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Theory of Heavy-ion Elastic Collisions Revisited

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Abstract

The literature on the strong absorption theory of heavy-ion elastic collisions contains a discrepancy involving a singularity in the scattering amplitude at the critical angle θ_c. The formulae concerned have been carefully derived *ab initio* and the obscurity removed. The validity of the approximations for the scattering amplitude at angles less than and greater than θ_c has been examined. The insight gained has made it possible to investigate further the effect of refraction, and to consider the behaviour of the real part of the nuclear phase.

1. Introduction

There have been various approaches to the theory of elastic collisions of heavy ions. In the optical model the parameters of the nuclear potential are found by obtaining best fits to experimental angular distributions; but the values found by different workers for the parameters cover a wide range, and no clear picture emerges.

Accounts based on diffraction theory have led ultimately to formulae which have become complicated and tiresome to evaluate, and which again provide no clear picture.

More insight has been provided by the strong absorption model (Frahn and Venter 1963) in which the term of order l in the partial wave series for the scattering amplitude $f(\theta)$ contains a factor $S(l)$ which cuts off the smaller values of l. A key element in the method is the separation of $f(\theta)$ into two parts $f^+(\theta)$ and $f^-(\theta)$. Then $f^\pm(\theta)$ is given in terms of $S(l)$. Glendenning (1975) has shown how the form of $S(l)$ varies with the values of the parameters adopted for the nuclear interaction. We showed (Mohr 1979) that values of $f^+(\theta)$ and $f^-(\theta)$ may be extracted from experimental angular distributions of elastic scattering, so as to obtain $S(l)$ for a wide range of pairs of colliding nuclei, taking $S(l)$ to be of Woods–Saxon form.

This method was then extended to transfer reactions (Mohr 1980) and inelastic collisions (Mohr 1982), taking $S(l)$ to have the form of a bell-shaped peak, and it was shown how transfer and inelastic collisions result in greater nuclear penetration than elastic collisions.

An important element in the theory is the effect of refraction, which depends on the real part of the nuclear phase, and while this dependence has hardly been investigated, we found it less important for transfer and inelastic collisions than for elastic collisions.

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At this stage we began to have suspicions about the treatment of elastic collisions in the strong absorption theory, and in fact we found a discrepancy which obscures a proper understanding of the theory. Having cleared up the obscurity, it became possible to investigate further the effect of refraction.

2. Scattering Amplitudes $f^{\pm}$ for Pure Coulomb Field

The scattering amplitude $f(\theta)$ is given by

$$2\mathrm{i}\,k\,f(\theta) = \sum_{l=0}^{\infty}(2l+1)S(l)\exp(2\mathrm{i}\,\sigma_l)\,\mathrm{P}_l(\cos\theta)\,, \tag{1}$$

where σ_l is the Coulomb phase shift of the partial wave of order l. Since

$$\mathrm{P}_l(\cos\theta) \sim (2/\pi l\sin\theta)^{\frac{1}{2}}\sin\{(l+\tfrac{1}{2})\theta+\tfrac{1}{4}\pi\}\,, \tag{2}$$

we may break up the $\sin\theta$ term into two exponentials, and so break up $f(\theta)$ into $f^{+}(\theta)$ and $f^{-}(\theta)$, to give

$$k\sin^{\frac{1}{2}}\theta\,f^{\pm}(\theta) = -(2\pi)^{-\frac{1}{2}}\sum_{l=0}^{\infty} l^{\frac{1}{2}}\,S(l)\exp\{\pm\mathrm{i}(l\theta\pm2\sigma_l)\}\,, \tag{3}$$

disregarding the quantity $\frac{1}{2}$ in comparison with l. Once $f^{+}(\theta)$ and $f^{-}(\theta)$ are separated out, the phase factors $\exp(\pm\mathrm{i}\frac{1}{4}\pi)$ which are associated with them may be dropped.

For a pure Coulomb field $S(l)$ is 1 for all l, and we have previously shown (Mohr 1979, Section 3) that the terms in the partial wave series for $f^{-}(\theta)$ form a Cornu-like double spiral in the complex plane, specified in terms of Fresnel integrals of argument w, where

$$w = (l_{\mathrm{c}}/\pi\sin\theta_{\mathrm{c}})^{\frac{1}{2}}2\sin\{\tfrac{1}{2}(\theta-\theta_{\mathrm{c}})\}\,, \tag{4}$$

$$l_0 = n\cot(\tfrac{1}{2}\theta)\,, \tag{5}$$

$$l_{\mathrm{c}} = n\cot(\tfrac{1}{2}\theta_{\mathrm{c}})\,, \tag{6}$$

and where

$$l_{\mathrm{c}} \approx k(R_1+R_2)\,, \tag{7}$$

$$n = Z_1Z_2e^2/\hbar v\,. \tag{8}$$

We have shown (Mohr 1976) also that

$$w = (2\pi/n)^{\frac{1}{2}}(l_{\mathrm{c}}-l_0)\sin(\tfrac{1}{2}\theta)\,. \tag{9}$$

These results follow from the semiclassical (stationary phase) approximation in which, at small angles, f^{+} is small compared with f^{-}, so that $f \approx f^{-}$. Also, θ_{c} is the quarter-point angle, at which $f = \frac{1}{2}f_{\mathrm{C}}$.

Figs 1 and 2 show how the spirals change with θ, for the two extreme cases $n = 20$ (heavy nuclei) and $n = 2$ (light nuclei) respectively. For easy comparison each is scaled by the factor $\sin^2(\frac{1}{2}\theta)$. Here f^{-} is given by the length of the straight line from O′ to a particular l value on the spiral, and f_{C} approximately by the length

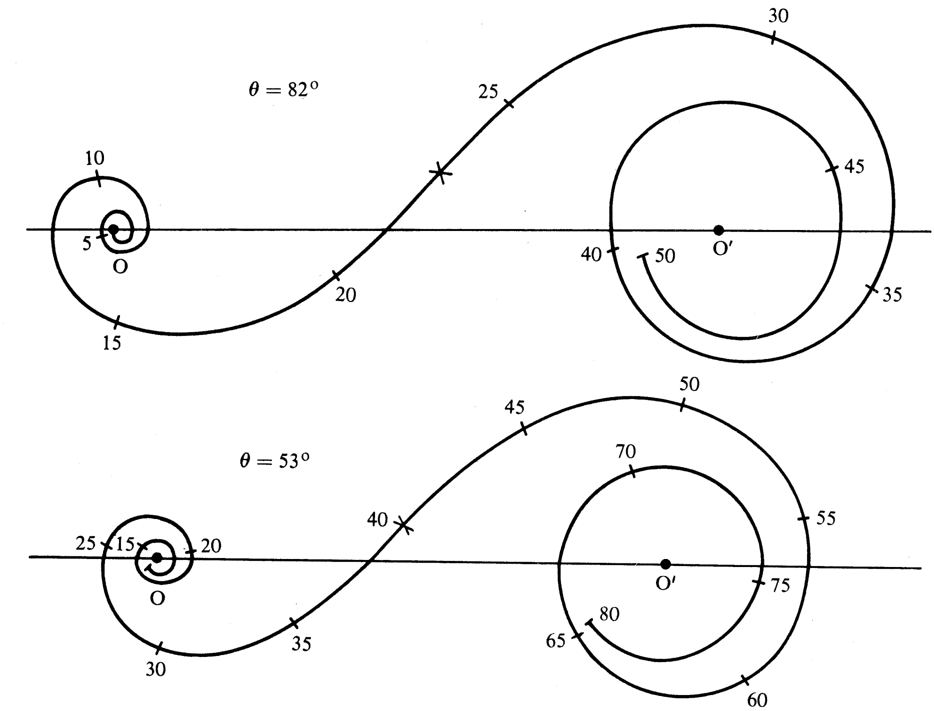

Fig. 1. Phase–amplitude diagrams for scattering by a Coulomb potential through an angle θ for $n = 20$. The successive terms in the partial wave series for $f^{-}(\theta)$ are represented by lines starting at the point O ($l = 0$) and ending at the point O′ ($l = \infty$). The numbers denote the values of l. For ease in drawing, a continuous curve without breaks in direction is shown. The cross indicates the value $l_0 = n\cot(\frac{1}{2}\theta)$ (point of stationary phase) which is nearly at the midpoint of the straight line OO′.

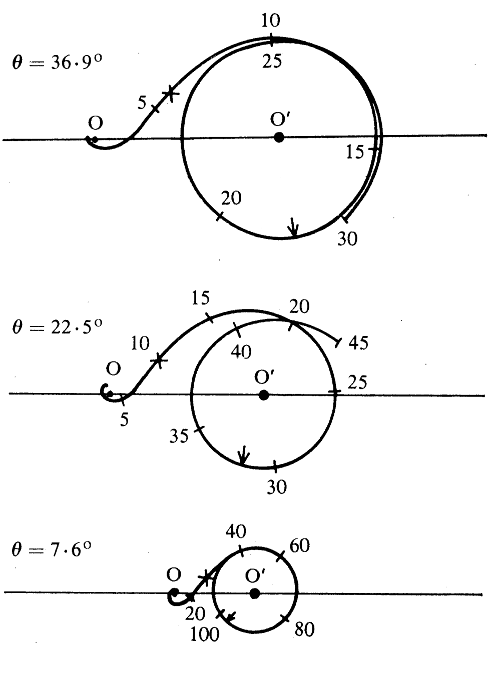

Fig. 2. As for Fig. 1, but with $n = 2$. The arrows indicate the point at which the spiral stops contracting and starts expanding.

OO′ in the case of Fig. 1. As θ decreases the l values move from the spiral about O′ to the spiral about O, in such a way that $\theta(l-l_0)$, and hence the value of w in (9), remains constant. The spirals shown were calculated from (3), but they agree well with those calculated by the semiclassical approximation using Fresnel integrals, for $n = 20$. For $n = 2$ the agreement is not as good.

We therefore expect this approximation also to give a fair account of the variation of f^- with θ, for $l < l_c$, with l_c given by (7), so that $\theta < \theta_c$ from (5) and (6). The form of the cutoff in l values due to absorption in close collisions is of vital importance, however, for $l > l_c$ and $\theta > \theta_c$, and this is dealt with in the next section.

For small n, however, classical considerations may be expected to break down, and this may be seen in Fig. 2 for $n = 2$. The spiral about O′ has only about one turn before expanding indefinitely. For a point P on the spiral with $l \gg l_0$ so that $S \approx 1$, we have from (3) and the discussion of our previous paper (Mohr 1979, Section 3)

$$\mathrm{O'P} \propto l^{\frac{1}{2}}/(\theta-2\rho_l) \approx l^{\frac{1}{2}}/(\theta-2n/l)\,,$$

which is a minimum for $l = 6n/\theta$, and these points of minimum radius are indicated by arrows in Fig. 2. Lack of convergence is a well-known difficulty with the long-range Coulomb field, and is worst for small n (weak fields).

Other features of Fig. 2 are: The point $l = l_0$ (shown by a cross) is no longer a point of symmetry; as θ decreases the diagrams shrink, and since they incorporate a factor $\sin^2(\frac{1}{2}\theta)$, this implies that f^- varies at a slower rate than for Coulomb scattering.

3. Strong Absorption Approximation

In this approximation one takes $l = l_c$ and $\sigma_l = \frac{1}{2}l\theta_c$ for all values of l, and then (3) becomes

$$k\sin^{\frac{1}{2}}\theta\, f^-(\theta) = -(l_c/2\pi)^{\frac{1}{2}} \sum_{l=0}^{\infty} S(l)\exp(-\mathrm{i}\, l\theta^-)\,, \tag{10}$$

where $\theta^- = \theta-\theta_c$. Putting $\lambda = l-l_c$ and replacing the summation by an integral gives

$$k\sin^{\frac{1}{2}}\theta\, f^-(\theta) = c\int_{\lambda=-\infty}^{\infty} S(\lambda)\exp(-\mathrm{i}\,\lambda\theta^-)\,\mathrm{d}\lambda\,, \tag{11}$$

where $c = -(l_c/2\pi)^{\frac{1}{2}}$. It is usual to take $S(\lambda)$ to be of the Woods–Saxon form $1/\{1+\exp(-\lambda/\Delta)\}$, and then $S = \frac{1}{2}$ at $\lambda = 0$ or $l = l_c$, where $\theta = \theta_c$ the quarter-point angle at which $f = \frac{1}{2}f_C$. For values of λ large enough to make $S(\lambda) \approx 1$, the term $\exp(-\mathrm{i}\,\lambda\theta^-)$ in (11) for given θ^- and varying λ can be represented by a circle of unit radius, and we note from Figs 1 and 2 that the spirals about O′ tend to circles at large l.

For $l \approx l_c$, or $\lambda \approx 0$, we have $S(\lambda) \approx \frac{1}{2}\exp(\lambda/\Delta)$, so in (11) we have a phase factor $\exp(-\mathrm{i}\,\lambda\theta^-)$ arising from the Coulomb phase σ_l, and a 'phase factor' $\exp(\lambda/\Delta)$ which can be interpreted as arising from an imaginary nuclear phase. The real part of the nuclear phase takes account of the effect of refraction, to be considered in the next section.

We now evaluate (11), separating the real and imaginary parts, and for convenience in notation put $\beta = 1/\Delta$, so that $S(\lambda) = 1/\{1+\exp(-\beta\lambda)\}$. We start from the known integral (Gradshteyn and Ryzhik 1965, equation 3.911(1))

$$\int_0^\infty \sin(\theta^-\lambda)/\{1+\exp(\beta\lambda)\}\,\mathrm{d}\lambda = 1/2\theta^- - \pi/2\beta\sinh(\theta^-\pi/\beta)\,. \tag{12}$$

We also require use of

$$\int_0^\infty \exp(-\mu\lambda)\exp(-\mathrm{i}\,\theta^-\lambda)\,\mathrm{d}\lambda = 1/(\mu+\mathrm{i}\,\theta^-)\,, \tag{13}$$

where $\exp(-\mu\lambda)$ is a convergence factor which makes the spiral about O′ converge to the point O′, and take $\mu \to 0$, giving in the limit

$$\int_0^\infty \cos(\theta^-\lambda)\,\mathrm{d}\lambda = 0 = \int_{-\infty}^0 \cos(\theta^-\lambda)\,\mathrm{d}\lambda\,, \tag{14}$$

$$\int_0^\infty \sin(\theta^-\lambda)\,\mathrm{d}\lambda = 1/\theta^-\,. \tag{15}$$

Subtracting (12) from (15) gives

$$\int_0^\infty \sin(\theta^-\lambda)/\{1+\exp(-\beta\lambda)\}\,\mathrm{d}\lambda = 1/2\theta^- + \pi/2\beta\sinh(\theta^-\pi/\beta)\,. \tag{16}$$

Replacing λ by $-\lambda$ in (12) gives

$$\int_{-\infty}^0 \sin(\theta^-\lambda)/\{1+\exp(-\beta\lambda)\}\,\mathrm{d}\lambda = -1/2\theta^- + \pi/2\beta\sinh(\theta^-\pi/\beta)\,. \tag{17}$$

Adding (16) and (17) gives

$$\int_{-\infty}^\infty \sin(\theta^-\lambda)/\{1+\exp(-\beta\lambda)\}\,\mathrm{d}\lambda = \pi/\beta\sinh(\theta^-\pi/\beta)\,. \tag{18}$$

The same procedure, having regard to signs and using (14) instead of (15), gives

$$\int_{-\infty}^\infty \cos(\theta^-\lambda)/\{1+\exp(-\beta\lambda)\}\,\mathrm{d}\lambda = 0\,. \tag{19}$$

Finally, from (11), (18) and (19) we have

$$k\sin^{\frac{1}{2}}\theta\, f^-(\theta) = \mathrm{i}\,c\pi\Delta/\sinh(\pi\Delta\theta^-)\,. \tag{20}$$

A different result is implied by Bassichis and Dar (1966, equation 43) and by Hartmann (1976, equation 22), namely

$$k\sin^{\frac{1}{2}}\theta\, f^-(\theta) = \mathrm{i}\,c\pi\Delta\theta^-/\sinh(\pi\Delta\theta^-)\,, \tag{21}$$

which differs in an important respect from (20) in its behaviour as $\theta \to \theta_c$, or $\theta^- \to 0$: the result (21) tends to a constant, whereas (20) tends to infinity. The original work of Frahn and Venter (1963, equations 52a and 77) calls (21) a 'form factor' which has to be multiplied by the sharp cutoff amplitude O′P, where P is a point on the spiral about O′. From our earlier paper (Mohr 1979, Section 3) we have, for θ^- small, $\mathrm{O'P} \propto 1/\theta^-$, which brings us back to the correct form (20).

The divergence in the result (20) for $\theta \to \theta_c$ comes from the approximations made in the derivation. The question arises: for what values of θ does the result fail to hold? For $\theta < \theta_c$ we have the semiclassical approximation, for $\theta > \theta_c$ we have (20), and the curves for f^- in these two regions must join smoothly together as $\theta \to \theta_c$ from above. This procedure will indicate where (20) ceases to hold, and of course will leave a gap just above θ_c. This gap is smallest for large n, for then f^- is largest for $\theta < \theta_c$. Thus the divergence in (20) provides the flexibility required to bridge the region $\theta \approx \theta_c$, even though a little arbitrariness is involved.

A result similar to (20) holds for f^+ with θ^+ in place of θ^-, but f^+ never diverges because $\theta^+ > 0$.

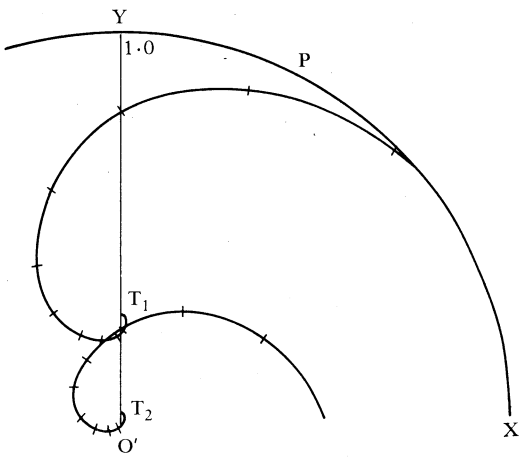

Fig. 3. Phase–amplitude diagram for scattering by a Coulomb potential with strong absorption. The imaginary axis O′Y corresponds to $\lambda = 0$, and the angle YO′P has the value $\lambda\theta^-$. The curve ending at T_1 is for $\Delta = 1$, $\theta^- = 1$, and the curve ending at T_2 for $\Delta = 1$, $\theta^- = 2$. The bars mark off equal intervals of l.

Later we shall require the form of the phase–amplitude diagram when the term $S(\lambda)$ in (11) is taken into account. It is shown in Fig. 3 for two values of θ^-. For large enough λ, when $S(\lambda) \to 1$, the form is the circle $\exp(\mathrm{i}\lambda\theta^-)$. For smaller λ it is given by $\int_\lambda^\infty S(\lambda)\exp(-\mathrm{i}\lambda\theta^-)\,\mathrm{d}\lambda$, the integral being evaluated numerically for suitably spaced values of λ. For $\lambda = -\infty$ the integral has the value (20), indicated by the lengths $\mathrm{O'T_1}$ and $\mathrm{O'T_2}$ for $\Delta\theta^- = 1$ and 2 respectively.

In the early part of this section we were concerned mainly with the behaviour of f^- for $\theta^- \to 0$, but (10), (11) and (20) hold also with f^+ in place of f^- and θ^+ in place of θ^-, so that Fig. 3 is valid for both f^+ and f^-, if we consider only the Coulomb interaction. A basic assumption was to take the difference between successive Coulomb phases to be a constant θ_c. And we recall that f is given by the vector sum of the amplitudes f^+ and f^-.

4. Effect of Refraction

Refraction gives rise to a difference between f^+ and f^-, and involves the real nuclear phase due to the nuclear interaction. If we assume that this phase is appreciable only for $l < l_c$, and that the difference between successive nuclear phases is a constant θ_{N} for the first few values of $l < l_{\mathrm{c}}$, we obtain an appropriate modification of (20), and of the left side of the axis O′Y in Fig. 3 where $\lambda < 0$. For f^{-} the spiral winds up more tightly, and f^{+} less tightly, so that the terminal point T is raised and lowered respectively, and f^{+} and f^{-} increased and decreased correspondingly.

The nuclear phases are best calculated using the WKB method for a complex potential, for then we obtain insight into the behaviour of the real and imaginary parts of the nuclear phase, the differential equations for which are strongly coupled in this method.

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