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Quantum Electrodynamics in Strong Magnetic Fields. I Electron States

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Abstract

Dirac's equation in the presence of a static magnetic field is solved in terms of both cartesian and cylindrical coordinates, and solutions are found for three different spin operators. Choosing the spin to correspond to the parallel component μ_z of the magnetic moment operator leads to wavefunctions (a) which are symmetric between electron and positron states and (b) which are eigenfunctions of the Hamiltonian including radiative corrections. A vertex function $[\gamma^{\varepsilon' \varepsilon}_{q' q}(\boldsymbol{k})]^\mu$ is defined and shown to be proportional to a gauge independent quantity $[\Gamma^{\varepsilon' \varepsilon}_{q' q}(\boldsymbol{k})]^\mu$. Symmetry properties of $[\Gamma^{\varepsilon' \varepsilon}_{q' q}(\boldsymbol{k})]^\mu$ are derived in the case where the spin corresponds to μ_z. The use of the vertex function is illustrated by deriving the electron propagator in coordinate space from the vacuum expectation value. Properties of functions $J^n_{n',-n}(x)$ which appear extensively and are related to generalized Laguerre polynomials are derived and summarized in the Appendix.

1. Introduction

This is the first of a series of papers in which we present a systematic development of quantum electrodynamics (QED) with the effect of a magnetostatic field taken into account exactly. The method we used is based on an approach due originally to Svetozarova and Tsytovich (1962), and developed and applied to various problems in a number of earlier papers (Melrose 1974; Melrose and Stoneham 1976, 1977; Melrose and Zheleznyakov 1981; Melrose and Parle 1981). However, these earlier developments were piecemeal and contained some omissions and other weaknesses. One weakness concerns the covariance and gauge invariance of the theory. A covariant and gauge invariant theory for wave dispersion has been developed in a classical context (Melrose 1973, 1981, 1982) and in this series of papers it is combined with QED in the formulation of a relativistic quantum version of the kinetic theory of plasmas. Intrinsically, relativistic quantum effects tend to be important only at high energies ($\hbar\omega \gtrsim 1$ MeV), at high temperatures ($T_e \gtrsim 10^{10}$ K), or in strong magnetic fields close to the critical field $B_c = 4{\cdot}4 \times 10^9$ T ($=4{\cdot}4 \times 10^{13}$ G). An important application of our synthesized version of QED and the kinetic theory of plasmas is to the treatment of processes in a magnetized vacuum; the wave dispersion is then determined by the properties of the birefringent vacuum and photon–photon splitting and other nonlinear effects are also attributed to nonlinear electromagnetic responses of the magnetized vacuum. It is straightforward to include the responses of both the vacuum and of a material medium simultaneously in the theory. In applications to

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pulsars the contributions from both the vacuum and the ambient electron gas need to be taken into consideration; see, for example, Pavlov *et al.* (1980). Our synthesized theory describes processes in classical plasmas in one limit and processes in a magnetized vacuum in another limit.

In this paper we are concerned with solutions of Dirac's equation in the presence of a magnetic field and in the use of these wavefunctions in the construction of a vertex function $[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^{\mu}$ which plays an important role in our theory. The solution of Dirac's equation in a magnetostatic field is well known (e.g. Johnson and Lippmann 1949). However, the wavefunctions usually used lack symmetry between electron and positron states, and this reflects an underlying weakness in that the wavefunctions are eigenstates of a 'spin' operator which is not particularly meaningful. It is highly desirable to have a theory which is symmetric between positron and electron states. Indeed, incorrect treatment of the symmetry has led to errors in the literature: for example, the impossibility of spontaneous pair creation in a magnetic field as discussed by Canuto and Chiu (1971) and the erroneous relations suggested by Klepikov (1954) and corrected by Kaminker and Yakovlev (1982). Sokolov and Ternov (1968) chose wavefunctions which are eigenfunctions of an operator μ_z which may be interpreted as the component of the magnetic moment operator along the magnetostatic field $\boldsymbol{B}$. They showed that μ_z commutes not only with the Dirac Hamiltonian but also with the radiation corrections to it, so that eigenstates of μ_z remain eigenstates in the absence of external interactions. This is *not* the case for eigenfunctions of other 'spin' operators; this point was made recently by Herold *et al.* (1982). Sokolov and Ternov (1968) used cylindrical coordinates whereas most other authors use cartesian coordinates in solving Dirac's equations. This raises a question as to how the form of our vertex function depends on the choice of coordinates and of the gauge for the magnetostatic field.

In Section 2 we solve Dirac's equation in both cartesian and cylindrical polar coordinates, and in Section 3 we construct eigenfunctions of various 'spin' operators including μ_z. A vertex function $[\gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^{\mu}$ is defined in Section 4, and is shown to be proportional to $[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^{\mu}$ with the gauge and coordinate dependent factors included in the constant of proportionality. (Our definition here of $[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^{\mu}$ has the labels $\varepsilon'\varepsilon$ and $q'q$ interchanged compared with the definition used in earlier papers.) The symmetry properties of $[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^{\mu}$ are written down in Section 5. In Section 6 we digress somewhat from our systematic development by using the results of Sections 2 and 3 to calculate the electron propagator from the vacuum expectation value. What we show is that a known result for the electron propagator in coordinate space is reproduced. This calculation illustrates some of the techniques and results we require including several mathematical identities which are summarized in the Appendix. The electron propagator is gauge dependent and has no momentum space representations in the usual sense. (This is because $G(x_1, x_2)$ depends separately on x_1 and x_2 rather than only on $x_1 - x_2$.) This fact implies that the usual procedure for deriving momentum space expressions for S matrix elements does not work in our case. As we show in an accompanying paper (Melrose and Parle 1983; present issue p. 799) this does not preclude one formulating a momentum space version of the rules for writing down amplitudes for Feynman diagrams.

The notation used is that of Berestetskii *et al.* (1971) and of Itzykson and Zuber (1980) with minor modifications explained where relevant. We use natural units ($\hbar = c = 1$).

2. Solutions of Dirac's Equation

In the presence of a magnetostatic field $\boldsymbol{B}$ with vector potential $\boldsymbol{A}(\boldsymbol{x})$, the Hamiltonian in Dirac's equation

$$\{\mathrm{i}\,\partial/\partial t - \hat{H}(\boldsymbol{x})\}\psi(\boldsymbol{x}, t) = 0 \tag{1}$$

has the form

$$\hat{H}(\boldsymbol{x}) = \boldsymbol{\alpha}.\{\hat{\boldsymbol{p}} + e\boldsymbol{A}(\boldsymbol{x})\} + \beta m. \tag{2}$$

In the Schrödinger representation we have $\hat{\boldsymbol{p}} = -\mathrm{i}\,\mathrm{grad}$, and in the standard representation of the Dirac algebra we have

$$\boldsymbol{\alpha} = \rho_x\boldsymbol{\sigma}, \quad \boldsymbol{\gamma} = \mathrm{i}\rho_y\boldsymbol{\sigma}, \quad \beta = \rho_z, \tag{3}$$

where $\boldsymbol{\sigma}$ denotes the usual Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

with $\boldsymbol{\rho}$ denoting corresponding forms with the entries 0 and 1 replaced respectively by the null and unit 2×2 matrices.

(*a*) *Cartesian Coordinates*

We first solve (1) choosing the gauge

$$\boldsymbol{A}(\boldsymbol{x}) = (0, Bx, 0), \tag{5}$$

where $\boldsymbol{B}$ is along the z-axis. In this case it is convenient to use cartesian coordinates. The Hamiltonian does not depend on t, y or z and hence we may choose a solution which varies as $\exp(-\mathrm{i}p^0t + \mathrm{i}p_y y + \mathrm{i}p_z z)$. However, we know that p^0 has both positive and negative energy solutions, and it is convenient to write $p^0 = \varepsilon\mathscr{E}_q$ where $\varepsilon = \pm 1$ denotes the sign of the energy and q denotes the other quantum numbers collectively, with $\mathscr{E}_q > 0$. We also know that if p_z is interpreted as the z component of momentum for an electron ($\varepsilon = 1$) then p_z represents minus the z component of momentum for a positron ($\varepsilon = -1$). Consequently it is convenient to seek a solution to (1) of the form

$$\psi(\boldsymbol{x}, t) = f(x)\exp(-\mathrm{i}\varepsilon\mathscr{E}_q t + \mathrm{i}\varepsilon p_y y + \mathrm{i}\varepsilon p_z z), \tag{6}$$

and one finds

$$\begin{bmatrix} -\varepsilon\mathscr{E}_q + m & 0 & \varepsilon p_z & \mathscr{O}_1 \\ 0 & -\varepsilon\mathscr{E}_q + m & \mathscr{O}_2 & -\varepsilon p_z \\ \varepsilon p_z & \mathscr{O}_1 & -\varepsilon\mathscr{E}_q - m & 0 \\ \mathscr{O}_2 & -\varepsilon p_z & 0 & -\varepsilon\mathscr{E}_q - m \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \end{bmatrix} = 0, \tag{7}$$

where for convenience we write $\mathscr{O}_1 = -\mathrm{i}(\partial/\partial x + \varepsilon p_y + eBx)$ and $\mathscr{O}_2 = \mathrm{i}(\partial/\partial x - \varepsilon p_y - eBx)$. Here ψ and hence f are written as column matrices with components 1 to 4. By writing

$$\xi \equiv (eB)^{\frac{1}{2}}(x + \varepsilon p_y/eB), \tag{8}$$

where $\equiv$ denotes a definition, (7) may be reduced to

$$(-\varepsilon\mathscr{E}_q \pm m)f_{1,3}(x) + \varepsilon p_z f_{3,1}(x) - \mathrm{i}(eB)^{\frac{1}{2}}(\xi + \mathrm{d}/\mathrm{d}\xi)f_{4,2}(x) = 0, \tag{9a}$$

$$(-\varepsilon\mathscr{E}_q \pm m)f_{2,4}(x) - \varepsilon p_z f_{4,2}(x) - \mathrm{i}(eB)^{\frac{1}{2}}(\xi - \mathrm{d}/\mathrm{d}\xi)f_{3,1}(x) = 0. \tag{9b}$$

After operating on (9a) with $\xi - \mathrm{d}/\mathrm{d}\xi$ and on (9b) with $\xi + \mathrm{d}/\mathrm{d}\xi$ one finds

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \frac{\mathscr{E}_q^2 - m^2 - p_z^2}{eB} - (\xi^2 \pm 1)\right) f(x) = 0, \tag{10}$$

with the upper sign for $f_1(x)$ and $f_3(x)$ and the lower sign for $f_2(x)$ and $f_4(x)$. Equation (10) is of the same form as Schrödinger's equation for a simple harmonic oscillator. Introducing n by

$$2neB \equiv \mathscr{E}_q^2 - m^2 - p_z^2, \tag{11}$$

the physically acceptable solutions have

$$2n \mp 1 = 2l + 1, \tag{12}$$

with $l = 0, 1, 2, \ldots$. The ground state $n = 0$ is nondegenerate and each excited state $n = 1, 2, \ldots$ is doubly degenerate corresponding to the $\pm$ signs in (10). The solutions to (10) are the normalized oscillator wavefunctions

$$v_l(\xi) \equiv \mathrm{H}_l(\xi)\exp(-\tfrac{1}{2}\xi^2)/(\pi^{\frac{1}{2}}2^l l!)^{\frac{1}{2}}, \tag{13}$$

where $\mathrm{H}_l(\xi)$ is a Hermite polynomial. One finds

$$f(x) = \begin{bmatrix} C_1\, v_{n-1}(\xi) \\ C_2\, v_n(\xi) \\ C_3\, v_{n-1}(\xi) \\ C_4\, v_n(\xi) \end{bmatrix}, \tag{14}$$

where C_1 to C_4 are normalization constants, with C_1 and C_3 identically zero for $n = 0$.

By using the identities

$$(\xi + \mathrm{d}/\mathrm{d}\xi)v_l(\xi) = (2l)^{\frac{1}{2}}v_{l-1}(\xi), \tag{15a}$$

$$(\xi - \mathrm{d}/\mathrm{d}\xi)v_l(\xi) = \{2(l+1)\}^{\frac{1}{2}}v_{l+1}(\xi), \tag{15b}$$

equation (7) implies that the C satisfy

$$\begin{bmatrix} -\varepsilon\mathscr{E}_q + m & 0 & \varepsilon p_z & -\mathrm{i}p_n \\ 0 & -\varepsilon\mathscr{E}_q + m & \mathrm{i}p_n & -\varepsilon p_z \\ \varepsilon p_z & -\mathrm{i}p_n & -\varepsilon\mathscr{E}_q - m & 0 \\ \mathrm{i}p_n & -\varepsilon p_z & 0 & -\varepsilon\mathscr{E}_q - m \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = 0, \tag{16}$$

with

$$p_n \equiv (2neB)^{\frac{1}{2}}. \tag{17}$$

The matrix of coefficients (16) is of rank two due to the solutions being pairwise degenerate. The solutions to (16) involve only two independent constants, D_1 and D_2 say:

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \frac{1}{\{2\varepsilon\mathscr{E}_q(\varepsilon\mathscr{E}_q+m)\}^{\frac{1}{2}}} \left[\begin{bmatrix} \varepsilon\mathscr{E}_q+m \\ 0 \\ \varepsilon p_z \\ \mathrm{i}p_n \end{bmatrix} D_1 + \begin{bmatrix} 0 \\ \varepsilon\mathscr{E}_q+m \\ -\mathrm{i}p_n \\ -\varepsilon p_z \end{bmatrix} D_2 \right], \tag{18}$$

with

$$\sum_{i=1}^{4} |C_i|^2 = \sum_{i=1}^{2} |D_i|^2 = 1 \tag{19}$$

in the following. The Johnson and Lippmann (1949) wavefunctions correspond to $D_1 = 1$, $D_2 = 0$, and $D_1 = 0$, $D_2 = 1$ in (18); these and other solutions are discussed in Section 3 below.

We choose to normalize our wavefunctions without introducing factors such as $(eB)^{\frac{1}{2}}$ which normally appear in say the Johnson and Lippmann wavefunctions. The normalization here corresponds to

$$\tfrac{1}{2}\sum_{\sigma}\sum_{n=0}^{\infty}\int\frac{\mathrm{d}p_y}{2\pi}\int\frac{\mathrm{d}p_z}{2\pi}\psi_q^+(x)\psi_q(x') = (eB)^{-\frac{1}{2}}\delta^3(x-x'). \tag{20}$$

In deriving (20) we have averaged over two spin states $\sigma = \pm 1$ and used the identity

$$\sum_{n=0}^{\infty} v_n(\xi)v_n(\xi') = \delta(\xi-\xi'). \tag{21}$$

The normalization (20) is implied by (19), and because of it we must introduce a density of initial states factor to take account of normalization constants, cf. Section 2*c* below.

(*b*) *Cylindrical Coordinates*

If we choose, in place of (5), the gauge

$$A(x) = \tfrac{1}{2}(-By, Bx, 0), \tag{22}$$

then it is convenient to use cylindrical polar coordinates:

$$r \equiv (x^2+y^2)^{\frac{1}{2}}; \qquad x = r\cos\phi, \quad y = r\sin\phi. \tag{23}$$

In this case we seek a solution of (1) of the form

$$\psi(x,t) = g(r,\phi)\exp(-\mathrm{i}\varepsilon\mathscr{E}_q t + \mathrm{i}\varepsilon p_z z). \tag{24}$$

In place of (7) we require

$$\begin{bmatrix} -\varepsilon\mathscr{E}_q+m & 0 & \varepsilon p_z & \mathscr{D}_1 \\ 0 & -\varepsilon\mathscr{E}_q+m & \mathscr{D}_2 & -\varepsilon p_z \\ \varepsilon p_z & \mathscr{D}_1 & -\varepsilon\mathscr{E}_q-m & 0 \\ \mathscr{D}_2 & -\varepsilon p_z & 0 & -\varepsilon\mathscr{E}_q-m \end{bmatrix} \begin{bmatrix} g_1(r,\phi) \\ g_2(r,\phi) \\ g_3(r,\phi) \\ g_4(r,\phi) \end{bmatrix} = 0, \tag{25}$$

where for convenience we write

$$\mathscr{D}_1 = -\mathrm{i}\exp(-\mathrm{i}\phi)\left(\frac{\partial}{\partial r} - \frac{\mathrm{i}}{r}\frac{\partial}{\partial \phi} + \tfrac{1}{2}eBr\right),$$

$$\mathscr{D}_2 = -\mathrm{i}\exp(\mathrm{i}\phi)\left(\frac{\partial}{\partial r} + \frac{\mathrm{i}}{r}\frac{\partial}{\partial \phi} - \tfrac{1}{2}eBr\right).$$

The ϕ dependences are satisfied by the choices

$$g_1(r,\phi) = g_1(r)\exp\{\mathrm{i}(m-1)\phi\}, \qquad g_2(r,\phi) = g_2(r)\exp(\mathrm{i}m\phi), \tag{26a, b}$$

$$g_3(r,\phi) = g_3(r)\exp\{\mathrm{i}(m-1)\phi\}, \qquad g_4(r,\phi) = g_4(r)\exp(\mathrm{i}m\phi), \tag{26c, d}$$

where $m = 0, \pm 1, \pm 2, \ldots$ should not be confused with the rest mass of the electron, also denoted by m.

In place of equations (9) we have

$$(-\varepsilon\mathscr{E}_q \pm m)g_{1,3}(r) + \varepsilon p_z g_{3,1}(r) - \mathrm{i}\left(\frac{\partial}{\partial r} + \frac{m}{r} + \frac{eBr}{2}\right)g_{4,2}(r) = 0, \tag{27a}$$

$$(-\varepsilon\mathscr{E}_q \pm m)g_{2,4}(r) - \varepsilon p_z g_{4,2}(r) - \mathrm{i}\left(\frac{\partial}{\partial r} - \frac{m-1}{r} - \frac{eBr}{2}\right)g_{3,1}(r) = 0, \tag{27b}$$

and with n defined as in (11). In place of (10) we have

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{(m-\frac{1}{2}\mp\frac{1}{2})^2}{r^2} + eB(2n-m+\tfrac{1}{2}\mp\tfrac{1}{2}) - \tfrac{1}{4}e^2B^2r^2\right)g(r) = 0. \tag{28}$$

The normalizable solutions of (28) are related to the generalized Laguerre polynomials $\mathrm{L}_n^{\nu}(x)$. We introduce the functions (Svetozarova and Tsytovich 1962; Melrose 1974)

$$\mathrm{J}_{\nu}^{n}(x) \equiv \{n!/(n+\nu)!\}^{\frac{1}{2}}\exp(-\tfrac{1}{2}x)x^{\frac{1}{2}\nu}\mathrm{L}_n^{\nu}(x). \tag{29}$$

The properties of these functions, which appear frequently below, are summarized in the Appendix. Our final result is

$$g(r,\phi) = \begin{bmatrix} C_1\,\mathrm{J}^{s}_{n-s-1}(\frac{1}{2}eBr^2)\exp\{\mathrm{i}(n-s-1)\phi\} \\ C_2\,\mathrm{J}^{s}_{n-s}(\frac{1}{2}eBr^2)\exp\{\mathrm{i}(n-s)\phi\} \\ C_3\,\mathrm{J}^{s}_{n-s-1}(\frac{1}{2}eBr^2)\exp\{\mathrm{i}(n-s-1)\phi\} \\ C_4\,\mathrm{J}^{s}_{n-s}(\frac{1}{2}eBr^2)\exp\{\mathrm{i}(n-s)\phi\} \end{bmatrix}, \tag{30}$$

where

$$s \equiv n-m \tag{31}$$

is referred to as the radial quantum number by Sokolov and Ternov (1968). The coefficients C_1 to C_4 again satisfy (16) and may be written in the form (18).

In this case our normalization (19) corresponds to

$$\tfrac{1}{2}\sum_{\sigma}\sum_{n=0}^{\infty}\sum_{s=-\infty}^{\infty}\int\frac{\mathrm{d}p_z}{2\pi}\psi_q^{+}(\boldsymbol{x})\psi_q(\boldsymbol{x}') = (2\pi/eB)\delta^3(\boldsymbol{x}-\boldsymbol{x}'). \tag{32}$$

In deriving (32) the addition theorem (A22) is used to perform the sum over s and the sum over n is then over $J_0^n(\frac{1}{2}eBR^2)$, with R defined by (A23). This latter sum is of the form $J_0^n(0) = 1$. Hence the sum is zero except for $R = 0$, which requires $\phi = \phi'$ and $r = r'$.

(*c*) *Density of States*

Our normalization (19) implies density of state factors which are different in the cases of cartesian and cylindrical coordinates.

In the case of cartesian coordinates let us replace p_y and p_z by discrete quantum numbers corresponding to the system being confined to a box in the y–z plane with sides L_y and L_z. Then (20) implies that our normalization corresponds to $L_y L_z/V(eB)^{\frac{1}{2}}$ electrons in this box, where V is the total volume of our system. Hence we need to include a density of states factor

$$D_i = (eB)^{\frac{1}{2}} V/L_y L_z \tag{33a}$$

for each initial electron or positron. The corresponding factor for cylindrical coordinates is

$$D_i = eBV/2\pi L_z . \tag{33b}$$

The density of final states involves a factor

$$D_f = \{V(eB)^{\frac{1}{2}}/(2\pi)^2\}\, dp_y\, dp_z \tag{34a}$$

in cartesian coordinates, or

$$D_f = \{VeB/(2\pi)^2\}\, dp_z \tag{34b}$$

in cylindrical coordinates, for each final electron or positron.

3. Spin States

Clearly from (12) we may write

$$n = l + \tfrac{1}{2}(\sigma + 1), \tag{35}$$

and interpret $\sigma = \pm 1$ as a spin eigenvalue. However, there is a variety of ways in which the 'spin' operator may be chosen.

(*a*) *Johnson and Lippmann Wavefunctions*

The most widely adopted choice is essentially that made by Johnson and Lippmann (1949). This choice corresponds to $\varepsilon = \pm 1$ and either $D_1 = 1$, $D_2 = 0$ or $D_1 = 0$, $D_2 = 1$ in (18). With q denoting n, σ, p_z and p_y, and with

$$\mathscr{E}_q \equiv (m^2 + p_z^2 + 2neB)^{\frac{1}{2}}, \tag{36}$$

this choice corresponds to

$$\psi_q(\boldsymbol{x}, t) = \frac{\exp(-i\varepsilon\mathscr{E}_q t + i\varepsilon p_y y + i\varepsilon p_z z)}{2\varepsilon\mathscr{E}_q(\varepsilon\mathscr{E}_q + m)} \left[\delta_{\sigma,1} \begin{bmatrix} (\varepsilon\mathscr{E}_q + m)v_{n-1}(\xi) \\ 0 \\ \varepsilon p_z v_{n-1}(\xi) \\ i p_n v_n(\xi) \end{bmatrix} + \delta_{\sigma,-1} \begin{bmatrix} 0 \\ (\varepsilon\mathscr{E}_q + m)v_n(\xi) \\ -i p_n v_{n-1}(\xi) \\ -\varepsilon p_z v_n(\xi) \end{bmatrix} \right]. \tag{37}$$

This choice of wavefunctions has been adopted by Svetozarova and Tsytovich (1962), Melrose (1974), Daugherty and Ventura (1977, 1978), Daugherty and Bussard (1980) and Bussard (1980), amongst others.

The wavefunctions (37) are eigenfunctions of a spin operator which is relatively easily found: It is

$$s_z = \rho_z \sigma_z(\hat{H}+m) + \hat{p}_z \rho_x(1+\rho_z). \tag{38}$$

However, this operator seems to be of no physical significance. Its meaning is obvious only for nonrelativistic electrons when one has $s_z \approx 2m\beta\sigma_z$, giving the electron eigenvalues $\pm 2m$ corresponding to the usual spin eigenvalues $\pm\frac{1}{2}$. However, for nonrelativistic positrons (38) does not give easily interpretable results. We have been led to reject the wavefunctions (37) for many practical purposes because they fail to give easily interpretable results for the spin states of nonrelativistic positrons.

The wavefunctions (37) do not exhibit a simple symmetry between the electron and positron states, as do the other wavefunctions derived below. This is a particularly unsatisfactory feature for our purposes because it obscures a crossing symmetry property. We considered an alternative set of wavefunctions which corresponds to (37) for $\varepsilon = 1$ and involves column matrices

$$(p_z v_{n-1}(\xi) - \mathrm{i} p_n v_n(\xi), (\mathscr{E}_q+m)v_n(\xi), 0),$$
$$(\mathrm{i} p_n v_{n-1}(\xi), -p_z v_n(\xi), 0, (\mathscr{E}_q+m)v_{n-1}(\xi))$$

for $\varepsilon = -1$ and $\sigma = \pm 1$. However, although symmetric between positron and electron states, these wavefunctions are not eigenvalues of any particular spin operator. Indeed the electron states are eigenfunctions of (38) and the positron states are eigenfunctions of a different spin operator. (We refer loosely to the wavefunctions for $\varepsilon = 1$ and $\varepsilon = -1$ as electron and positron states respectively.)

(b) *Helicity States*

Sokolov and Ternov (1968) considered eigenfunctions of the operator $\boldsymbol{\sigma}.\{\hat{\boldsymbol{p}}+e\boldsymbol{A}(\boldsymbol{x})\}$ which does commute with the Hamiltonian. They referred to the corresponding eigenstates as describing 'longitudinal' polarization. Herold (1979) derived analogous 'helicity' eigenstates using a somewhat different line of reasoning.

It is not difficult to construct the operator $\boldsymbol{\sigma}.\{\hat{\boldsymbol{p}}+e\boldsymbol{A}(\boldsymbol{x})\}$ and show that it has eigenvalues σh_q with

$$h_q \equiv (p_z^2+p_n^2)^{\frac{1}{2}}. \tag{39}$$

The solutions correspond to

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \frac{\exp\{\mathrm{i}\Phi(\varepsilon,\sigma)\}}{\{4\varepsilon\sigma\mathscr{E}_q h_q(\varepsilon\mathscr{E}_q+m)(\sigma h_q+\varepsilon p_z)\}^{\frac{1}{2}}} \begin{bmatrix} (\varepsilon\mathscr{E}_q+m)(\sigma h_q+\varepsilon p_z) \\ \mathrm{i}\varepsilon p_z(\varepsilon\mathscr{E}_q+m) \\ \sigma h_q(\sigma h_q+\varepsilon p_z) \\ \mathrm{i}\sigma h_q p_n \end{bmatrix}, \tag{40}$$

or to

$$D_1 = \exp\{\mathrm{i}\Phi(\varepsilon,\sigma)\}\left(\frac{\sigma h_q+\varepsilon p_z}{2\sigma h_q}\right)^{\frac{1}{2}}, \qquad \frac{D_2}{D_1} = \frac{\mathrm{i} p_n}{\sigma h_q+\varepsilon p_z} \tag{41}$$

in (18), where $\Phi(\varepsilon,\sigma)$ is an arbitrary phase factor for each choice of $\varepsilon = \pm 1$ and $\sigma = \pm 1$.

(c) *Eigenstates of* μ_z

The operator

$$\hat{\boldsymbol{\mu}} \equiv m\boldsymbol{\sigma} + \rho_y \boldsymbol{\sigma} \times \{\boldsymbol{p} + e\boldsymbol{A}(\boldsymbol{x})\} \tag{42}$$

was identified as the magnetic moment operator by Sokolov and Ternov (1968). It is straightforward to show that $\hat{\mu}_z$ commutes with the Hamiltonian. Sokolov and Ternov referred to the corresponding eigenstates as describing 'transverse' polarization. They showed that $\hat{\mu}_z$ also commutes with the radiative corrections to the Hamiltonian. Herold *et al.* (1982) arrived at analogous eigenstates by finding the eigenfunctions of the self-energy operator.

The eigenstates in this case correspond to

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \frac{\exp\{\mathrm{i}\,\Phi(\varepsilon,\sigma)\}}{\{4\varepsilon\sigma\mathscr{E}_q\mathscr{E}_q^0(\varepsilon\mathscr{E}_q+\sigma\mathscr{E}_q^0)(\sigma\mathscr{E}_q^0+m)\}^{\frac{1}{2}}} \begin{bmatrix} (\varepsilon\mathscr{E}_q+\sigma\mathscr{E}_q^0)(\sigma\mathscr{E}_q^0+m) \\ -\mathrm{i}\,\varepsilon p_z p_n \\ (\sigma\mathscr{E}_q^0+m)\varepsilon p_z \\ \mathrm{i}(\varepsilon\mathscr{E}_q+\sigma\mathscr{E}_q^0)p_n \end{bmatrix}, \tag{43}$$

or to

$$D_1 = \frac{\exp\{\mathrm{i}\,\Phi(\varepsilon,\sigma)_f(\varepsilon\mathscr{E}_q+\sigma\mathscr{E}_q^0)(\sigma\mathscr{E}_q^0+m)}{\{4\varepsilon\sigma\mathscr{E}_q\mathscr{E}_q^0(\varepsilon\mathscr{E}_q+\sigma\mathscr{E}_q^0)(\sigma\mathscr{E}_q^0+m)\}^{\frac{1}{2}}}, \qquad \frac{D_2}{D_1} = \frac{-\mathrm{i}\,\varepsilon p_z p_n}{(\varepsilon\mathscr{E}_q+\sigma\mathscr{E}_q^0)(\sigma\mathscr{E}_q^0+m)} \tag{44a, b}$$

in (18), where the phases are again arbitrary, with

$$\mathscr{E}_q^0 \equiv (m^2+2neB)^{\frac{1}{2}}.$$

A convenient choice of the phases gives

$$\begin{aligned}\psi_q^\varepsilon(\boldsymbol{x},t) = {} & \frac{\exp(-\mathrm{i}\,\varepsilon\mathscr{E}_q t + \mathrm{i}\,\varepsilon p_y y + \mathrm{i}\,\varepsilon p_z z)}{\{4\mathscr{E}_q\mathscr{E}_q^0(\mathscr{E}_q+m)(\mathscr{E}_q^0+m)\}^{\frac{1}{2}}} \\ & \times \begin{bmatrix} \delta_{\varepsilon,1}\begin{bmatrix} \delta_{\sigma,1}\begin{bmatrix} (\mathscr{E}_q+\mathscr{E}_q^0)(\mathscr{E}_q^0+m)v_{n-1}(\xi) \\ -\mathrm{i}\,p_n p_z v_n(\xi) \\ p_z(\mathscr{E}_q^0+m)v_{n-1}(\xi) \\ \mathrm{i}\,p_n(\mathscr{E}_q+\mathscr{E}_q^0)v_n(\xi) \end{bmatrix} + \delta_{\sigma,-1}\begin{bmatrix} -\mathrm{i}\,p_z p_n v_{n-1}(\xi) \\ (\mathscr{E}_q+\mathscr{E}_q^0)(\mathscr{E}_q^0+m)v_n(\xi) \\ -\mathrm{i}\,p_n(\mathscr{E}_q+\mathscr{E}_q^0)v_{n-1}(\xi) \\ -p_z(\mathscr{E}_q^0+m)v_n(\xi) \end{bmatrix} \end{bmatrix} \\ +\delta_{\varepsilon,-1}\begin{bmatrix} \delta_{\sigma,1}\begin{bmatrix} p_z(\mathscr{E}_q^0+m)v_{n-1}(\xi) \\ -\mathrm{i}\,p_n(\mathscr{E}_q+\mathscr{E}_q^0)v_n(\xi) \\ (\mathscr{E}_q+\mathscr{E}_q^0)(\mathscr{E}_q^0+m)v_{n-1}(\xi) \\ \mathrm{i}\,p_z p_n v_n(\xi) \end{bmatrix} + \delta_{\sigma,-1}\begin{bmatrix} \mathrm{i}\,p_n(\mathscr{E}_q+\mathscr{E}_q^0)v_{n-1}(\xi) \\ -p_z(\mathscr{E}_q^0+m)v_n(\xi) \\ \mathrm{i}\,p_z p_n v_{n-1}(\xi) \\ (\mathscr{E}_q+\mathscr{E}_q^0)(\mathscr{E}_q^0+m)v_n(\xi) \end{bmatrix} \end{bmatrix} \end{bmatrix}. \end{aligned} \tag{45}$$

We use the form (45) in the following.

4. Vertex Function

A quantity which plays an important role in our theory is the vertex function

$$[\gamma_{q'q}^{\varepsilon'\varepsilon}(\boldsymbol{k})]^\mu \equiv \frac{1}{V}\int \mathrm{d}\boldsymbol{x}\,\exp(-\mathrm{i}\,\boldsymbol{k}.\boldsymbol{x})\,\bar{\psi}_{q'}^{\varepsilon'}(\boldsymbol{x})\gamma^\mu\psi_q^\varepsilon(\boldsymbol{x}), \tag{46}$$

where V is the volume of the system. In (46) it is convenient to separate the sign ε of the energy from the other quantum numbers q. In this section we evaluate this function for both cartesian and cylindrical coordinates and show that in both cases the result is proportional to a gauge independent quantity $[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^\mu$, which we evaluate explicitly for the wavefunctions (45).

Note that we do not invert the order of the quantum numbers between the left- and right-hand sides of (46), contrary to the notation adopted earlier (Melrose 1974; Melrose and Stoneham 1977). Consequently, in evaluating the amplitude for a Feynman diagram, the q appear in the same order as the wavefunctions. This corresponds to matrix multiplication starting from the final state on the left to the initial state on the right for an electron line.

(*a*) *Cartesian Coordinates*

On substituting the wavefunctions (6) with (14) in (46), the y and z integrals give δ functions. The remaining x integral may be reduced to a standard integral (e.g. Gradshteyn and Ryzhik 1965, 7·377), which implies

$$\int_{-\infty}^{\infty} \mathrm{d}x \exp(-\mathrm{i}k_x x)\, v_{n'}(\xi')\, v_n(\xi) = (eB)^{-\frac{1}{2}} \exp\{\mathrm{i}k_x(\varepsilon p_y + \varepsilon' p'_y)/2eB\} \times \{-\mathrm{i}\exp(-\mathrm{i}\psi)\}^{n'-n} \mathrm{J}^n_{n'-n}(k_\perp^2/2eB), \quad (47)$$

where we introduce ψ by writing

$$\boldsymbol{k} = (k_\perp \cos\psi, k_\perp \sin\psi, k_z), \quad (48)$$

and with $\xi' = (eB)^{\frac{1}{2}}(x + \varepsilon' p'_y/eB)$. The y integral implies $\varepsilon' p'_y = \varepsilon p_y - k_y$. One then finds

$$[\gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^\mu = \{(2\pi)^2/V(eB)^{\frac{1}{2}}\}\exp\{\mathrm{i}k_x(\varepsilon p_y + \varepsilon' p'_y)/2eB\}\delta(\varepsilon p_y - \varepsilon' p'_y - k_y) \times \delta(\varepsilon p_z - \varepsilon' p'_z - k_z)[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^\mu, \quad (49)$$

with

$$\begin{aligned}[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^\mu = {} & \{-\mathrm{i}\exp(-\mathrm{i}\psi)\}^{n'-n} \\ & \times \{(C_1'^* C_1 + C_3'^* C_3)\mathrm{J}^{n-1}_{n'-n} + (C_2'^* C_2 + C_4'^* C_4)\mathrm{J}^n_{n'-n}, \\ & \mathrm{i}(C_1'^* C_4 + C_3'^* C_2)\exp(\mathrm{i}\psi)\,\mathrm{J}^n_{n'-n-1} - \mathrm{i}(C_2'^* C_3 + C_4'^* C_1)\exp(-\mathrm{i}\psi)\,\mathrm{J}^{n-1}_{n'-n+1}, \\ & (C_1'^* C_4 + C_3'^* C_2)\exp(\mathrm{i}\psi)\,\mathrm{J}^n_{n'-n-1} + (C_2'^* C_3 + C_4'^* C_1)\exp(-\mathrm{i}\psi)\,\mathrm{J}^{n-1}_{n'-n+1}, \\ & (C_1'^* C_3 + C_3'^* C_1)\mathrm{J}^{n-1}_{n'-n} - (C_2'^* C_4 + C_4'^* C_2)\mathrm{J}^n_{n'-n}\}, \end{aligned} \quad (50)$$

where the argument $k_\perp^2/2eB$ of the J functions is omitted for brevity.

(*b*) *Cylindrical Coordinates*

In the case of cylindrical coordinates we substitute (24) with (30) in (46) and write the integral in terms of cylindrical polar coordinates. The z integral again gives a δ function. The ϕ integral gives a Bessel function and the r integral is then evaluated

using a result derived by Sokolov and Ternov (1968, p. 81). The net result of these two steps is the identity

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{\infty} \mathrm{d}r\, r \exp\{-\mathrm{i}\, k_\perp r \cos(\psi-\phi)\}\, \mathrm{J}^{s'}_{n'-s'}(\tfrac{1}{2}eBr^2) \exp\{-\mathrm{i}(n'-s')\phi\}$$

$$\times\, \mathrm{J}^{s}_{n-s}(\tfrac{1}{2}eBr^2) \exp\{\mathrm{i}(n-s)\phi\} = (2\pi/eB)\{-\mathrm{i}\exp(-\mathrm{i}\psi)\}^{s-s'}$$

$$\times\, \mathrm{J}^{s}_{s'-s}(k_\perp^2/2eB)\{-\mathrm{i}\exp(-\mathrm{i}\psi)\}^{n'-n}\, \mathrm{J}^{n}_{n'-n}(k_\perp^2/2eB). \quad (51)$$

With this result we obtain

$$[\gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^\mu = \{(2\pi)^2/VeB\}\{-\mathrm{i}\exp(-\mathrm{i}\psi)\}^{s-s'}\, \mathrm{J}^{s}_{s'-s}(k_\perp^2/2eB)$$

$$\times\, \delta(\varepsilon p_z - \varepsilon' p'_z - k_z)[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^\mu, \quad (52)$$

with $[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^\mu$ again given by (50).

(*c*) *Explicit Form for* $[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]$

An explicit form for $[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^\mu$ was written down by Melrose and Stoneham (1977) for the wavefunctions (37). It is straightforward to derive their result using (50) and the values of C_1 to C_4 implied by setting $D_1 = 1$, $D_2 = 0$ and $D_1 = 0$, $D_2 = 1$ in (18). (Recall that we change the conventions for the ordering of the labels $\varepsilon'\varepsilon$ and $q'q$ here.) Similarly one may evaluate (50) explicitly for the wavefunctions (37), (40) or (43). For later purposes the most appropriate, but unfortunately the most cumbersome, form is that obtained for the wavefunctions (45). We write the explicit form only for the wavefunctions (45).

We introduce l and l', which may be referred to as orbital quantum numbers, by writing

$$n = l + \tfrac{1}{2}(1+\sigma), \qquad n' = l' + \tfrac{1}{2}(1+\sigma'). \quad (53)$$

We also introduce normalization factors C_q and $C_{q'}$ with

$$C_q \equiv \{(\mathscr{E}_q + \mathscr{E}^0_q)(\mathscr{E}^0_q + m)/4\mathscr{E}_q \mathscr{E}^0_q\}^{\frac{1}{2}}\{\mathrm{i}\exp(\mathrm{i}\psi)\}^l \quad (54)$$

and similarly for $C_{q'}$. It is also convenient to introduce

$$\rho_z \equiv p_z/(\mathscr{E}_q + \mathscr{E}^0_q), \qquad \rho_n \equiv p_n/(\mathscr{E}^0_q + m), \quad (55a, b)$$

and similarly

$$\rho'_z \equiv p'_z/(\mathscr{E}_{q'} + \mathscr{E}^0_{q'}), \qquad \rho'_{n'} \equiv p_{n'}/(\mathscr{E}^0_{q'} + m). \quad (56a, b)$$

We find that

$$[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^\mu = C^*_{q'} C_q [\delta_{\sigma'\sigma}\{\alpha^{\varepsilon'\varepsilon}_{q'q}(\mathrm{J}^l_{l'-l} + \rho'_{n'}\rho_n \mathrm{J}^{l+\sigma}_{l'-l}),$$

$$\varepsilon\beta^{\varepsilon'\varepsilon}_{q'q}(-\rho_n \exp(\mathrm{i}\,\sigma\psi)\, \mathrm{J}^{l+\sigma}_{l'-l-\sigma} - \rho'_{n'} \exp(-\mathrm{i}\,\sigma\psi)\, \mathrm{J}^{l}_{l'-l+\sigma}),$$

$$\mathrm{i}\varepsilon\sigma\beta^{\varepsilon'\varepsilon}_{q'q}(\rho_n \exp(\mathrm{i}\,\sigma\psi)\, \mathrm{J}^{l+\sigma}_{l'-l-\sigma} - \rho'_{n'} \exp(-\mathrm{i}\,\sigma\psi)\, \mathrm{J}^{l}_{l'-l+\sigma}),$$

$$\eta^{\varepsilon'\varepsilon}_{q'q}(\mathrm{J}^l_{l'-l} + \rho'_{n'}\rho_n \mathrm{J}^{l+\sigma}_{l'-l})\}$$

$$-\varepsilon\sigma\delta_{\sigma',-\sigma}\{a^{\varepsilon'\varepsilon}_{q'q}(-\rho_n\exp(\mathrm{i}\sigma\psi)\,\mathrm{J}^{l+\sigma}_{l'-l-\sigma}+\rho'_{n'}\exp(\mathrm{i}\sigma\psi)\,\mathrm{J}^{l}_{l'-l-\sigma}),$$

$$\varepsilon b^{\varepsilon'\varepsilon}_{q'q}(\mathrm{J}^{l}_{l'-l}-\rho'_{n'}\rho_n\exp(2\mathrm{i}\sigma\psi)\,\mathrm{J}^{l+\sigma}_{l'-l-2\sigma}),$$

$$\mathrm{i}\varepsilon\sigma b^{\varepsilon'\varepsilon}_{q'q}(\mathrm{J}^{l}_{l'-l}+\rho'_{n'}\rho_n\exp(2\mathrm{i}\sigma\psi)\,\mathrm{J}^{l+\sigma}_{l'-l-2\sigma}),$$

$$d^{\varepsilon'\varepsilon}_{q'q}(-\rho_n\exp(\mathrm{i}\sigma\psi)\,\mathrm{J}^{l+\sigma}_{l'-l-\sigma}+\rho'_{n'}\exp(\mathrm{i}\sigma\psi)\,\mathrm{J}^{l}_{l'-l-\sigma})\}], \tag{57}$$

where the argument of the J functions is $k_\perp^2/2eB$ and with

$$\alpha^{\varepsilon'\varepsilon}_{q'q} = \delta_{\varepsilon'\varepsilon}(1+\rho'_z\rho_z)+\sigma\delta_{\varepsilon'-\varepsilon}(\rho'_z+\rho_z), \tag{58a}$$

$$\beta^{\varepsilon'\varepsilon}_{q'q} = \delta_{\varepsilon'\varepsilon}(1-\rho'_z\rho_z)+\sigma\delta_{\varepsilon'-\varepsilon}(\rho'_z-\rho_z), \tag{58b}$$

$$\eta^{\varepsilon'\varepsilon}_{q'q} = \delta_{\varepsilon'\varepsilon}(\rho'_z+\rho_z)+\sigma\delta_{\varepsilon'-\varepsilon}(1+\rho'_z\rho_z), \tag{58c}$$

$$a^{\varepsilon'\varepsilon}_{q'q} = \delta_{\varepsilon'\varepsilon}(\rho'_z+\rho_z)-\sigma\delta_{\varepsilon'-\varepsilon}(1+\rho'_z\rho_z), \tag{58d}$$

$$b^{\varepsilon'\varepsilon}_{q'q} = \delta_{\varepsilon'\varepsilon}(\rho'_z-\rho_z)-\sigma\delta_{\varepsilon'-\varepsilon}(1-\rho'_z\rho_z), \tag{58e}$$

$$d^{\varepsilon'\varepsilon}_{q'q} = \delta_{\varepsilon'\varepsilon}(1+\rho'_z\rho_z)-\sigma\delta_{\varepsilon'-\varepsilon}(\rho'_z+\rho_z). \tag{58f}$$

In deriving (57) with (58) we have used (45) which involves a specific choice of the relative phases. The specific choice made ensures that $[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^\mu$ exhibits a 'crossing' symmetry property relating to the changes of signs of $\varepsilon, \varepsilon'$ and $\boldsymbol{k}$. Other choices of phases in (45) affect (57) by changing the relative phases between the terms with $\varepsilon'\varepsilon = \pm 1$ and $\sigma'\sigma = \pm 1$. More generally $[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^\mu$ could be defined by (57) with four arbitrary phase factors multiplying the terms proportional to $\delta_{\varepsilon'\varepsilon}\delta_{\sigma'\sigma}$, $\delta_{\varepsilon'\varepsilon}\delta_{\sigma'-\sigma}$, $\delta_{\varepsilon'-\varepsilon}\delta_{\sigma'\sigma}$ and $\delta_{\varepsilon'-\varepsilon}\delta_{\sigma'-\sigma}$.

5. Properties of the Vertex Function

In this section we discuss two features of the vertex function which are important in the subsequent development of the theory. One concerns symmetry properties, and the other concerns the role and interpretation of the factors multiplying $[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^\mu$ in (49) and (52).

(a) *Symmetry Properties*

From the definition (46) of $[\gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^\mu$ one may obtain an identity by changing the sign of $\boldsymbol{k}$ and then complex conjugating. One has $\gamma^{\dagger\mu} = \gamma^0\gamma^\mu\gamma^0$ with $(\gamma^0)^2 = 1$, and then with $\bar{\psi}_q(\boldsymbol{x}) \equiv \psi^\dagger_q(\boldsymbol{x})\gamma^0$ one finds

$$[\gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^{*\mu} = [\gamma^{\varepsilon\varepsilon'}_{qq'}(-\boldsymbol{k})]^\mu. \tag{59}$$

Now applying this symmetry to (49) or (52) [note that $\boldsymbol{k} \to -\boldsymbol{k}$ implies $\psi \to \psi'+\pi$ and that $q \leftrightarrow q'$ implies $s \leftrightarrow s'$, with $\mathrm{J}^s_{s'-s} = (-)^{s'-s}\mathrm{J}^{s'}_{s-s'}$ in the latter], one finds

$$[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^{*\mu} = [\Gamma^{\varepsilon\varepsilon'}_{qq'}(-\boldsymbol{k})]^\mu. \tag{60}$$

The other symmetry property may be seen by inspection of (57) with (58). On changing the signs of $\varepsilon, \varepsilon'$ and $\boldsymbol{k}$ there are sign changes from $\varepsilon \to -\varepsilon$ and $\exp(\mathrm{i}\sigma\psi) \to$

$-\exp(\mathrm{i}\sigma\psi)$. The net effect is an overall change in sign of $(-)^{l'-l}$ and a change in sign of the terms proportional to $\delta_{\sigma',-\sigma}$ relative to those proportional to $\delta_{\sigma'\sigma}$. Hence we have

$$[\Gamma^{-\varepsilon'-\varepsilon}_{q'q}(-\boldsymbol{k})]^{\mu} = (-)^{l'-l}[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^{\mu}. \tag{61}$$

The simple form of the symmetry (61) is due in part to the particular choice of the phase factors in (45). For other choices which involve relative phase factors between the eigenfunctions in (45), equation (61) is satisfied separately for the four parts with $\varepsilon'\varepsilon = \pm 1$ and $\sigma'\sigma = \pm 1$ with phase factors which differ from $(-)^{l'-l}$ in general.

The particular choice in (45) corresponds to $\psi^{-\varepsilon}_{q}(\boldsymbol{x}) = (-)^{l}\gamma^5\psi^{\varepsilon}_{q}(-\boldsymbol{x})$, with $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ and where we use $v_n(-\xi) = (-)^n v_n(\xi)$. Then using $\gamma^5\gamma^{\mu}+\gamma^{\mu}\gamma^5 = 0$ one may rederive (61) directly from (46) with (49) or (52).

The form of $[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^{\mu}$ derived by Melrose and Stoneham (1977) using the wavefunctions (37) does not satisfy the symmetry property (61). As already remarked, this is due to a lack of symmetry between the electron and positron states in (37).

(b) Sum over Intermediate States

Our choice of gauge appears in the vertex functions $[\gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^{\mu}$ only in the factors premultiplying $[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^{\mu}$. These factors are taken into account in the density of states factors discussed in Section 2*c* for electrons or positrons in the initial or final states. We show here that these factors need appear *only* relating the initial and final states. This is simplest to show for the factor $2\pi\delta(\varepsilon p_z-\varepsilon' p'_z-k_z)$ which is interpreted as describing conservation of momentum along $\boldsymbol{B}$. When summing over an intermediate state q'' one has

$$\int \frac{\mathrm{d}p''_z}{2\pi}\, 2\pi\delta(\varepsilon p_z-\varepsilon'' p''_z-k_{1z})\, 2\pi\delta(\varepsilon'' p''_z-\varepsilon' p'_z-k_{2z}) = 2\pi\delta(\varepsilon p_z-\varepsilon' p'_z-k_{1z}-k_{2z}). \tag{62}$$

One interprets this by stating that the component of momentum along $\boldsymbol{B}$ is conserved at each vertex.

In (49) the factor $\delta(\varepsilon p_y-\varepsilon' p'_y-k_y)$ is interpreted as describing the change in the x coordinate of the centre of gyration (speaking classically) of the electron. This interpretation is based on the fact that the mean value of x for the electron (or positron) is found to be

$$\langle x\rangle = \int_{-\infty}^{\infty} \mathrm{d}\xi\, x\, v_n^2(\xi) = -\varepsilon p_y/eB. \tag{63}$$

As in (62) the sum over intermediate states in this case is also trivial. The remaining factor in (49) is $(eB)^{-\frac{1}{2}}\exp\{\mathrm{i}k_x(\varepsilon p_y+\varepsilon' p'_y)/2eB\}$. Now the sum over q'' includes a factor $(eB)^{\frac{1}{2}}$ according to (34a), and hence along with (62) and the corresponding result for p_y we have

$$(eB)^{\frac{1}{2}}[(eB)^{-\frac{1}{2}}\exp\{\mathrm{i}k_{1x}(\varepsilon p_y+\varepsilon'' p''_y)/2eB\}][(eB)^{-\frac{1}{2}}\exp\{\mathrm{i}k_{2x}(\varepsilon'' p''_y+\varepsilon' p'_y)/2eB\}]$$
$$= (eB)^{-\frac{1}{2}}\exp\{\mathrm{i}(k_{1x}+k_{2x})(\varepsilon p_y+\varepsilon' p'_y)/2eB\}\exp\{\mathrm{i}(\boldsymbol{k}_1\times\boldsymbol{k}_2)_z/2eB\}, \tag{64}$$

with $\varepsilon p_y-\varepsilon' p'_y-k_{1y}-k_{2y} = 0$ from the p''_y integral. It follows that the sum over the intermediate states q'' reduces to one involving the Γ and the quantum numbers ε'', n'' and σ''.

An analogous result applies to cylindrical coordinates, cf. equation (52). The interpretation of the radial quantum number s is in terms of the radial distance of the centre of gyration from the origin. Sokolov and Ternov (1968, p. 73) showed that the mean value of r^2 is given by

$$\langle r^2\rangle = (2/eB)(n+s+\tfrac{1}{2}). \tag{65}$$

Classically, an electron gyrating about a centre a radial distance a from the origin has $\langle r^2\rangle = R^2+a^2$, where R is its radius of gyration, and the classical limit corresponds to $R = (2n/eB)^{\frac{1}{2}}$. Hence $(2s/eB)^{\frac{1}{2}}$ is related to the distance of the centre of gyration from the origin. When summing over an intermediate state q'' in this case we have (62) and, from (34b) and (52),

$$\frac{eB}{2\pi}\sum_{s''=0}^{\infty}\left(\frac{2\pi}{eB}\{-\mathrm{i}\exp(-\mathrm{i}\psi_1)\}^{s-s''}J^{s}_{s''-s}(k_{1\perp}^2/2eB)\right)$$

$$\times\left(\frac{2\pi}{eB}\{-\mathrm{i}\exp(-\mathrm{i}\psi_2)\}^{s''-s'}J^{s''}_{s'-s''}(k_{2\perp}^2/2eB)\right)$$

$$=\frac{2\pi}{eB}\{-\mathrm{i}\exp(-\mathrm{i}\Psi)\}^{s-s'}J^{s}_{s'-s}(K_\perp^2/2eB)\exp\{\mathrm{i}(\boldsymbol{k}_1\times\boldsymbol{k}_2)_z/2eB\}, \tag{66}$$

with $\boldsymbol{K}\equiv\boldsymbol{k}_1+\boldsymbol{k}_2 = (K_\perp\cos\Psi, K_\perp\sin\Psi, K_z)$. Comparing (64) and (66), we note that both are of the form of a product of a gauge dependent phase factor, and a factor involving a phase $(\boldsymbol{k}_1\times\boldsymbol{k}_2)_z/2eB$. An analogous form results for sums over any number of intermediate states. Note that the sum over an intermediate state q'' reduces to one involving the Γ and ε'', n'' and σ''.

6. Propagator in Coordinate Space

We digress somewhat from the systematic development of our theory to rederive a known result for the electron propagator in coordinate space. The method involves evaluation of the vacuum expectation value using the wavefunctions (45). This calculation illustrates some of the methods involved in the present approach.

The prescription for evaluating the electron propagator from the vacuum expectation value is as follows. First, the Dirac wavefunctions are second quantized by writing

$$\hat{\psi}(x) = \sum_q\{\hat{a}_q\psi_q^+(x)\exp(-\mathrm{i}\mathscr{E}_q t)+\hat{b}_q^\dagger\psi_q^-(x)\exp(\mathrm{i}\mathscr{E}_q t)\}, \tag{67a}$$

$$\hat{\bar{\psi}}(x) = \sum_q\{\hat{a}_q^\dagger\bar{\psi}_q^+(x)\exp(\mathrm{i}\mathscr{E}_q t)+\hat{b}_q\bar{\psi}_q^-(x)\exp(-\mathrm{i}\mathscr{E}_q t)\}, \tag{67b}$$

where we now add explicit labels with $\varepsilon = \pm 1$ for the wavefunctions. The creation and annihilation operators satisfy anticommutation relations

$$[\hat{a}_q,\hat{a}_{q'}^\dagger]_+ = [\hat{b}_q,\hat{b}_{q'}^\dagger]_+ = \delta_{qq'}, \tag{68}$$

with all other anticommutators vanishing. The propagator is then given by

$$G(x,x') = -\mathrm{i}\langle 0|T\{\hat{\psi}(x)\hat{\bar{\psi}}(x')\}|0\rangle, \tag{69}$$

where T denotes the chronological ordering operator. Thus, one has

$$G(x,x') = -\mathrm{i}\sum_q \left[\theta(t-t')\,\psi_q^+(\boldsymbol{x})\,\bar{\psi}_q^+(\boldsymbol{x}')\exp\{-\mathrm{i}\,\mathscr{E}_q(t-t')\}\right.$$
$$\left.-\theta(t'-t)\,\psi_q^-(\boldsymbol{x})\,\bar{\psi}_q^-(\boldsymbol{x}')\exp\{\mathrm{i}\,\mathscr{E}_q(t-t')\}\right], \tag{70}$$

where

$$\theta(t) = \int \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{i}\exp(-\mathrm{i}\,\omega t)}{\omega+\mathrm{i}0} = \begin{cases} 1, & t>0 \qquad (71a)\\ 0, & t<0, \qquad (71b)\end{cases}$$

denotes the step function.

When using cartesian coordinates we have

$$\sum_q = \sum_{\sigma=\pm 1}\sum_{n=0}^{\infty} V(eB)^{\frac{1}{2}}\int\frac{\mathrm{d}p_y}{2\pi}\frac{\mathrm{d}p_z}{2\pi}, \tag{72a}$$

and when using cylindrical coordinates we have

$$\sum_q = \sum_{\sigma=\pm 1}\sum_{n=0}^{\infty}\sum_{s=0}^{\infty} V\frac{eB}{2\pi}\int\frac{\mathrm{d}p_z}{2\pi}. \tag{72b}$$

For any choice of the spin operator the sum over σ leads to equivalent results. In cartesian and cylindrical coordinates we find

$$\sum_{\sigma=\pm 1}\psi_q^{\varepsilon}(\boldsymbol{x})\,\bar{\psi}_q^{\varepsilon}(\boldsymbol{x}')\exp\{-\mathrm{i}\,\varepsilon\mathscr{E}_q(t-t')\} = \varepsilon(\mathrm{i}\,\partial_\mu\gamma^\mu - eA_\mu\gamma^\mu + m)$$
$$\times\frac{1}{2\mathscr{E}_q}\begin{cases}\{\frac{1}{2}(1+\sigma_z)v_{n-1}(\xi)\,v_{n-1}(\xi')+\frac{1}{2}(1-\sigma_z)v_n(\xi)\,v_n(\xi')\}\\ \times\exp\{-\mathrm{i}\,\varepsilon\mathscr{E}_q(t-t')+\mathrm{i}\,\varepsilon p_y(y-y')+\mathrm{i}\,\varepsilon p_z(z-z')\}, \qquad (73a)\\ [\frac{1}{2}(1+\sigma_z)\mathrm{J}_{n-s-1}^s(\frac{1}{2}eBr^2)\,\mathrm{J}_{n-s-1}^s(\frac{1}{2}eBr'^2)\exp\{-\mathrm{i}(\phi-\phi')\}\\ \qquad +\frac{1}{2}(1-\sigma_z)\mathrm{J}_{n-s}^s(\frac{1}{2}eBr^2)\,\mathrm{J}_{n-s}^s(\frac{1}{2}eBr'^2)]\\ \times\exp\{-\mathrm{i}\,\varepsilon\mathscr{E}_q(t-t')+\mathrm{i}(n-s)(\phi-\phi')+\mathrm{i}\,\varepsilon p_z(z-z')\}. \qquad (73b)\end{cases}$$

The integral over p_y and the sum over s are performed next. It is convenient to introduce the gauge dependent phase factor

$$\phi(x,x') \equiv \exp\left(-\mathrm{i}\,e\int_x^{x'}\mathrm{d}x_\mu\,A^\mu(x)\right)$$
$$= \begin{cases}\exp\{-\frac{1}{2}\mathrm{i}\,eB(x+x')(y-y')\} & (74a)\\ \exp\{\frac{1}{2}\mathrm{i}\,eBrr'\sin(\phi-\phi')\}. & (74b)\end{cases}$$

One finds

$$(eB)^{\frac{1}{2}}\int\frac{\mathrm{d}p_y}{2\pi}\exp\{\mathrm{i}\,\varepsilon p_y(y-y')\}\,v_n(\xi)\,v_n(\xi') = \frac{eB}{2\pi}\phi(x,x')\exp(-\tfrac{1}{4}\lambda^2)\,\mathrm{L}_n(\tfrac{1}{2}\lambda^2), \tag{75a}$$

$$\frac{eB}{2\pi}\sum_{s=0}^{\infty}\mathrm{J}_{n-s}^s(\tfrac{1}{2}eBr^2)\,\mathrm{J}_{n-s}^s(\tfrac{1}{2}eBr'^2)\exp\{\mathrm{i}(n-s)(\phi-\phi')\}$$
$$= \frac{eB}{2\pi}\phi(x,x')\exp(-\tfrac{1}{4}\lambda^2)\,\mathrm{L}_n(\tfrac{1}{2}\lambda^2), \tag{75b}$$

with

$$\lambda^2 = eB\{(x-x')^2+(y-y')^2\}. \tag{76}$$

The remaining integral over p_z is of the form of an integral representation of a Hankel function and may be rewritten using an alternative integral representation (see e.g. Gradshteyn and Ryzhik 1965, 8·421):

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}p_z}{\mathscr{E}_q} \exp\{\mp \mathrm{i}\,\mathscr{E}_q(t-t') \mp \mathrm{i}\,p_z(z-z')\}$$

$$= \int_0^{\infty} \frac{\mathrm{d}\mu}{\mu} \exp\{\mp \tfrac{1}{2}\mathrm{i}(t/|\,t\,|)(\rho^2\mu + \mathscr{E}_q^{02}/\mu)\}, \tag{77}$$

with

$$\rho^2 \equiv (t-t')^2-(z-z')^2. \tag{78}$$

The right-hand side involves n only through $(\mathscr{E}_q^0)^2 = m^2+2neB$ and the sum over n is then of the form

$$\sum_{n=0}^{\infty} \exp(\mathrm{i}\,\alpha n)\,\mathrm{L}_n(x) = \frac{\mathrm{i}\exp(-\frac{1}{2}\mathrm{i}\,\alpha)}{2\sin\frac{1}{2}\alpha} \exp(\tfrac{1}{2}x - \tfrac{1}{2}\mathrm{i}\,x\cot\tfrac{1}{2}\alpha), \tag{79}$$

which follows directly from a generating function for the Laguerre polynomials.

Finally, using

$$(\mathrm{i}\,\partial_\mu\gamma^\mu - eA_\mu\gamma^\mu + m)\phi(x,x') = \phi(x,x')(\mathrm{i}\,\partial_\mu\gamma^\mu + eb_\mu\gamma^\mu + m), \tag{80}$$

with

$$b^\mu \equiv (0, \tfrac{1}{2}\boldsymbol{B}\times\boldsymbol{x}), \tag{81}$$

one finds

$$G(x,x') = -\phi(x,x')(\mathrm{i}\,\partial_\mu\gamma^\mu + eb_\mu\gamma^\mu + m)\int_0^{\infty} \frac{\mathrm{d}\lambda}{8\pi^2} \frac{1-\mathrm{i}\,\sigma_z\tan(eB/2\lambda)}{(2\lambda/eB)\tan(eB/2\lambda)}$$

$$\times \exp\left(\frac{\mathrm{i}\,eB\{(x-x')^2+(y-y')^2\}}{4\tan(eB/2\lambda)} + \frac{\mathrm{i}\lambda}{2}\{(z-z')^2-(t-t')^2\} - \frac{\mathrm{i}m^2}{2\lambda}\right), \tag{82}$$

which is a well-known result (Géhéniau 1950; Géhéniau and Demeur 1951; Schwinger 1951; Källén 1958).

The choice of gauge appears only in the phase factor $\phi(x,x')$ in (82), and the remaining terms depend only on $x^\mu - x'^\mu$. Consequently, $G(x,x')/\phi(x,x')$ has a momentum space representation. However, in practice it is much more convenient to develop the theory in momentum space in terms of the vertex functions introduced in Section 4. Then the propagator between two vertices, for example a term $\gamma^\mu G(x,x')\gamma^\nu$, is evaluated using (70) with the sum over q'' rather than q; this is combined with an $\boldsymbol{x}$ dependent wavefunction $\bar{\psi}_q^\varepsilon(\boldsymbol{x})$ to the left of γ^μ and with an $\boldsymbol{x}'$ dependent wavefunction $\psi_{q'}^{\varepsilon'}(\boldsymbol{x}')$ to the right of γ^μ to form two vertex functions. The gauge dependent phase factors appear only in the factor discussed in Section 5*b*.

7. Discussion

The main result in this paper is the explicit form for the vertex function $[\Gamma_{q'q}^{\varepsilon'\varepsilon}(\boldsymbol{k})]^\mu$ in terms of spin eigenfunctions which have simple symmetry properties between electron ($\varepsilon = 1$) and positron ($\varepsilon = -1$) states. In contrast the form for the vertex

function used in earlier work (see e.g. Melrose 1974; Melrose and Stoneham 1977) exhibits no obvious symmetry between electron and positron states due to the lack of an obvious symmetry in the underlying wavefunctions. By way of illustration, Melrose (1974) wrote down a probability for gyromagnetic emission which is proportional to $|\boldsymbol{e}_{\mathrm{M}}(\boldsymbol{k}).\boldsymbol{\Gamma}^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})|^2$, where $\boldsymbol{e}_{\mathrm{M}}(\boldsymbol{k})$ is the polarization vector for waves in a mode M and $\boldsymbol{\Gamma}$ is the 3-vector part of Γ^{μ}. This probability has the unsatisfactory feature, when the asymmetric wavefunctions are used, that the procedure which gives the nonrelativistic formula for gyromagnetic emission by electrons (Melrose and Zheleznyakov 1981) does not give sensible results for positrons. With the use of the vertex function derived here, the symmetry property (61) ensures that the crossing symmetry relation between gyromagnetic emission by electrons and positrons has a simple form, as is shown in Part III in this series (Melrose and Parle 1983).

It should be emphasized that the use of the vertex function based on the Johnson and Lippmann (1949) wavefunctions does not lead to any intrinsic errors, but merely leads to difficulties in interpretation and tends to obscure some crossing symmetries. Any choice of wavefunctions must lead to equivalent results when one sums or averages over all spin states. Specifically, for example, the quantity

$$\sum_{\sigma'\sigma}[\Gamma^{\varepsilon'\varepsilon}_{q'\varepsilon}(\boldsymbol{k})]^{\mu}[\Gamma^{\varepsilon'\varepsilon}_{q'\varepsilon}(\boldsymbol{k})]^{*\nu}$$

has the same form irrespective of the choice of spin eigenfunctions. This fact may be used as a check on specific forms for the vertex function.

The vertex function in the form (57) is used extensively in Part III.

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Appendix. Properties of $J_\nu^n(x)$

The functions $J_\nu^n(x)$ are defined by (29), namely

$$J_\nu^n(x) = \{n!/(n+\nu)!\}^{\frac{1}{2}} \exp(-\tfrac{1}{2}x)\, x^{\frac{1}{2}\nu}\, L_n^\nu(x) \tag{A1}$$

$$= (-)^\nu J_{-\nu}^{n+\nu}(x), \tag{A2}$$

where the generalized Laguerre polynomials are as defined by Abramowitz and Stegun (1965, p. 775) and Gradshteyn and Ryzhik (1965, 8·970). The function $I_{n,n'}(x)$ defined by Sokolov and Ternov (1968) is related to $J_n^n(x)$ by

$$I_{n,n'}(x) = J_{n-n'}^{n'}(x). \tag{A3}$$

We summarize some elementary properties of the $J_\nu^n(x)$:

Recursion Formulae

$$J_\nu^{n+1}(x) = \left(\frac{n+\nu+1}{n+1}\right)^{\frac{1}{2}} J_\nu^n(x) - \left(\frac{x}{n+1}\right)^{\frac{1}{2}} J_{\nu+1}^n(x) \tag{A4a}$$

$$= \frac{-x+n+1}{(n+1)^{\frac{1}{2}}(n+\nu+1)^{\frac{1}{2}}} J_\nu^n(x) + \left(\frac{x(n+\nu)}{(n+1)(n+\nu+1)}\right)^{\frac{1}{2}} J_{\nu-1}^n(x), \tag{A4b}$$

$$J_\nu^{n-1}(x) = \left(\frac{n+\nu}{n}\right)^{\frac{1}{2}} J_\nu^n(x) - \left(\frac{x}{n}\right)^{\frac{1}{2}} J_{\nu-1}^n(x) \tag{A5a}$$

$$= \frac{-x+n}{n^{\frac{1}{2}}(n+\nu)^{\frac{1}{2}}} J_\nu^n(x) + \left(\frac{x(n+\nu+1)}{n(n+\nu)}\right)^{\frac{1}{2}} J_{\nu+1}^n(x), \tag{A5b}$$

$$(x+\nu)J_\nu^n(x) = \{x(n+\nu)\}^{\frac{1}{2}} J_{\nu-1}^n(x) + \{x(n+\nu+1)\}^{\frac{1}{2}} J_{\nu+1}^n(x), \tag{A6}$$

$$2x(\mathrm{d}/\mathrm{d}x)J_\nu^n(x) = \{x(n+\nu)\}^{\frac{1}{2}} J_{\nu-1}^n(x) - \{x(n+\nu+1)\}^{\frac{1}{2}} J_{\nu+1}^n(x). \tag{A7}$$

Sum Rules (Sokolov and Ternov 1968)

$$\sum_{s'=0}^{\infty} J_{s'-s}^{s}(x)\, J_{s'-s''}^{s''}(x) = \delta_{ss''}, \tag{A8}$$

$$\sum_{s'=0}^{\infty} (s'-s)\{J_{s-s'}^{s'}(x)\}^2 = x. \tag{A9}$$

Integral Identities (Sokolov and Ternov 1968)

$$\int_0^\infty \mathrm{d}x\, J_\nu^n(x)\, J_\nu^{n'}(x) = \delta_{nn'}, \tag{A10}$$

$$\int_0^\infty dx\, x^{\frac{1}{2}}\{J_\nu^n(x)\}^2 = (n+\nu+1)^{\frac{1}{2}}\left(1+\frac{n+\frac{1}{2}}{4(n+\nu+1)}\right), \tag{A11}$$

$$\int_0^\infty dx\, x\{J_\nu^n(x)\}^2 = 2n+\nu+\tfrac{3}{2}. \tag{A12}$$

Closure Relation (for the Laguerre polynomials)

$$\sum_{n=0}^{\infty} J_0^n(x)\,J_0^n(x') = \delta(x-x'). \tag{A13}$$

Particular Values

$$J_\nu^0(x) = (-)^\nu J_{-\nu}^\nu(x) = x^{\frac{1}{2}\nu}\exp(-\tfrac{1}{2}x)/(\nu!)^{\frac{1}{2}}, \tag{A14}$$

$$J_\nu^1(x) = (-)^\nu J_{-\nu}^{\nu+1}(x) = \frac{x^{\frac{1}{2}\nu}\exp(-\frac{1}{2}x)}{\{(\nu+1)!\}^{\frac{1}{2}}}(\nu+1-x), \tag{A15}$$

$$J_\nu^2(x) = (-)^\nu J_\nu^{\nu+2}(x) = \frac{x^{\frac{1}{2}\nu}\exp(-\frac{1}{2}x)}{2^{\frac{1}{2}}\{(\nu+2)!\}^{\frac{1}{2}}}\{(\nu+1)(\nu+2)-2(\nu+2)x+x^2\}, \tag{A16}$$

$$J_\nu^3(x) = (-)^\nu J_{-\nu}^{\nu+3}(x) = \frac{x^{\frac{1}{2}\nu}\exp(-\frac{1}{2}x)}{(3!)^{\frac{1}{2}}\{(\nu+3)!\}^{\frac{1}{2}}}\{(\nu+1)(\nu+2)(\nu+3)$$
$$-3(\nu+2)(\nu+3)x+3(\nu+3)x^2-x^3\}. \tag{A17}$$

Limiting Values

(i) Small x ($x \ll 1$)

$$J_\nu^n(x) = \{(n+\nu)!/n!\}^{\frac{1}{2}}x^{\frac{1}{2}\nu}/\nu!, \qquad \nu > 0; \tag{A18a}$$

$$= (-)^\nu\{n!/(n-|\nu|)!\}^{\frac{1}{2}}x^{\frac{1}{2}|\nu|}/|\nu|!, \quad \nu < 0. \tag{A18b}$$

(ii) Classical limit $n \to \infty$

$$J_\nu^n(z^2/4n) = \left(\frac{(n+\nu)!}{n!\,n^\nu}\right)^{\frac{1}{2}}\sum_{r=0}^{\infty} C_r(z/2n)^r J_{\nu+r}(z); \tag{A19}$$

$$C_0 = 1, \quad C_1 = -\tfrac{1}{2}(\nu+1), \quad C_2 = \tfrac{1}{8}(\nu+1)(\nu+2);$$

$$(r+1)C_{r+1} = -\tfrac{1}{2}(\nu+1)C_r+\tfrac{1}{4}(\nu+r)C_{r-1}-\tfrac{1}{2}nC_{r-2}.$$

Airy Integral Approximation (Sokolov and Ternov 1968, p. 87)

(i) $0 < x < x_0 = (\sqrt{n}-\sqrt{n'})^2$:

$$J_{n-n'}^{n'}(x) = (1/\pi\sqrt{3})(1-x/x_0)^{\frac{1}{2}}K_{1/3}\left(\tfrac{2}{3}(nn'x_0^2)^{1/4}(1-x/x_0)^{3/2}\right); \tag{A20}$$

(ii) $x > x_0' \equiv (\sqrt{n}+\sqrt{n'})^2$

$$J_{n-n'}^{n'}(x) = (1/\pi\sqrt{3})(x/x_0'-1)^{\frac{1}{2}}K_{1/3}\left(\tfrac{2}{3}(nn'x_0'^2)^{1/4}(x/x_0'-1)^{3/2}\right). \tag{A21}$$

Addition Theorem

$$\{-\mathrm{i}\exp(-\mathrm{i}\Phi)\}^{n''-n}\mathrm{J}^{n}_{n''-n}(\tfrac{1}{2}R^2) = \sum_{n'=0}^{\infty}\exp\{-\tfrac{1}{2}\mathrm{i}\,rr'\sin(\phi-\phi')\}$$

$$\times\{-\mathrm{i}\exp(-\mathrm{i}\phi)\}^{n'-n}\{-\mathrm{i}\exp(-\mathrm{i}\phi')\}^{n''-n'}\mathrm{J}^{n}_{n'-n}(\tfrac{1}{2}r^2)\,\mathrm{J}^{n'}_{n''-n'}(\tfrac{1}{2}r'^2), \quad \text{(A22)}$$

for

$$R\cos\Phi = r\cos\phi + r'\cos\phi', \qquad R\sin\Phi = r\sin\phi + r'\sin\phi'. \quad \text{(A23)}$$

[To our knowledge the result (A22) has not been written down previously; we have derived it by evaluating the integral

$$\sum_{n'=0}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\xi\int_{-\infty}^{\infty}\mathrm{d}\xi'\,\exp(-\mathrm{i}x\xi)\exp(-\mathrm{i}x'\xi')\,v_n(\xi+\xi_0)\,v_{n'}(\xi+\xi_0-y)$$

$$\times\, v_{n'}(\xi'+\xi'_0)\,v_{n''}(\xi'+\xi'_0-y''),$$

firstly directly using (47) to evaluate the two integrals separately, and secondly using (21) and then (47).]

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