^{2n}$ in the central region, using computer techniques, Feigenbaum (1978, 1979) was able to demonstrate the existence of a universal function

$$g_1(x) = \lim_{n\to\infty} (-\alpha)^n [f]^{2^n} (\lambda_{2^{n+1}}, x/(-\alpha)^n).$$

He also showed that one could define a whole sequence of functions

$$g_r(x) = \lim_{n\to\infty} (-\alpha)^n [f]^{2^n} (\lambda_{2^{n+r}}, x/(-\alpha)^n), \tag{9}$$

satisfying

$$g_{r-1}(x) = -\alpha g_r(g_r(-x/\alpha)). \tag{10}$$

Feigenbaum then conjectured that this sequence of functions converged to a unique limit

$$g(x) = \lim_{r\to\infty} g_r(x) = \lim_{n\to\infty} (-\alpha)^n [f]^{2^n} (\lambda_{2^\infty}, x/(-\alpha)^n), \tag{11}$$

which satisfied the fixed point Feigenbaum–Cvitanovic relation

$$g(x) = -\alpha g(g(-x/\alpha)). \tag{12}$$

[The existence of g in certain cases was in fact proved by Collet *et al.* (1980) and Lanford (1982), who also proved existence and uniqueness for the mappings $x \to 1-\mu x^{1+\varepsilon}$, with ε small.] The scale of g is arbitrary and is set through the normalization condition $g(0) = 1$. In an effort to make the present paper self-contained, as well as for later comparison with other universal functions, we give g_0, g_1 and g in Fig. 1.

Associated with the cycles k born by tangent bifurcation is a cascade of harmonics $k\cdot 2^n$ emerging by subsequent period doubling (forward bifurcation). One may again abstract the same universal function $g_1(x)$—and indeed the entire sequence $g_r(x)$ culminating in $g(x)$—by approaching the accumulation point $\lambda_{k\cdot 2^\infty}$. This is shown

Table 4*a*. Backward bifurcations $2^n \cdot 3$ for mappings A and B

Cycle	Map A		Map B	
	λ	Dx	λ	Dx
3	3·831874056	0·345710203	3·116700451	−2·343057611
6a	3·627557530	−0·140860795	2·772639937	0·266331112
12a	3·582229836	0·056600411	2·709628054	−0·189584964
24a	3·572577293	−0·022642507	2·696053119	0·061965036
48a	3·570509238	0·009049220	2·693157769	−0·026946032
96a	3·570066370	−0·003615723	2·692537798	0·010418610
192a	3·569971522	0·001444641	2·692405037	−0·004218240
384a	3·569951208	−0·000577183	2·692376604	0·001676493
768a	3·569946858	0·000230607	2·692370514	−0·000671228

Table 4*b*. Ratios of successive $D\lambda$ and Dx for mappings A and B

Cycle	Map A		Map B	
	$R\lambda$	Rx	$R\lambda$	Rx
6a	4·508	−2·454	5·460	−8·798
12a	4·696	−2·498	4·642	−1·405
24a	4·667	−2·500	4·689	−3·060
48a	4·700	−2·502	4·670	−2·300
96a	4·669	−2·503	4·670	−2·586
192a	4·669	−2·503	4·669	−2·470
384a	4·669	−2·503	4·669	−2·516
768a	—	−2·503	—	−2·498

Table 5. Backward bifurcations for mapping A

Cycle	λ	Dx	$R\lambda$	Rx
		(*a*) $2^n \cdot 4$		
4	3·960270127	0·351775477	—	—
8b	3·662192504	−0·146589904	4·095	−2·400
16b	3·589399844	0·059085959	4·763	−2·481
32b	3·574118089	−0·023662325	4·660	−2·497
64b	3·570839054	0·009458540	—	−2·502
		(*b*) $2^n \cdot 5$a		
5a	3·738914930	−0·158342067	—	—
10	3·605385838	0·064326385	4·797	−2·462
20	3·577549811	−0·025819307	4·658	−2·491
40	3·571573647	0·010325161	4·671	−2·501
80	3·570294339	−0·004126144	—	−2·502
		(*c*) $2^n \cdot 5$b		
5b	3·905706470	0·180442003	—	—
10	3·647048802	−0·076515812	4·257	−2·358
20	3·586281315	0·030775652	4·735	−2·486
40	3·573447578	−0·012325970	4·663	−2·497
80	3·570695539	0·004926708	—	−2·502
		(*d*) $2^n \cdot 5$c		
5c	3·990267047	0·353121938	—	—
10	3·673008246	−0·148321143	3·894	−2·381
20	3·591544528	0·059811996	4·802	−2·480
40	3·574581219	−0·023962174	4·656	−2·496
80	3·570938128	0·009578873	—	−2·502

in Fig. 2 for the particular case of $3 \cdot 2^n$ cycles and is fairly well understood; thus (the existence of g for $\varepsilon = 1$ has been proved by Campanino and Epstein 1981)

$$g_1(\mu x) = \mu \lim_{n \to \infty} (-\alpha)^n [f]^{k \cdot 2^n} (\lambda_{k \cdot 2^{n+1}}, x/(-\alpha)^n),$$

with the magnification μ being the only k-dependent ingredient. An appreciation of the scale μ may be gained by comparing Figs 1 and 2.

3. Reverse Bifurcations

In this paper we wish to draw attention to quite distinct limiting sequences in which 2^n multiples of the k cycle occur to the *left* of the basic k cycle, i.e. they are *not* harmonics of that cycle but are instead born by tangent bifurcation. For any $k > 2$ these sequences also approach λ_{2^∞} but in the *reverse* order

$$2^\infty \cdot k \leftarrow \dots \leftarrow 4 \cdot k \leftarrow 2 \cdot k \leftarrow k .$$

We call this the 'reverse' or 'backward' bifurcation sequence (Feigenbaum 1980; Kopylov and Sivac 1982). Tables 4 and 5 provide the superstable λ values for various low order cycles, mainly for map A. If one examines the $2^n \cdot 3$ order iterates of F (Fig. 3) one can pick out a copy, reduced in scale, of the basic 3 cycle in the vicinity of X. Fig. 4 shows that a similar pattern prevails for other cycles. Moreover one can establish numerically, beyond reasonable doubt (see Tables 4 and 5), that for this backward sequence the usual Feigenbaum constants arise:

$$\lim_{n \to \infty} R\lambda_{2^n \cdot k} \; (\equiv D\lambda_{2^n \cdot k}/D\lambda_{2^{n+1} \cdot k}) = \delta = 4 \cdot 6692 \dots , \tag{13a}$$

$$\lim_{n \to \infty} Rx_{2^n \cdot k} \; (\equiv Dx_{2^{n-1} \cdot k}/Dx_{2^n \cdot k}) = -\alpha = -2 \cdot 5029 \dots . \tag{13b}$$

This suggests that a universal function of the type g_0 or g_1 may exist for *each* of the 'reverse bifurcation' sequences connected with a particular k cycle. Indeed, we see strong indications of this in Figs 3 and 4 by observing that a copy of the fundamental $[f]^k$ occurs in the vicinity of the central fixed point for the various iterates as the period doubles up in the reverse order.

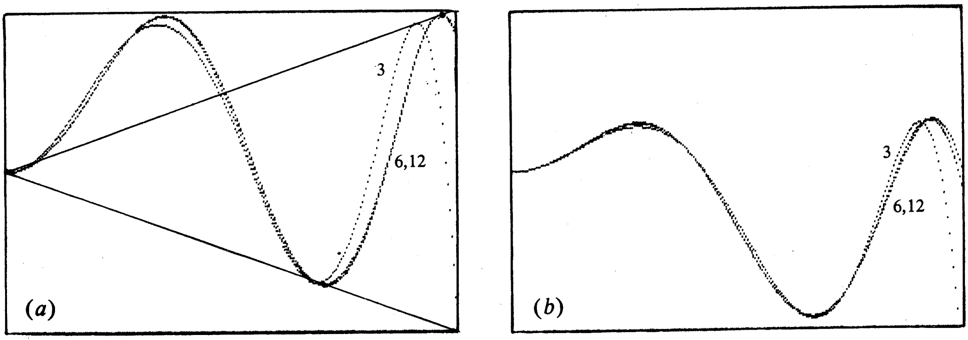
By appropriate computational procedures (enlarging and inverting the central region at each stage) we have established that the limiting function

$$g_0^k(x) = \lim_{n \to \infty} (-\alpha)^n [f]^{2^n \cdot k} (\lambda_{2^n \cdot k}, x/(-\alpha)^n)$$

exists and is *distinct for every cycle*. This is a totally new phenomenon and is quite different from what happens when the pitchfork sequence is studied.* It is in fact possible to define a whole sequence of functions

$$g_r^k(x) = \lim_{r \to \infty} (-\alpha)^n [f]^{2^n \cdot k} (\lambda_{2^{n+r} \cdot k}, x/(-\alpha)^n), \tag{14}$$

* We are careful to distinguish between the forward superstable values $\lambda_{k \cdot 2^n}$ and the backward superstable values $\lambda_{2^n \cdot k}$ in what follows. Of course, as far as functional iterates are concerned, there is no difference between $[f]^{2^n \cdot k}(\lambda, x)$ and $[f]^{k \cdot 2^n}(\lambda, x)$ at the *same* λ-parameter value.



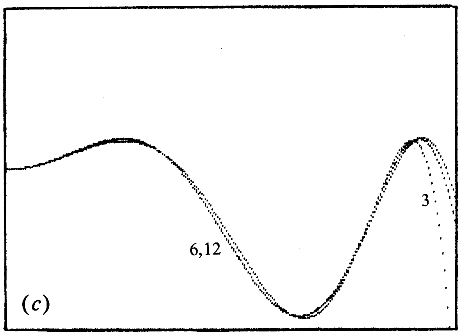


Fig. 5. Reverse sequence of functions tending to
(*a*) g_0^3,
(*b*) g_1^3,
(*c*) g^3,
for $2^n \cdot 3$ iterates. In (*a*) the fixed points nearest the origin occur where the $y = \pm x$ lines intersect the extrema.

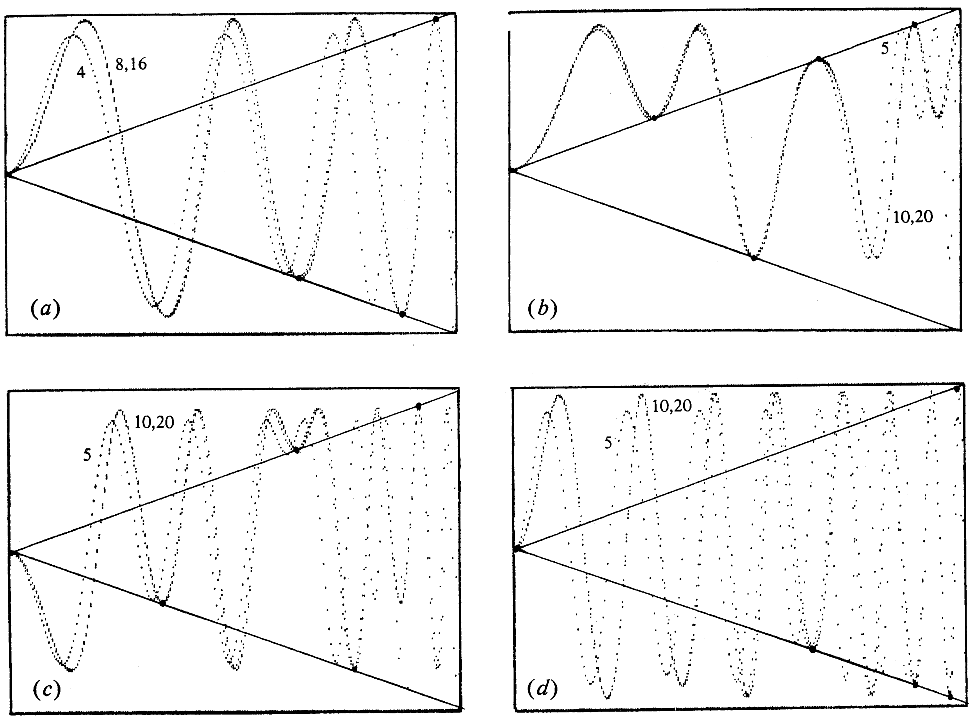


Fig. 6. Reverse sequence of functions tending to (*a*) g_0^4, (*b*) g_0^{5a}, (*c*) g_0^{5b} and (*d*) g_0^{5c} for $2^n \cdot 4$, $2^n \cdot 5a$, $2^n \cdot 5b$ and $2^n \cdot 5c$ iterates respectively. The fixed points are displayed similarly to Fig. 5*a*.

such that

$$g_{r-1}^k(x) = -\alpha g_r^k(g_r^k(-x/\alpha)). \tag{15}$$

Figs 5*a* and 5*b* evidently point to convergence toward a limit function

$$g^k(x) = \lim_{r\to\infty} g_r^k(x),$$

and it is clear that $g^k(x)$ obeys the standard fixed point equation

$$g^k(x) = -\alpha g^k(g^k(-x/\alpha)), \tag{16}$$

where again we have the freedom to set the scale through $g^k(0) = 1$.

Thus it appears that we have an *infinite* class of universal functions (McCarthy 1983) satisfying the standard fixed point equation. In order to distinguish between these functions we may utilize the characteristic structures of the universal functions g_0^k. It is obvious that even when k is specified, there is a variety of functions associated with the classification of Metropolis *et al.* (1973). This is illustrated by the four distinct universal functions corresponding to the 4, 5a, 5b and 5c cycles as shown in Figs 6*a*–6*d*.

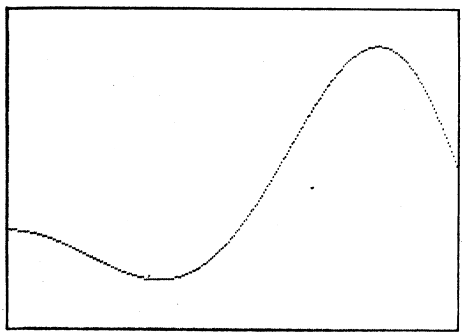


Fig. 7. Third iterate of the standard Feigenbaum universal function $g(x)$. (Compare this with Fig. 5*c*.)

Note, however, that from equation (14), if we go to the limit $r \to \infty$, then an alternative definition is

$$g^k(x) = \lim_{n\to\infty} (-\alpha)^n [f]^{2^n \cdot k}(\lambda_{2^\infty}, x/(-\alpha)^n), \tag{17}$$

assuming as always that the orders of limits can be reliably interchanged. This indicates that the *different* function sequences for fixed k all converge to the *same* limit g^k which depends *only* on the order k of the cycle and *not* on its structure! Further, since the standard universal function is defined by equation (11), we infer that

$$g^k(\mu x) = \mu[g]^k(x). \tag{18}$$

(This receives numerical support in Fig. 7 for the case $k = 3$.) Certainly when $g(g(x)) = -g(-\alpha x)/\alpha$, it is straightforward to verify that

$$[g]^k[g]^k = -[g]^k(-\alpha x)/\alpha. \tag{19}$$

It is worth observing, though, that for every k we are allowed independently to set the scale* by $g^k(0) = 1$, which means that $g^k(x)$ cannot simply be the kth iterate of the standard function $g(x)$ scaled to $g(0) = 1$; indeed, (18) is only correct up to a scaling μ. In any event, each of these g^k satisfies the familiar fixed point relation (16), indicating that an *infinite* number of solutions to that renormalization group equation exists.

4. Triplications

We now turn to a systematic study of cycle sequences of the type $k\cdot 3^n$, where the basic cycle has period k. We call these triplications of the k cycle and they correspond to a particular type of tangent bifurcation. First, we shall distinguish between two distinct sequences which we denote by $k\cdot 3^n$ and $3^n\cdot k$ associated with forward and backward (or reverse) triplications. The forward sequence arises to the right of every k cycle and converges to an accumulation point which depends on k and its cycle structure—in many ways it is analogous to the pitchfork sequence $k\cdot 2^n$. (As a special instance the 3 cycle spawns the sequence 3^n.) Such cycles may be identified by studying the three bands associated with the chaotic region to the right of the pitchfork accumulation point $\lambda_{3\cdot 2^\infty}$. As far as $k = 3$ is concerned, there is the distinct point of accumulation

$$\lambda_{3^\infty} = 3\cdot 854077963591, \qquad \text{map A;}$$
$$= 3\cdot 216164774983, \qquad \text{map B.}$$

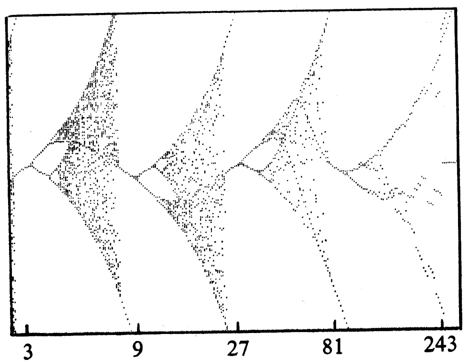


Fig. 8. Repeating triplication pattern as λ_{3^∞} is approached (see text). The density of points along the vertical axis thins out since the total number of iterations is fixed as λ varies along the horizontal axis.

The backward or reverse sequence, which we have denoted by $3^n\cdot k$, is characterized by the fact that it always converges to λ_{3^∞}, irrespective of k; indeed, this sequence is very similar to the reverse bifurcations discussed in the previous section. However, whether these triplings converge to λ_{3^∞} from the left or right depends upon whether or not the basic k cycle lies to the left or right of the 3 cycle. Thus $3^n\cdot 5$a converges from the left—so the terminology 'backward' is rather a misnomer for it— whereas $3^n\cdot 5$b and $3^n\cdot 5$c converge from the right. This triplication pattern is exhibited in Fig. 8 for $x(\lambda-\lambda_{3^\infty})^{-\ln A/\ln \Delta}$ against $\log(\lambda-\lambda_{3^\infty})$ with the constants Δ and A in (21a) and (21b) already anticipated.

* One must be careful not to confuse $g^k(0)$ with the related quantity $Dx_{2^n\cdot k} = [f]^{2^n\cdot C}(\lambda_{2^n\cdot k}, 0)$, where C is the number of iterations needed to bring x to X for $n = 0$.

Table 6a. Forward trifurcations $1 \cdot 3^n$ for mappings A and B

Cycle	Map A λ	Map A Dx	Map B λ	Map B Dx
3	3·831874055283	0·3457102029	3·116700451066	−2·343405761
9	3·853675276839	−0·0356580611	3·214601114697	0·129432449
27	3·854070677510	0·0038524177	3·216136205149	−0·015927717
81	3·854077831706	−0·0004151813	3·216164258172	0·001690728
243	3·854077961203	0·0000447527	3·216164765631	−0·000182565

Table 6b. Ratios of successive $D\lambda$ and Dx for mappings A and B

Cycle	Map A $R\lambda$	Map A Rx	Map B $R\lambda$	Map B Rx
9	55·13	−9·695	63·78	−18·105
27	55·27	−9·256	54·72	−8·126
81	55·25	−9·279	55·28	−9·421
243	—	−9·277	—	−9·261

Table 7. Forward trifurcations for mapping A

Cycle	λ	Dx	$R\lambda$	Rx
		(*a*) $4 \cdot 3^n$		
4	3·9602701272212	0·3517754767	—	—
12	3·9614314419566	−0·0084164470	56·00	−41·796
36	3·9614521815369	0·0008811366	54·99	−9·552
108	3·9614525586728	−0·0000951532	55·26	−9·260
324	3·9614525654981	0·0000102551	—	−9·279
		(*b*) $5\text{a} \cdot 3^n$		
5a	3·738914912970	−0·1583420673	—	—
15	3·744016873483	0·0232593167	58·05	−6·808
45	3·744104768920	−0·0024341389	54·93	−9·556
135	3·744106369092	0·0002628481	55·27	−9·261
405	3·744106398046	−0·0000283256	—	−9·280
		(*c*) $5\text{b} \cdot 3^n$		
5b	3·905706469831	0·1804420034	—	—
15	3·906641328957	−0·0097690137	56·51	−18·471
45	3·906657872652	0·0010263518	54·96	−9·518
135	3·906658173687	−0·0001108003	55·26	−9·263
405	3·906658179135	0·0000119443	—	−9·276
		(*d*) $5\text{c} \cdot 3^n$		
5c	3·990267046974	0·3531219383	—	—
15	3·990335169048	−0·0020549297	56·45	−17·184
45	3·990336375733	0·0002136359	54·94	−9·619
135	3·990336397698	−0·0000230798	55·19	−9·256
405	3·990336398096	0·0000024874	—	−9·279

In Tables 6–8 the numerical results for both trifurcating sequences are presented, chiefly for map A, in a similar way to the tabulation for bifurcations. For the forward sequence we define the relevant superstable values by $\lambda_{k \cdot 3^n}$ and for the backward sequence by $\lambda_{3^n \cdot k}$. As pointed out, we have

$$\lim_{n\to\infty} \lambda_{k\cdot 3^n} \equiv \lambda_{k\cdot 3^\infty} \neq \lambda_{3^\infty}, \quad \text{except for } k = 3; \tag{20a}$$

$$\lim_{n\to\infty} \lambda_{3^n\cdot k} = \lambda_{3^\infty}. \tag{20b}$$

Table 8. Backward trifurcations for mapping A

Cycle	λ	Dx	$R\lambda$	Rx
		(*a*) $3^n \cdot 4$		
4	3·960270127221	0·3517754767	—	—
12	3·855993729675	−0·0374971999	55·43	−9·381
36	3·854112685005	0·0040525443	55·17	−9·253
108	3·854078592024	−0·0004367589	55·25	−9·279
324	3·854077974966	0·0000470753	—	−9·278
		(*b*) $3^n \cdot 5$a		
5a	3·738914912971	0·158342067319		
15	3·852099410353	−0·015978208764	58·21	−9·910
45	3·854042076559	0·001725644420	55·13	−9·259
135	3·854077314123	−0·000185974316	55·26	−9·279
405	3·854077951835	0·000020045340	55·25	−9·278
		(*c*) $3^n \cdot 5$b		
5b	3·905706469831	0·1804420034	—	—
15	3·854991046674	−0·0190731288	56·57	−9·461
45	3·854094524136	0·0020604249	55·13	−9·257
135	3·854078263307	−0·0002220630	55·25	−9·279
405	3·854077969016	0·0000239337	—	−9·278
		(*d*) $3^n \cdot 5$c		
5c	3·990267046974	0·3531219383	—	—
15	3·856587276680	−0·0379534384	54·26	−9·304
45	3·854123393250	0·0041020168	55·23	−9·252
135	3·854078785902	−0·0004420946	55·25	−9·279
405	3·854077978475	0·0000476519	—	−9·278

With *both* sequences we find that there is a scaling law determining the relative window sizes, analogous to (8b),

$$\lim_{n\to\infty} R\lambda_{k\cdot 3^n} \left(\equiv \frac{D\lambda_{k\cdot 3^n}}{D\lambda_{k\cdot 3^{n+1}}} = \frac{\lambda_{k\cdot 3^n} - \lambda_{k\cdot 3^{n-1}}}{\lambda_{k\cdot 3^{n+1}} - \lambda_{k\cdot 3^n}}\right)$$

$$= \lim_{n\to\infty} R\lambda_{3^n\cdot k} \left(\equiv \frac{D\lambda_{3^n\cdot k}}{D\lambda_{3^{n+1}\cdot k}} = \frac{\lambda_{3^n\cdot k} - \lambda_{3^{n-1}\cdot k}}{\lambda_{3^{n+1}\cdot k} - \lambda_{3^n\cdot k}}\right) = \quad = 55\cdot 26, \tag{21a}$$

with a universal constant Δ that is map-independent, apart from the quadratic maximum requirement. As well, there is a second scaling law determining the trident sizes, analogous to (8a),

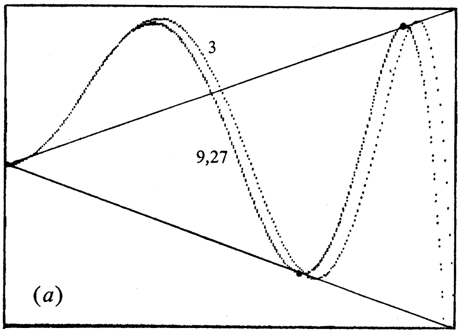
$$\lim_{n\to\infty} Rx_{k\cdot 3^n} \left(\equiv \frac{Dx_{k\cdot 3^{n-1}}}{Dx_{k\cdot 3^n}} = \frac{x^*_{k\cdot 3^{n-1}} - X}{x^*_{k\cdot 3^n} - X}\right)$$

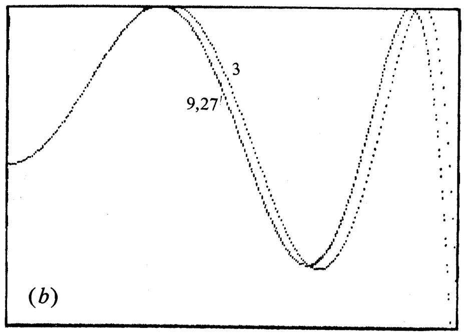
$$= \lim_{n\to\infty} Rx_{3^n\cdot k} \left(\equiv \frac{Dx_{3^{n-1}\cdot k}}{Dx_{3^n\cdot k}} = \frac{x^*_{3^{n-1}\cdot k} - X}{x^*_{3^n\cdot k} - X}\right) = -A = -9\cdot 277, \tag{21b}$$

governed by another universal constant A.†

There is a striking resemblance between pitchfork bifurcations and the forward triplications—both converge to separate points of accumulation $\lambda_{k\cdot 2^\infty}$ and $\lambda_{k\cdot 3^\infty}$

† After determining values for Δ and A, we realized that Derrida *et al.* (1979) had determined them for the first class of sequences. However, it is not entirely obvious that their work extends to the second class or that the numbers are truly universal (map-independent).





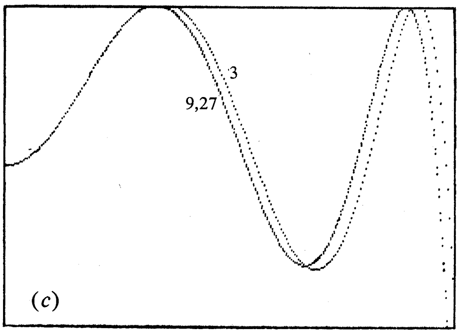


Fig. 9. Forward triplication sequence of functions tending to
(a) G_0,
(b) G_1,
(c) G.
The order of iteration is indicated by the integers. In (a) the fixed points nearest the origin occur where the $y = \pm x$ lines intersect the extrema.

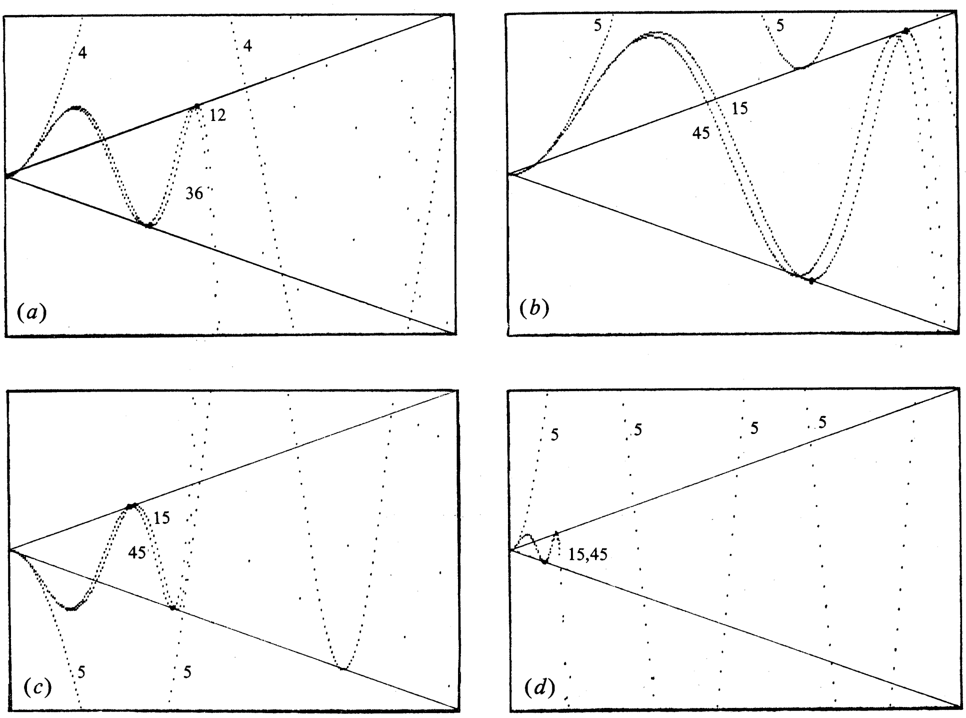


Fig. 10. Forward sequence of functions tending to G_0 for (a) $4 \cdot 3^n$, (b) $5a \cdot 3^n$, (c) $5b \cdot 3^n$ and (d) $5c \cdot 3^n$ iterates. Observe how (a) is a scaled version of Fig. 9a. The fixed points in each case are displayed similarly to Fig. 9a.

respectively—as well as between reverse bifurcations and trifurcations—both converge to the same point of accumulation $\lambda_{2\infty}$ and $\lambda_{3\infty}$ respectively. These analogies suggest that we should pursue the idea of universal trifurcation functions as we have already done for both kinds of cycle doublings.

5. Universal Triplication Functions

We begin by focussing on the analogue of the pitchfork sequence, namely forward period tripling $k\cdot 3^n$. By standard computational techniques we have shown that the limiting function

$$G_0(x) = \lim_{n\to\infty} (-A)^n [f]^{3^n} (\lambda_{3^n}, x/(-A)^n)$$

exists; it is depicted in Fig. 9*a*. Of course one can also define a series of functions via

$$G_r(x) = \lim_{n\to\infty} (-A)^n [f]^{3^n} (\lambda_{3^{n+r}}, x/(-A)^n), \qquad (22)$$

whereupon

$$G_{r-1}(x) = -AG_r(G_r(G_r(-x/A))). \qquad (23)$$

Computations (see Figs 9*b* and 9*c*) suppport the expectation that this sequence G_r converges to a limit function

$$G(x) = \lim_{r\to\infty} G_r(x) = \lim_{n\to\infty} (-A)^n [f]^{3^n} (\lambda_{3^\infty}, x/(-A)^n), \qquad (24)$$

such that

$$G(x) = -AG(G(G(-x/A))). \qquad (25)$$

Again we may fix the scale through $G(0) = 1$. (Actually more is necessary, as shown in the following section.) Equation (25) is in direct analogy to the Feigenbaum–Cvitanovic equation (12).

A study of the first sequence $k\cdot 3^n$ produces the same universal function (see Fig. 10). Thus, if

$$G_0(\mu x) = \mu \lim_{n\to\infty} (-A)^n [f]^{k\cdot 3^n} (\lambda_{k\cdot 3^n}, x/(-A)^n),$$

we find that $G(x)$ emerges (again up to some magnification) when we go to the accumulation point directly:

$$G(\mu x) = \mu \lim_{n\to\infty} (-A)^n [f]^{k\cdot 3^n} (\lambda_{k\cdot 3^\infty}, x/(-A)^n). \qquad (26)$$

This is shown for the case $k = 4$ in Fig. 11.

However, a study of the second reverse class of sequences $3^n\cdot k$ leads to *different* universal functions which we denote by G_0^k:

$$G_0^k(x) = \lim_{n\to\infty} (-A)^n [f]^{3^n\cdot k} (\lambda_{3^n\cdot k}, x/(-A)^n).$$

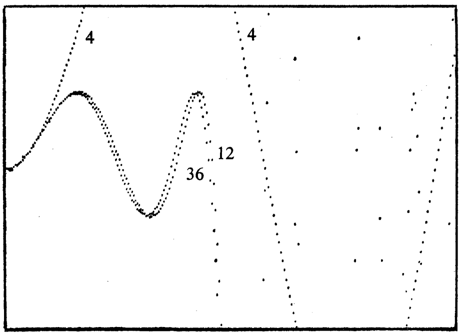


Fig. 11. Forward triplicating sequence $4 \cdot 3^n$ tending to scaled G.

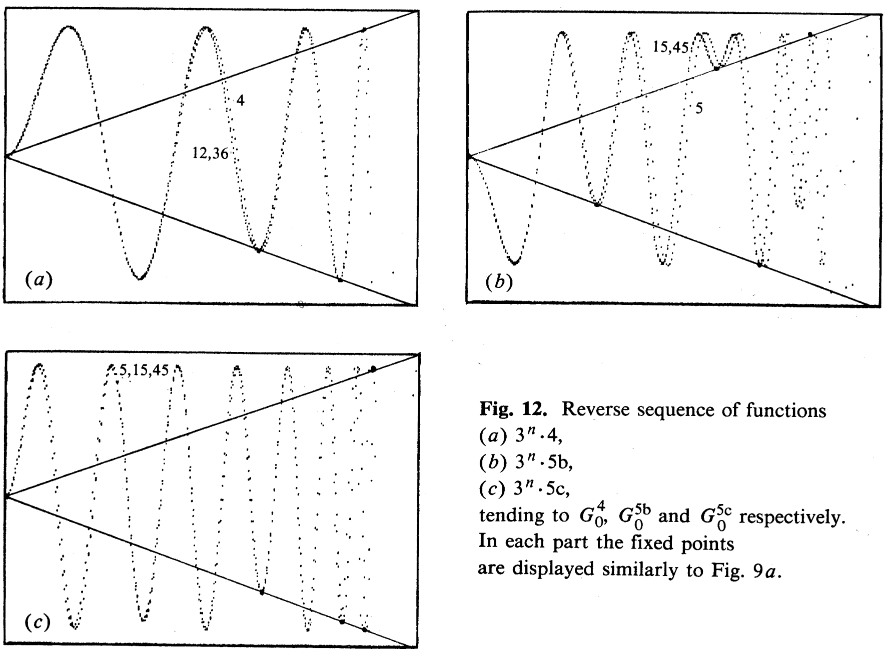


Fig. 12. Reverse sequence of functions
(*a*) $3^n \cdot 4$,
(*b*) $3^n \cdot 5b$,
(*c*) $3^n \cdot 5c$,
tending to G_0^4, G_0^{5b} and G_0^{5c} respectively. In each part the fixed points are displayed similarly to Fig. 9*a*.

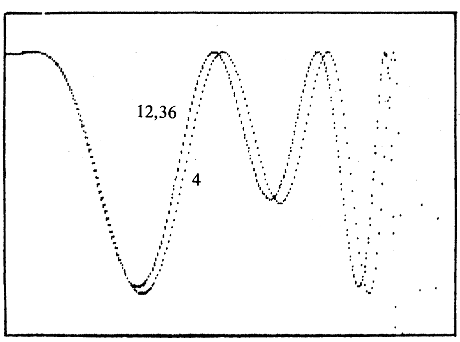


Fig. 13. Reverse sequences G^4 associated with $3^n \cdot 4$ cycles. We know this to be the fourth iterate of G, up to a scaling.

Figs 12*a*–12*c* show how this happens for $k = 4$, $5b$ and $5c$ respectively. Again one may define a sequence of functions

$$G_r^k(x) = \lim_{n\to\infty} (-A)^n [f]^{3^n \cdot k}(\lambda_{3^{n+r}\cdot k}, x/(-A)^n), \tag{27}$$

satisfying

$$G_{r-1}^k(x) = -AG_r^k(G_r^k(G_r^k(-x/A))). \tag{28}$$

There is graphic support (see Fig. 13) for the assumption that this sequence converges to a limit function

$$G^k(x) = \lim_{r\to\infty} G_r^k(x),$$

obeying the triplication equation

$$G^k(x) = -AG^k(G^k(G^k(-x/A))). \tag{29}$$

Alternatively, if we allow r to tend to infinity in (27) we see that formally

$$G^k(x) = \lim_{n\to\infty} (-A)^n [f]^{3^n \cdot k}(\lambda_{3^\infty}, x/(-A)^n). \tag{30}$$

Once more we come across an *infinite* class of functions obeying the same fixed point triplication equation (25). It would seem from (30) that one can identify

$$G^k(\mu x) = \mu[G]^k(x) \tag{31}$$

up to a magnification μ. Nonetheless, it is important to realize that, as was the case for $g^k(x)$ in Section 3, it is admissible to set the normalization $G^k(0) = 1$ for each k *a priori*. It is certainly true that when $G(x)$ obeys (25) so does $[G]^k(x)$.

The examination of these universal functions and sequences reinforces the parallel between the 2^n and 3^n harmonics. We have little doubt that these considerations apply to other kinds of period multiplications; for instance, the fourfold sequences associated with the 4-cycle window born by tangent bifurcation.

6. Fractional Universal Functions

The fact that the sequence

$$(-\alpha)^n f^{k\cdot 2^n}(\lambda_{k\cdot 2^\infty}, x/(-\alpha)^n)$$

converges to $g(x)$ as $n \to \infty$ suggests that the related sequence

$$g_n^{1/k}(x) \equiv (-\alpha)^n f^{2^n}(\lambda_{k\cdot 2^\infty}, x/(-\alpha)^n)$$

may converge in the limit $n \to \infty$ to some 'fractional' universal function which we denote by $g^{1/k}(x)$. Even if $g^{1/k}(x)$ is not unique, the kth iterate clearly is unique and it would be very surprising if no sort of convergence occurred as $n \to \infty$. If $g^{1/k}$ exists, obviously

$$[g^{1/k}]^k = g(x), \qquad g^{1/k}(g^{1/k}(x)) = -g^{1/k}(-\alpha x)/\alpha. \tag{32a, b}$$

When $k = 4$ there are solid grounds for anticipating such convergence because, *up to a scaling*, we know that

$$g(x) = \lim_{n\to\infty} (-\alpha)^n f^{4\cdot 2^n}(\lambda_{4\cdot 2^\infty}, x/(-\alpha)^n),$$

and therefore

$$\begin{aligned} g^{1/4}(x) &= \lim_{n\to\infty} (-\alpha)^n f^{2^n}(\lambda_{4\cdot 2^\infty}, x/(-\alpha)^n) \\ &= \lim_{n\to\infty} (-\alpha)^{n+2} f^{4\cdot 2^n}(\lambda_{4\cdot 2^\infty}, x/(-\alpha)^{n+2}) \\ &= \alpha^2 g(x/\alpha^2) \end{aligned} \tag{33}$$

evidently exists. Direct computation (see Fig. 14) bears out this assertion. Contrary to what is commonly believed (Feigenbaum 1983), this argument shows that *there are other points of accumulation besides* $\lambda = \lambda_{2^\infty}$ at which

$$\lim_{n\to\infty} (-\alpha)^n f^{2^n}(\lambda, x/(-\alpha)^n)$$

exists. More generally, we would expect convergence of this function sequence for all $k = 2^m$ (m integer > 1) *provided we fix* $\lambda = \lambda_{2^m\cdot 2^\infty}$, in which case the limit function is of the type

$$g^{2^{-m}} = (-\alpha)^m g(x/(-\alpha)^m), \tag{34}$$

and satisfies (32).

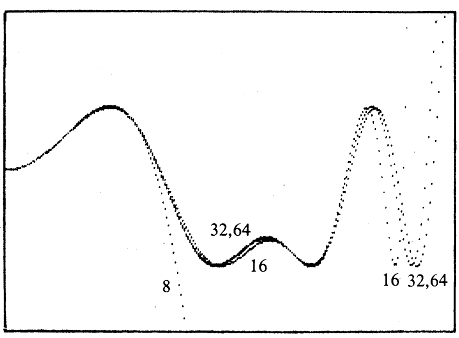


Fig. 14. Sequence of functions leading to $g^{1/4}$ at the accumulation point $\lambda_{4\cdot 2^\infty} = 3\cdot 96119824$.

The idea can be further generalized in a straightforward way to universal functions associated with period tripling. Thus, there must exist fractional functions such that

$$[G^{1/k}]^k = G, \qquad G^{1/k}(G^{1/k}(G^{1/k}(x))) = -G^{1/k}(-Ax)/A, \tag{35a, b}$$

for $k = 3^m$ and $m > 1$. For instance, when $m = 2$ we must have

$$\begin{aligned} G^{1/9}(x) &= A^2 G(x/A^2) \\ &= \lim_{n\to\infty} (-A)^n f^{3^n}(\lambda_{9\cdot 3^\infty}, x/(-A)^n), \end{aligned} \tag{36}$$

where $\lambda_{9\cdot 3^\infty}$ is a point of accumulation ($\neq \lambda_{3^\infty}$) associated with one of the many 9 cycles.

We conclude with some remarks on the normalization of the universal functions. The basic $G_0(x)$ has three superstable fixed points near the origin, one of which may be taken to be $x = 0$; see Fig. 9*a*. The function $G_1(x)$ is, of course, related to $G_0(x)$ through

$$G_0(x) = -AG_1(G_1(G_1(-x/A))).$$

Now, in the case of pitchfork bifurcation, the curve associated with $g_1(x)$ supports a circulation square such that

$$g_1(0) = 1, \qquad g_1(1) = 0.$$

Alternatively we may regard $x = 0, 1$ as fixed points of the iterate $[g_1]^2$, i.e. $[g_1]^2(0) = 0$ and $[g_1]^2(1) = 1$. However, for period tripling the curve associated with $G_1(x)$ supports a circulation polygon, corresponding to three superstable fixed points of $[G_1]^3(x)$, which we may take to be 1, 0 and γ_1; having set the scale with the first two fixed points, γ_1 is some new universal constant† lying between 0 and -1. Here we have

$$G_1(0) = 1, \qquad G_1(1) = \gamma_1, \qquad G_1(\gamma_1) = 0, \tag{37a, b, c}$$

thereby setting the scale for the first function in the sequence G_r. The limiting function G as $r \to \infty$ must then satisfy

$$G(0) = 1, \qquad G(1) = \gamma, \qquad G(\gamma) = -1/A, \tag{38a, b, c}$$

in contrast to (37), because of the fixed point relation (29). The need for three boundary or normalization conditions (38) for G is dictated by G being a solution of a period-tripling functional equation. While the origin of the third boundary condition is clear, it is not obvious to us at present whether or not it is a truly independent condition.

Note added in proof: A good approximation to $G(x)$ in lowest order is given by

$$G(x) = 1 - \mu_{3\infty} x^2.$$

For curves with a quadratic maximum μ is related to λ via equation (6a), so the value $\lambda_{3\infty} = 3\cdot 8541$ determines $\mu_{3\infty} = 1\cdot 7864$. To this order we find that equations (38) become approximately

$$G(0) = 1, \quad G(1) = \gamma = 1 - \mu_{3\infty}, \quad G(\gamma) = 1 - \mu_{3\infty}\gamma^2 = -1/A,$$

† In relation to this, and by examining our various universal curves which exhibit nontrivial local fixed point structure (Figs 5 and 6), we see that in every case one can choose the centremost fixed point to be $x^* = 0$ and (by scaling the horizontal axis) the rightmost fixed point to be $x^* = 1$. The values x^* of all other fixed points are then determined to lie between -1 and 1, at universal locations.

which illustrates that the second boundary condition is *not* independent. In numerical terms this gives

$$\gamma = -0{\cdot}7864, \qquad A = 9{\cdot}534,$$

yielding a value for A within 3% of the experimental value. Higher order (in x^2) approximations to $G(x)$ shift the above numerical value of γ by less than 1%, while A agrees with the experimental value to rather better than 1% already at second order.

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