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Effective Weak Non-leptonic Hamiltonians for Flavour Changing Processes

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Abstract

We present a complete analysis of the W mediated effective weak non-leptonic theory for s, c, b and t flavour changing processes. Calculations are based upon the standard model with n generations. The theory is developed using a matrix formalism which leads to compact theoretical results and a simpler numerical analysis. In our analysis we take account of scheme and effective theory dependence of the QCD parameters. Explicit numerical calculations are performed for effective weak non-leptonic Hamiltonians describing s, c, b and t decay, in both the penguin free and penguin generating sectors of the theory, for the modified minimal subtraction ($\overline{\text{ms}}$) scheme and $\alpha = 0$ gauge. Our approach is to display the calculations as a detailed application of the Appelquist–Carazzone decoupling theorem incorporating some of the latest theoretical developments in this area.

1. Introduction

The principal aim of this paper is to develop and extend the approach and techniques introduced by Gilman and Wise (1979) and Wise (1980) for systematically deriving effective weak non-leptonic Hamiltonians.

The purpose of this paper is twofold. Firstly, we will employ a revised estimate of $\Lambda_{5(\overline{\text{ms}})} = 0{\cdot}250$ GeV (five flavours) as the basic numerical input. This smaller value of Λ is motivated by growing evidence (see MacKenzie and Lepage 1981 or Ali 1981 for two recent examples) that the previously accepted value $\Lambda_{5(\overline{\text{ms}})} \approx 0{\cdot}4$ GeV (Roberts 1981) is a little too large. It is important to realize that here we are talking about the standard definition of Λ (Bardeen *et al.* 1978). The summary of our numerical work presented in a recent Letter (Miller and McKellar 1982) actually dealt with a non-standard definition of Λ (a fact we were unaware of at the time). We point out here that the tables presented in that Letter remain valid but *we must interpret those results as corresponding to modified minimal subtraction* $\overline{\text{ms}}$ (*and not minimal subtraction* ms) *with a standard definition of* Λ, *numerically given by* $\Lambda_{5(\overline{\text{ms}})} = 0{\cdot}350$ GeV. This Letter and the present paper thus present the numerical work for two different (but both acceptable) values of $\Lambda_{5(\overline{\text{ms}})}$; hence an indication of the sensitivity of the work on Λ is displayed.

Secondly, our (main) purpose is to detail calculational methods, paying particular attention to many subtleties which arise and that have not been fully dealt with by other authors.

It has long been realized that strong interaction effects on weak non-leptonic processes are very significant. With the general acceptance of QCD as the theory

of strong interactions it has become possible in principle (limited by one's endurance) to calculate strong interaction effects to any order.

The early pioneering works of Gaillard and Lee (1974) and Altarelli and Maiani (1974) showed that strong effects naturally lead to an enhanced $\Delta I = \frac{1}{2}$ amplitude in strange decays (but insufficient to explain the data completely). Vainshtein *et al.* (1975, 1977) pointed out how QCD could alter the L–L chiral structure of the unrenormalized four quark operator, generating L–R penguin operators. They illustrated how penguin operators offered a means of fully explaining the $\Delta I = \frac{1}{2}$ rule. The debate on the size of penguin effects continues (for example see Hill and Ross 1980; Guberina and Peccei 1980; Miller and McKellar 1981*d*).

That QCD produces enhancements and generates operators with unusual chiral structures is now a fact of life; QCD effects must be incorporated if we can ever hope to achieve satisfactory agreement with experiment.

Clearly, a better understanding will only be achieved by looking at a broader range of processes. The aim of this paper is to present, within a uniform treatment, effective Hamiltonians describing s, c, b and t decay. These eliminate the need of phenomenologists to calculate hard gluon effects. Soft gluon effects must still in principle be investigated.

In this Introduction we would like to give the reader a broader view of what is involved in the calculations.

The standard model of $SU(3)_c \times SU(2)_L \times U(1)_Y$ we shall call the (3, 2, 1) model; with QCD switched off it becomes the (2, 1) model. The calculations we present here are a detailed application of the Appelquist–Carazzone (1975) decoupling theorem applied to the (3, 2, 1) model with our attention focussed on the behaviour of the W boson–quark interaction.

Recall the decoupling theorem. It assures us that, given a renormalizable field theory described by a Lagrangian containing at least one heavy field, we may decouple the heavy field to obtain an effective Lagrangian correctly describing all light field processes. The effects of the heavy particle are contained in finite (decoupling) renormalizations of all the light field parameters and fields, and in the generation of (individually) non-renormalizable operators inversely proportional to powers of the square of the heavy field mass.

When the W is decoupled one remnant of the W boson–quark interactions is the non-leptonic four quark (non-renormalizable) interaction, suppressed by an inverse power of m_W^2. The essential method for decoupling W in the (3, 2, 1) model (using a Wilson (1969) short distance expansion) was developed by both Gaillard and Lee (1974) and Altarelli and Maiani (1974). The method was extended by Gilman and Wise (1979) and Wise (1980) who displayed in addition decoupling of heavy quarks lighter than W. We refer to this extended method as the GLAM–GW method of decoupling. We formulate it as a set of 'rules' to be applied to the weak non-leptonic sector of the (3, 2, 1) model.

Procedural Rules of GLAM–GW Method

(1) Only bother to decouple *explicitly* those heavy fields which appear *explicitly*. (Thus for example we shall decouple all quarks heavier than the W, the W boson itself, and when appropriate t, b and c quarks. We shall not explicitly decouple the Higgs, the Z or the heavy leptons which should be done in a full calculation.)

(2) (a) Decouple quarks and the W in the (2, 1) non-leptonic sector ('easily' accomplished, according to rule 3)—this is referred to as establishing 'free field' results.

(b) Establish equivalent results in the (3, 2, 1) theory using (i) the renormalization group (RG) equation of QCD; (ii) asymptotic freedom of QCD; and (iii) the leading log approximation (LLA) (we shall always be calculating at a one loop level).

(3) As long as the decoupled result in the (2, 1) theory is not to be used beyond the tree level then μ dependence in the pure (2, 1) theory parameters, such as m_W and the gauge coupling g_W, can be ignored. Consequently:

(a) In the non-leptonic sector of the (3, 2, 1) theory we encounter only the RG of QCD.

(b) We need finite decoupling renormalization constants only to zero loop order in the (2, 1) model; these are all equal to one (hence rule 2a).

(c) We may use the 'operator insertion' technique to calculate the anomalous dimensions of operators.

We apply this method to consider all aspects of W mediated weak non-leptonic flavour changing processes of s, c, b and t. Our work goes beyond the similar works of Gilman and Wise (1979), Wise (1980) and Guberina and Peccei (1980) for $\Delta s = 1$ processes, Hayashi *et al.* (1980) for $\Delta c = 1$ processes, Ponce (1981) for $\Delta b = 1$ processes, and in certain respects, the two loop work of Altarelli *et al.* (1981*a*, 1981*b*). (The reader is cautioned that these works contain different methods, notation and numerical inputs; comparing their results is therefore quite a task.)

We have developed a matrix formulation for the calculations enabling us to express all final theoretical effective Hamiltonians in a compact and general form. This is particularly convenient for the numerical computations. To ensure consistency in our approach we must include scheme dependence in our calculations. This implies use of a second order parametrization of the running coupling constant and a first order parametrization of running quark mass. Since we work in the $\overline{\text{ms}}$ scheme we calculate all RG invariants appropriately.

In earlier papers, effective theory dependence was only partially taken into account. The effective theory dependence of RG invariants has generally been ignored. Recent theoretical developments in decoupling (see Ovrut and Schnitzer 1981*a*, 1981*b*; Kazama and Yao 1981; Weisberger 1981 and references therein) have paved the way for the calculation of such effects. (See for example Miller and McKellar (1981*a*, 1981*b*) and observe that the methods of the (1981*b*) reference are *reapplied here* to calculate RG invariants for effective QCD in the $\overline{\text{ms}}$ scheme, based upon our revised estimate of the standard definition of $\Lambda^2_{5(\overline{\text{ms}})}$ (Bardeen *et al.* 1978).) We incorporate these new results in this paper.

Finally we mention that we have gone to particular trouble to ensure that running parameters are all meaningfully defined at their relevant mass scales.

We point out that although a path integral formulation of decoupling (Ovrut and Schnitzer 1981*a*) makes it clear that the *order* in which heavy particles are decoupled is not important, one finds that with the perturbative GLAM–GW method the results do depend on the order of decoupling. An explicit example is given in Appendix 2. Our approach is to decouple heavy fields sequentially in a mass ordered fashion. We start with the heaviest, the W boson being decoupled *following* the decoupling of all generations $n, \ldots, 4$ (we assume b′ is more massive than the W).

The content of the paper is as followed. In Section 2 the basic problem is set up. Quarks heavier than the W are decoupled. This is done by two methods, showing that the GLAM–GW method gives the expected answer for a known case. Then the W is decoupled in the (2, 1) theory. A mathematical and physical decomposition is performed at this stage before calculations can proceed. In Section 3 decoupling in the (3, 2, 1) model of the W and t, b and c (when appropriate) is performed. Final theoretical results are then obtained. In Section 4 theoretical results are reduced to their simplest numerical form appropriate for phenomenological application. A few comparisons with other works are made. All the work that follows employs $\overline{\text{ms}}$ and the Landau ($\alpha = 0$) gauge. We shall show in Appendix 3 that operator coefficients and physical results are the *same* for both ms and $\overline{\text{ms}}$ schemes when sensible (*and consistent*) subtraction point choices are made for either scheme.

2. Setting up the Problem

The basic model is the standard (3, 2, 1). As mentioned in the Introduction, we assume the first three quark generations lie below the W in mass and all higher generations lie above.

We shall explicitly decouple heavy quarks and the W sequentially according to the order of the masses, for the reasons discussed above. (In Appendix 2 we explicitly show that the GLAM–GW method gives results which depend on the order of decoupling the W and the heavy quarks.) We write the W–quark current interaction term as

$$\mathscr{H}_{\mathrm{I}}(x) = \sqrt{\tfrac{1}{2}}\, g_{\mathrm{W}}\, J_{\mu}^{+}(x)\, W^{-\mu}(x) + \mathrm{h.c.}\,, \tag{1}$$

in terms of the electro-weak gauge coupling constant g_{W} and the current

$$J_{\mu}^{+} = \bar{\boldsymbol{q}}_{-}^{\prime i}\, \gamma_{\mu}\, a \boldsymbol{q}_{+}^{i} \equiv (\bar{q}_{-}^{\prime i}\, q_{+}^{i})_{\mathrm{L}}\,. \tag{2}$$

Here $a(a') = \frac{1}{2}(1 \mp \gamma_5)$ and i is a colour index with implied summation over repeated indices. The quark fields $\boldsymbol{q}_{\pm}$ are vectors in flavour space defined by

$$\boldsymbol{q}_{-}^{\mathrm{T}} = (\mathrm{d\,s\,b} \ldots q_{-n})\,, \qquad \boldsymbol{q}_{+}^{\mathrm{T}} = (\mathrm{u\,c\,t} \ldots q_{+n}) \tag{3a, b}$$

(where the superscript T indicates the transpose) and the 'Cabibbo rotated' vector is

$$\boldsymbol{q}_{-}^{\prime i} = V \boldsymbol{q}_{-}^{i}\,, \tag{4}$$

where V is a unitary $n \times n$ mixing matrix. The explicit parametrization is unimportant for this paper. For the six quark model V is the Kobayashi–Maskawa (1973) matrix and for an eight quark model a parametrization of V may be found in Bose and Paschos (1980).

(a) *Decoupling Heavy Quarks with Mass* $> m_{\mathrm{W}}$

After the decoupling of these heavy quarks, the W–quark interactions of interest still belong to the renormalizable part of the resulting effective Lagrangian. As a result, from a structural point of view, this stage of decoupling is trivial. However, in this section we perform the decoupling by the GLAM–GW method. This will introduce the techniques involved in a case in which the final results are already known.

We define a set of operators (we avoid attaching effective theory indices to fields as they are understood to be present)

$$ {}_{2n}\mathcal{O}^{\theta\phi} = (\bar{q}^i_{-\theta}\, q^i_{+\phi})_L\, W^-, \tag{5} $$

where θ, ϕ are vector indices of vectors $\boldsymbol{q}_\pm$ in (3) and $2n$ is the number of flavours in the complete theory. Then from (1)–(4) we may write

$$ \mathcal{H}_I = \sqrt{\tfrac{1}{2}}\, g_W \sum_{\substack{\theta,\phi \\ =1}}^{n} V^+_{\theta\phi}\, {}_{2n}\mathcal{O}^{\theta\phi} + \text{h.c.} \tag{6} $$

Since we are concerned with the mass ordering of the quarks it is useful to represent them as a $2n$ dimensional flavour space vector

$$ \boldsymbol{q}^T = (q_1 \dots q_{2n}) \equiv (q_{-1}\, q_{+1} \cdots q_{-n}\, q_{+n}), \tag{7} $$

where in writing the second of equations (7) we have assumed that the usual mass pattern (for generations $n = 2$ and 3) persists in higher generations. At this level of decoupling we will not need the LLA, and hence we may decouple heavy quarks $q_{2n}, \dots, q_7$ simultaneously.

The free field result (i.e. decoupling in the (2, 1) model) is

$$ {}_{2n}\mathcal{O}^{\theta\phi} = B_f^{\theta\phi}\, {}_6\mathcal{O}^{\theta\phi} + O(1/m_{2n}^{(7)2}). \tag{8} $$

Here $m_k^{(l)}$ is the mass of the lth flavour in an effective theory with the k fermion flavours. The effect of decoupling has been to renormalize the operators finitely and to generate non-renormalizable terms, the most significant of which is proportional to $(m_{2n}^{(7)})^{-2}$. The free field coefficients are given by

$$ B_f^{\theta\phi} = \left(1 - \sum_{i=4}^{n} \delta_{\theta i}\right)\left(1 - \sum_{i=4}^{n} \delta_{\phi i}\right), \tag{9} $$

being easy to write down since, as mentioned in the Introduction, in the GLAM–GW method we decouple at a zero loop level in the (2, 1) model. The effective theory indexing is still important though, for we now derive the analogous results for decoupling in the (3, 2, 1) model in which the finite renormalizations due to QCD are significant.

Including QCD interactions, we have for the analogue of (8) (decoupling in the (3, 2, 1) model)

$$ {}_{2n}\mathcal{O}^{\theta\phi} = B^{\theta\phi}\, {}_6\mathcal{O}^{\theta\phi} + O(1/m_{2n}^{(7)2}), \tag{10} $$

where the coefficients $B^{\theta\phi}$ may be related to the free field coefficients $B_f^{\theta\phi}$. The anomalous dimension of the operators ${}_k\mathcal{O}^{\theta\phi}$ may be calculated by insertion into the amputated Green function $G^{(W,2\psi)}$ (one external W, two external quarks). This is illustrated in Fig. 1. Since gluons can never couple to the external W line (we need at least *two* W–quark vertices to achieve this), one sees that the anomalous dimension of ${}_k\mathcal{O}^{\theta\phi}$ (from equation 5) is just that of the left-handed current γ_J. Because of the colour singlet nature of the current one finds by explicit calculation that γ_J vanishes at the one loop level in $\overline{\text{ms}}$ and all gauges, i.e.

$$ \gamma_J = 0 + O(g_k^4). \tag{11} $$

Gross (1975) argues that anomalous dimensions of conserved and partially conserved currents vanish identically. The anomalous dimension of a current in $\overline{\text{ms}}$ is flavour independent, hence Gross's argument applies and γ_J vanishes to all orders. Note that γ_J in other renormalization schemes (in a massive quark model) need not vanish (Altarelli *et al.* 1981*a*). Now applying $\mu(\mathrm{d}/\mathrm{d}\mu)$ to both sides of (10) shows that, since

$$\mu(\mathrm{d}/\mathrm{d}\mu)\,{}_k\mathcal{O}^{\theta\phi} = -\gamma_J\,{}_k\mathcal{O}^{\theta\phi}, \qquad \text{then} \quad \mu(\mathrm{d}/\mathrm{d}\mu)B^{\theta\phi} = 0. \tag{12}$$

Thus the $B^{\theta\phi}$ are subtraction point independent, from which we may conclude that

$$B^{\theta\phi} = B_{\mathrm{f}}^{\theta\phi}. \tag{13}$$

Substituting (13) into (10), and the result into (6), we obtain

$$\mathscr{H}_{\mathrm{Ieff}} = \sqrt{\tfrac{1}{2}}\, g_{\mathrm{W}} \sum_{\theta,\phi=1}^{3} V_{\theta\phi}^{+}\,{}_6\mathcal{O}^{\theta\phi} + \mathrm{h.c.} \tag{14}$$

The simplicity of this result is a consequence of the vanishing anomalous dimension γ_J.

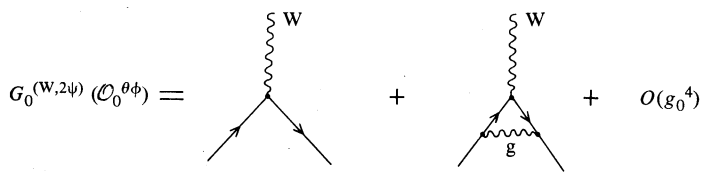

Fig. 1. Operators ${}_{2n}\mathcal{O}^{\theta\phi}$ (see equation 5) inserted into $G^{(\mathrm{W},2\psi)}$ to determine their anomalous dimension.

That equation (6) implies (14) is just a statement of the Appelquist–Carazzone (1975) decoupling theorem for this particular case, so we may regard our analysis as a proof of the decoupling theorem in this case, or as a verification of the GLAM–GW method.

Having decoupled the quarks heavier than the W, we may now write the W–quark interaction as

$$\mathscr{H}_{\mathrm{Ieff}} = \sqrt{\tfrac{1}{2}}\, g_{\mathrm{W}}\,{}_6J_{\mu}^{+}\, W^{-\mu} + \mathrm{h.c.}, \tag{15}$$

where

$${}_6J_{\mu}^{+} = \sum_{\theta,\phi=1}^{3} V_{\theta\phi}^{+}(\bar{q}^{i}_{-\theta}\, q^{i}_{+\phi})_{\mathrm{L}}. \tag{16}$$

In what follows non-vanishing anomalous dimensions and the use of the LLA force us to decouple the remaining heavy fields sequentially, beginning with W.

When the W is decoupled the leading remnant of the W–quark interaction is a non-renormalizable four quark interaction proportional to m_{W}^{-2}. In the GLAM–GW method the resulting effective Hamiltonian is analysed using the Wilson (1969) short distance expansion:

$$\begin{aligned}
\mathscr{H}_{\mathrm{eff}}(x) &= -\tfrac{1}{2}\mathrm{i}\int T\{\mathscr{H}_{\mathrm{Ieff}}(x), \mathscr{H}_{\mathrm{Ieff}}(x')\}\,\mathrm{d}x' \\
&= \tfrac{1}{4}g_{\mathrm{W}}^2 \int D_{\mathrm{W}}^{\mu\nu}(x,x')[T\{{}_6J_{\mu}^{+}(x)\,{}_6J_{\nu}^{-}(x')\} + T\{{}_6J_{\mu}^{-}(x)\,{}_6J_{\nu}^{+}(x')\}]\,\mathrm{d}x' \\
&= \sqrt{\tfrac{1}{2}}\,4G\,{}_6J_{\mu}^{+}(x)\,{}_6J^{-\mu}(x) + \text{higher dimensional terms},
\end{aligned} \tag{17}$$

where

$$\sqrt{\tfrac{1}{2}}\,G = g_W^2/8m_W^2 \tag{18}$$

is the Fermi coupling constant. Equation (17) is the free field result (zero loop decoupling in the (2, 1) model).

Using (15) and performing a Fierz transformation we obtain explicitly

$$\mathscr{H}_{\text{eff}} = \sqrt{\tfrac{1}{2}}\,4G \sum_{\substack{\psi,\chi \\ \theta,\phi=1}}^{3} V^{+}_{\psi\phi} V_{\theta\chi} (\bar{q}^{i}_{+\theta} q^{i}_{+\phi})_{\text{L}} (\bar{q}^{i}_{-\psi} q^{i}_{-\chi})_{\text{L}} . \tag{19}$$

Before we obtain the one loop analogue in the (3, 2, 1) model we must digress to discuss the flavour structure of the four quark operators in equation (19).

(*b*) *Decomposing* $\mathscr{H}_{\text{eff}}$ *with respect to Flavour Structure*

The anomalous dimension of a four quark operator depends upon its flavour space structure. For this reason we decompose (19) into three pieces, each piece containing four quark operators of the same anomalous dimension.

The summation in (19) may be written as

$$\sum_{\substack{\psi,\chi \\ \theta,\phi}} = \sum_{\substack{\theta,\phi \\ \phi\neq\theta}} \sum_{\substack{\psi,\chi \\ \psi\neq\chi}} + \sum_{\substack{\theta,\phi \\ \theta\neq\phi}} \sum_{\substack{\psi \\ (\chi=\psi)}} + \sum_{\substack{\theta \\ (\phi=\theta)}} \sum_{\substack{\psi,\chi \\ \psi\neq\chi}} + \sum_{\substack{\theta \\ (\phi=\theta)}} \sum_{\substack{\psi \\ (\chi=\psi)}} . \tag{20}$$

In the first term all flavour indices are distinct. This defines the penguin free (PF) flavour changing Hamiltonian

$$\mathscr{H}^{\Delta}_{\text{eff}}(\text{PF}) = \sqrt{\tfrac{1}{2}}\,4G \sum_{\substack{\theta,\phi=1 \\ \theta\neq\phi}}^{3} \sum_{\substack{\psi,\chi=1 \\ \psi\neq\chi}}^{3} V^{+}_{\psi\phi} V_{\theta\chi} (\bar{q}^{i}_{+\theta} q^{i}_{+\phi})_{\text{L}} (\bar{q}^{i}_{-\psi} q^{i}_{-\chi})_{\text{L}} . \tag{21}$$

The second term in (20) corresponds to a penguin generating (PG) q_+ flavour changing Hamiltonian. It may be further decomposed as

$$\sum_{\substack{\theta,\phi \\ \theta\neq\phi \\ =1}}^{3} \sum_{\substack{\psi=1 \\ (\chi=\psi)}}^{3} = \left(\sum_{\theta=2}^{3} \sum_{\phi=1}^{\theta-1} + \sum_{\phi=2}^{3} \sum_{\theta=1}^{\phi-1} \right) \sum_{\substack{\psi=1 \\ (\chi=\psi)}}^{3} . \tag{22}$$

The first term on the RHS of (22) describes weak non-leptonic decays of $\bar{q}_{+2}$ and $\bar{q}_{+3}$, i.e. $\bar{\text{c}}$ and $\bar{\text{t}}$ quarks. The second term is just the h.c. of the first. Hence the second term of (20) may be expressed as

$$\mathscr{H}^{\Delta}_{\text{eff}}(+, \text{PG}) = \sum_{\phi=2}^{3} \mathscr{H}^{\Delta}_{\text{eff}}(+\phi, \text{PG}) + \text{h.c.}, \tag{23}$$

where

$$\mathscr{H}^{\Delta}_{\text{eff}}(+\phi, \text{PG}) = \sqrt{\tfrac{1}{2}}\,4G \sum_{\theta=1}^{\phi-1} \sum_{\psi=1}^{3} V^{+}_{\psi\phi} V_{\theta\psi} (\bar{q}^{i}_{+\theta} q^{j}_{+\phi})_{\text{L}} (\bar{q}^{j}_{-\psi} q^{i}_{-\psi})_{\text{L}} \tag{24}$$

($\phi = 2, 3$) describes decays of the $q_{+\phi}$ quark. A similar argument leads us to express the third term in (20) in the form

$$\mathscr{H}^{\Delta}_{\text{eff}}(-, \text{PG}) = \sum_{\phi=2}^{3} \mathscr{H}^{\Delta}_{\text{eff}}(-\phi, \text{PG}) + \text{h.c.} \tag{25}$$

This describes the penguin generating q_- flavour changing decays in terms of ($\phi = 2, 3$)

$$\mathscr{H}^{\Delta}_{\text{eff}}(-\phi, \text{PG}) = \sqrt{\tfrac{1}{2}}4G \sum_{\theta=1}^{\phi-1} \sum_{\psi=1}^{3} V^{+}_{\theta\psi} V_{\psi\phi} (\bar{q}^{j}_{+\psi} q^{i}_{+\psi})_{\text{L}} (\bar{q}^{i}_{-\theta} q^{j}_{-\phi})_{\text{L}} . \tag{26}$$

The fourth term in (20) describes flavour conserving weak non-leptonic processes and is also penguin generating:

$$\mathscr{H}^{0}_{\text{eff}}(\text{PG}) = \sqrt{\tfrac{1}{2}}4G \sum_{\theta,\psi=1}^{3} |V_{\theta\psi}|^2 (\bar{q}^{i}_{+\theta} q^{j}_{+\theta})_{\text{L}} (\bar{q}^{j}_{-\psi} q^{i}_{-\psi})_{\text{L}} . \tag{27}$$

In summary the decomposition of $\mathscr{H}_{\text{eff}}$ is

$$\mathscr{H}_{\text{eff}} = \mathscr{H}^{\Delta}_{\text{eff}}(\text{PF}) + \mathscr{H}^{\Delta}_{\text{eff}}(+, \text{PG}) + \mathscr{H}^{\Delta}_{\text{eff}}(-, \text{PG}) + \mathscr{H}^{0}_{\text{eff}}(\text{PG}), \tag{28a}$$

$$\mathscr{H}^{\Delta}_{\text{eff}}(\pm, \text{PG}) = \sum_{\phi=2}^{3} \mathscr{H}^{\Delta}_{\text{eff}}(\pm\phi, \text{PG}) + \text{h.c.} \tag{28b}$$

In this paper we focus attention upon the flavour changing weak non-leptonic part of $\mathscr{H}_{\text{eff}}$. The flavour conserving part will be analysed in a separate publication.

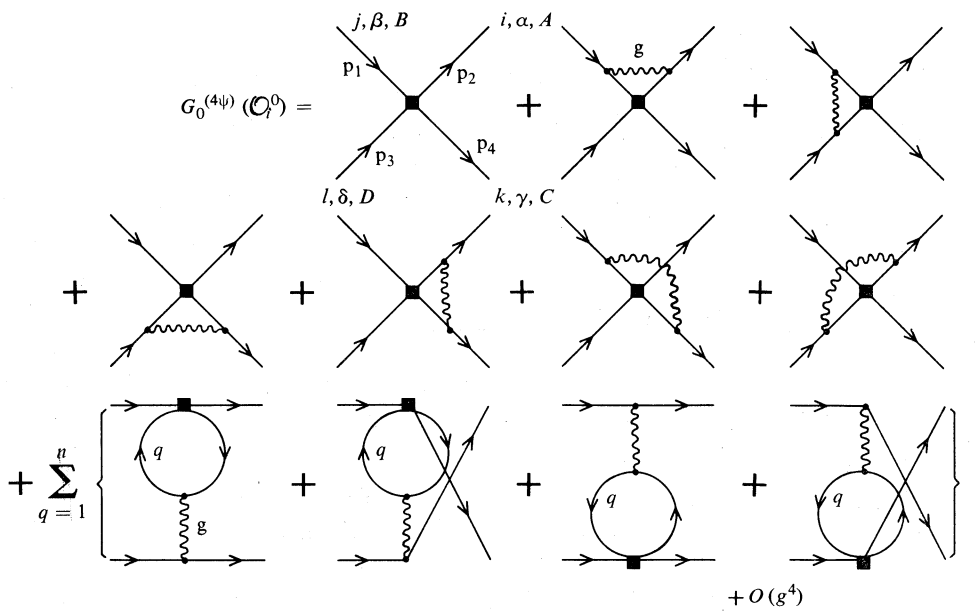

Fig. 2. Four quark operator $\mathcal{O}_i^0$ inserted into the unrenormalized amputated Green function $G_0^{(4\psi)}$, where i, j, k, l are colour indices, α, β, γ, δ are spinor indices and A, B, C, D are flavour indices on the external fermion lines. The solid squares represent the vertex factor defined by the four quark operator $\mathcal{O}_i^0$ relative to the arbitrary indexing of external lines indicated. All crossed diagrams have been made explicit. The penguin diagram vanishes when all four flavours in $\mathcal{O}_i^0$ are unequal, due to a flavour conserving δ at the quark–gluon vertex.

The calculation of the anomalous dimensions of four quark operators is described in Appendix 1 for the penguin free and generating flavour changing cases. (Full details of a general treatment may be found in Miller and McKellar 1981*d*.) The difference between the two calculations is the non-vanishing penguin diagram for

the penguin generating case, i.e. the last four graphs of Fig. 2. Four quark operators exhibit operator mixing and one associates an anomalous dimension matrix with a set of operators closed under renormalization, as discussed in Appendix 1.

We can now proceed to consider the decoupling of W and the other heavy quarks lighter than the W in the (3, 2, 1) model using the GLAM–GW method.

3. Decoupling W and Below in (3, 2, 1) Model

In this section we shall complete the discussion of decoupling for the flavour changing weak non-leptonic sector. From (28) we have two cases to consider: (i) $\mathscr{H}^{\Delta}_{\text{eff}}(\text{PF})$ and (ii) $\mathscr{H}^{\Delta}_{\text{eff}}(\pm, \text{PG})$. We begin with the simplest case:

(a) *Case of* $\mathscr{H}^{\Delta}_{\text{eff}}(\text{PF})$

In Section 2*b* the W was decoupled in the free field limit (zero loop level of the (2, 1) model), the relevant result here being given in (21). We now obtain the analogous one loop result in the (3, 2, 1) model. We make a further decomposition of the summation in (21):

$$\sum_{\substack{\theta\neq\phi \\ \theta,\phi=1}}^{3}\sum_{\substack{\psi\neq\chi \\ \psi,\chi=1}}^{3} = \left[\sum_{\phi=2}^{3}\sum_{\theta=1}^{\phi-1}\sum_{\substack{\psi\neq\chi \\ \psi,\chi=1}}^{\phi}\right]+[\phi\leftrightarrow\theta]$$

$$+\left[\sum_{\chi=2}^{3}\sum_{\psi=1}^{\chi-1}\sum_{\substack{\theta\neq\phi \\ \theta,\phi=1}}^{\chi-1}\right]+[\chi\leftrightarrow\psi], \tag{29}$$

which is easily proved by considering the sets of summation indices. The first term in (29) describes non-leptonic weak decays of the c and t. The second term describes the same for $\bar{c}$ and $\bar{t}$. The third term would describe the same for s and b and the fourth for $\bar{s}$ and $\bar{b}$, but of course it is easy to see that in the third term the $\chi = 2$ contribution vanishes and in the fourth term that from $\psi = 2$ vanishes, i.e. there is no penguin free $\Delta s = 1$ non-leptonic Hamiltonian (Gilman and Wise 1979).

No contributions from quarks heavier than the decaying flavour arise at a one loop level for this part of $\mathscr{H}_{\text{eff}}$. So here there is no need to decouple heavy quarks below the W threshold, and we remain in an effective theory of six flavours. (For example in c decay we could decouple b and t; the effect however will only be small, i.e. of QCD two loop order.) Our work in this subsection will be complete once W has been decoupled.

We will utilize the results of Appendix 1, and introduce a matrix notation which unifies the description of GLAM–GW calculations. We define a set of operators ${}_6\mathscr{O}^{\theta\phi\psi\chi}$ with components ($\theta \neq \phi, \psi \neq \chi$)

$${}_6\mathscr{O}_1^{\theta\phi\psi\chi} = (\bar{q}^i_{+\theta}\, q^i_{+\phi})_{\text{L}}(\bar{q}^j_{-\psi}\, q^j_{-\chi})_{\text{L}}, \qquad {}_6\mathscr{O}_2^{\theta\phi\psi\chi} = (\bar{q}^i_{+\theta}\, q^j_{+\phi})_{\text{L}}(\bar{q}^j_{-\psi}\, q^i_{-\chi})_{\text{L}}, \tag{30a, b}$$

the pre-suffix 6 indicating the number of flavours in the effective theory at this stage. These operators are penguin non-generating, and as seen in Appendix 1 they form a closed renormalization set at the one loop level. The anomalous dimension matrix for this set is, in Landau gauge and the ms (or $\overline{\text{ms}}$) scheme,

$$\gamma^{(6)}(g_6) = \frac{g_6^2}{8\pi^2}\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix} + O(g_6^4) \tag{31}$$

(see equation A9). Here

$$g_k \quad \text{and} \quad m_k^{(l)} \tag{32}$$

are respectively the renormalized gauge coupling constant and quark mass parameters in an effective theory of QCD with k flavours.

Using equations (30) the free field result in (21) may be written as

$$\mathscr{H}^{\Delta}_{\text{eff}}(\text{PF}) = \sqrt{\tfrac{1}{2}}4G \sum_{\substack{\theta,\phi=1 \\ \theta\neq\phi}}^{3} \sum_{\substack{\psi,\chi=1 \\ \psi\neq\chi}}^{3} [A^{\theta\phi\psi\chi}(1,0)]^{\mathrm{T}}\, {}_6\mathscr{O}^{\theta\phi\psi\chi}. \tag{33}$$

The free field coefficients are given by

$$[A^{\theta\phi\psi\chi}(1,0)]^{\mathrm{T}} = (0, V^{+}_{\psi\phi} V_{\theta\chi}). \tag{34}$$

Since the components of ${}_6\mathscr{O}^{\theta\phi\psi\chi}$ form a closed renormalization set, decoupling W in the presence of QCD implies that

$$\mathscr{H}^{\Delta}_{\text{eff}}(\text{PF}) = \sqrt{\tfrac{1}{2}}4G \sum_{\substack{\theta,\phi=1 \\ \theta\neq\phi}}^{3} \sum_{\substack{\psi,\chi=1 \\ \psi\neq\chi}}^{3} A^{\theta\phi\psi\chi}(y, g_6)^{\mathrm{T}}\, {}_6\mathscr{O}^{\theta\phi\psi\chi}. \tag{35}$$

Here we write

$$y = m_{\mathrm{W}}/\mu, \tag{36}$$

and the fact that $A^{\theta\phi\psi\chi}(y, g_6)$ depends on mass parameters only through y is a consequence of it being dimensionless. Equation (35) represents the leading term in the Wilson (1969) short distance expansion of (17) when performed in the (3, 2, 1) model. From (35) and the RHS of (17) it follows upon application of $\mu(\mathrm{d}/\mathrm{d}\mu)$ that the coefficients $A^{\theta\phi\psi\chi}$ satisfy the RG equation

$$[\{-y(\partial/\partial y)+\beta_6(g_6)(\partial/\partial g_6)+2\gamma_5(g_6)\}\mathbf{1}-\boldsymbol{\gamma}^{(6)}(g_6)^{\mathrm{T}}]A^{\theta\phi\psi\chi} = 0, \tag{37}$$

when the subtraction point dependence of m_{W} and $D^{\mu\nu}_{\mathrm{W}}$ is ignored. This is easy to solve if we use the one loop approximation to $\boldsymbol{\gamma}^{(6)}$ in (31), for at this level the matrix diagonalizing $\boldsymbol{\gamma}^{(6)}$ is subtraction point independent and hence commutes with $\mu(\mathrm{d}/\mathrm{d}\mu)\mathbf{1}$. Equation (37) may then be reduced to two mathematically decoupled scalar RG equations. This leads to the scaling solution

$$A^{\theta\phi\psi\chi}(y, g_6) = \exp\left(\int_{g_6}^{\bar{g}_6(Q)} \frac{(-)\boldsymbol{\gamma}^{(6)}(x)^{\mathrm{T}}\,\mathrm{d}x}{\beta_6(x)}\right) A^{\theta\phi\psi\chi}(m_{\mathrm{W}}/Q, \bar{g}_6(Q)). \tag{38}$$

Here $\bar{g}_k(Q)$ is the standard definition of the running coupling constant in an effective theory of QCD with k quark flavours, and

$$\ln(Q/\mu) = \int_{g_k}^{\bar{g}_k(Q)} \frac{\mathrm{d}x}{\beta_k(x)}. \tag{39}$$

To apply the LLA we choose $Q = m_{\mathrm{W}}$ (so no large logarithms can be generated) and approximate $A^{\theta\phi\psi\chi}(1, \bar{g}_6(m_{\mathrm{W}}))$ by $A^{\theta\phi\psi\chi}(1, 0)$, the free field coefficients. Thus in the LLA we have

$$A^{\theta\phi\psi\chi}(y, g_6) \approx \exp\left(\int_{g_6}^{\bar{g}_6(m_{\mathrm{W}})} \frac{(-)\boldsymbol{\gamma}^{(6)}(x)^{\mathrm{T}}\,\mathrm{d}x}{\beta_6(x)}\right) A^{\theta\phi\psi\chi}(1, 0). \tag{40}$$

The parameter μ is chosen (and m_W considered) sufficiently large to validate the use of a leading asymptotic expansion in (40). Defining scalars

$$s_k(P,Q) = (1/16\pi^2 b_0^{(k)}) \ln\{\bar{g}_k^2(P)/\bar{g}_k^2(Q)\}, \tag{41}$$

where

$$\beta_k(x) = -b_0^{(k)} x^3 + b_1^{(k)} x^5 + \ldots, \tag{42}$$

one obtains as the approximate scaling solution

$$\boldsymbol{A}^{\theta\phi\psi\chi}(y, g_6)^{\mathrm{T}} = \boldsymbol{A}^{\theta\phi\psi\chi}(1,0)^{\mathrm{T}} \exp\{s_6(m_W, \mu)\tilde{\gamma}_0^{(6)}\}. \tag{43}$$

Here for convenience we define

$$\tilde{\gamma}^{(k)} = 8\pi^2 \gamma^{(k)}; \qquad \gamma^{(k)}(x) = \gamma_0^{(k)} x^2 + \ldots. \tag{44a, b}$$

With (43) in (35) we obtain

$$\mathscr{H}_{\mathrm{eff}}^{\Delta}(\mathrm{PF}) = \sqrt{\tfrac{1}{2}}\, 4G \sum_{\substack{\theta,\phi=1 \\ \theta\neq\phi}}^{3} \sum_{\substack{\psi,\chi=1 \\ \psi\neq\chi}}^{3} [\boldsymbol{A}^{\theta\phi\psi\chi}(1,0)]^{\mathrm{T}} \exp\{s_6(m_W,\mu)\tilde{\gamma}_0^{(6)}\}\, {}_6\mathcal{O}^{\theta\phi\psi\chi}. \tag{45}$$

We defer the numerical evaluation of this equation to Section 4, when we discuss all the numerical results.

(b) Case of $\mathscr{H}_{\mathrm{eff}}^{\Delta}(\pm, \mathrm{PG})$

In Section 2*b* the W was decoupled at the zero loop level in the (2, 1) model (the free field result). Here we decouple W and other heavy quarks (where necessary) at a one loop level in the (3, 2, 1) model.

From (28) it is sufficient that we study $\mathscr{H}_{\mathrm{eff}}^{\Delta}(\pm\phi, \mathrm{PG})$. Each $(\pm)$ case is slightly different from the other and we will deal with each one separately.

The Case $\mathscr{H}_{\mathrm{eff}}^{\Delta}(-, \mathrm{PG})$. The relevant free field result is given in (26) in terms of penguin generating four quark operators. We begin by defining a set of operators, belonging to an effective theory with k flavours, ${}_k\mathcal{O}_{-\phi}^{\theta\psi}$ and ${}_k\mathcal{O}_{+\phi}^{\theta\psi}$ (which will be needed in the following case for $\mathscr{H}_{\mathrm{eff}}^{\Delta}(+, \mathrm{PG})$) by

$$[{}_k\mathcal{O}_{\pm\phi}^{\theta\psi}]_1 = (\bar{q}^i_{\pm\theta} q^i_{\pm\phi})_{\mathrm{L}} (\bar{q}^j_{\mp\psi} q^j_{\mp\psi})_{\mathrm{L}}, \qquad [{}_k\mathcal{O}_{\pm\phi}^{\theta\psi}]_2 = (\bar{q}^i_{\pm\theta} q^j_{\pm\phi})_{\mathrm{L}} (\bar{q}^j_{\mp\psi} q^i_{\mp\psi})_{\mathrm{L}},$$

$$[{}_k\mathcal{O}_{\pm\phi}^{\theta\psi}]_3 = \sum_{l=1}^{k} (\bar{q}^i_{\pm\theta} q^i_{\pm\phi})_{\mathrm{L}} (\bar{q}^j_l q^j_l)_{\mathrm{L}}, \qquad [{}_k\mathcal{O}_{\pm\phi}^{\theta\psi}]_4 = \sum_{l=1}^{k} (\bar{q}^i_{\pm\theta} q^j_{\pm\phi})_{\mathrm{L}} (\bar{q}^j_l q^i_l)_{\mathrm{L}},$$

$$[{}_k\mathcal{O}_{\pm\phi}^{\theta\psi}]_5 = \sum_{l=1}^{k} (\bar{q}^i_{\pm\theta} q^i_{\pm\phi})_{\mathrm{L}} (\bar{q}^j_l q^j_l)_{\mathrm{R}}, \qquad [{}_k\mathcal{O}_{\pm\phi}^{\theta\psi}]_6 = \sum_{l=1}^{k} (\bar{q}^i_{\pm\theta} q^j_{\pm\phi})_{\mathrm{L}} (\bar{q}^j_l q^i_l)_{\mathrm{R}}, \tag{46}$$

where q_l are the components of the quark vector $\boldsymbol{q}^{\mathrm{T}} = (q_{-1}\, q_{+1}\, q_{-2}\, q_{+2} \ldots)$ (mass ordered except for u and d).

Given that all the flavours on the RHS of equations (46) belong to the effective theory and that $\theta \neq \phi$ it follows from Appendix 1 that the components of ${}_k\mathcal{O}_{\pm\phi}^{\theta\psi}$

form a closed renormalization set whose anomalous dimension matrix is given at the one loop level in the ms (or $\overline{\text{ms}}$) scheme and the Landau gauge by

$$\gamma^{(k)}(g_k) = \frac{g_k^2}{8\pi^2}\begin{bmatrix} -1 & 3 & 0 & 0 & 0 & 0 \\ 3 & -1 & -\frac{1}{9} & \frac{1}{3} & -\frac{1}{9} & \frac{1}{3} \\ 0 & 0 & -\frac{11}{9} & \frac{11}{3} & -\frac{2}{9} & \frac{2}{3} \\ 0 & 0 & 3-\frac{1}{9}k & -1+\frac{1}{3}k & -\frac{1}{9}k & \frac{1}{3}k \\ 0 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & -\frac{1}{9}k & \frac{1}{3}k & -\frac{1}{9}k & -8+\frac{1}{3}k \end{bmatrix} + O(g_k^4). \quad (47)$$

In terms of (46) we may express (26) in the form

$$\mathscr{H}^{\Delta}_{\text{eff}}(-\phi, \text{PG}) = \sqrt{\tfrac{1}{2}}4G \sum_{\theta=1}^{\phi-1} \sum_{\psi=1}^{3} [A^{\theta\psi}_{-\phi}(1,0)]^{\text{T}}\, {}_6\mathscr{O}^{\theta\psi}_{-\phi}, \quad (48)$$

where the free field coefficients are given by

$$[A^{\theta\psi}_{-\phi}(1,0)]^{\text{T}} = (0, V^{+}_{\theta\psi} V_{\psi\phi}, 0, 0, 0, 0). \quad (49)$$

Since ${}_6\mathscr{O}^{\theta\psi}_{-\phi}$ form a closed renormalization set, decoupling in the (3, 2, 1) model must give

$$\mathscr{H}_{\text{eff}}(-\phi, \text{PG}) = \sqrt{\tfrac{1}{2}}4G \sum_{\theta=1}^{\phi-1} \sum_{\psi=1}^{3} A^{\theta\psi}_{-\phi}(y, g_6)^{\text{T}}\, {}_6\mathscr{O}^{\theta\psi}_{-\phi}, \quad (50)$$

y having been defined by (36).

In equation (50), $A^{\theta\psi}_{-\phi}(y, g_6)$ originate in the same Wilson (1969) short distance expansion as $A^{\theta\phi\psi\chi}(y, g_6)$ in (35) and hence satisfy the same RG equation (37). We can immediately deduce the structure of (50) in the LLA with a leading asymptotic expansion from (45). Hence at this stage we have

$$\mathscr{H}^{\Delta}_{\text{eff}}(-\phi, \text{PG}) = \sqrt{\tfrac{1}{2}}4G \sum_{\theta=1}^{\phi-1} \sum_{\psi=1}^{3} [A^{\theta\psi}_{-\phi}(1,0)]^{\text{T}} \exp\{s_6(m_{\text{W}}, \mu)\tilde{\gamma}_0^{(6)}\}\, {}_6\mathscr{O}^{\theta\psi}_{-\phi}. \quad (51)$$

The scalars $s_k(P, Q)$ are defined in (41) and $\tilde{\gamma}^{(k)}$ in (44), but of course using the 6×6 matrices in (47).

Depending upon the process we are to study, t, b or c might be considered as heavy quarks. We shall now proceed to remove k (= 1, 2 or 3) heavy quarks from (51) so as to obtain a *general expression* applicable to both b decay ($-\phi = -3$) and s decay ($-\phi = -2$). This simply makes programming easier. The heavy quarks are to be decoupled sequentially in the order q_{+3}, q_{-3}, q_{+2}.

We begin by decoupling q_{+3} from the operator set ${}_6\mathscr{O}^{\theta\psi}_{-\phi}$. In the free field limit this is simple, being a zero loop (2, 1) decoupling. From (46) we see that

$${}_6\mathscr{O}^{\theta\psi}_{-\phi} = \boldsymbol{B}^{\theta\psi}_{5,-\phi}(1,0)\, {}_5\mathscr{O}^{\theta\psi}_{-\phi} + O(1/m_6^{(6)2}), \quad (52)$$

where $\boldsymbol{B}^{\theta\psi}_{5,-\phi}(1,0)$, a matrix of free field coefficients, is given by

$$\boldsymbol{B}^{\theta\psi}_{5,-\phi}(1,0)_{ij} = \delta_{ij}\{1-\delta_{\psi 3}(\delta_{i1}+\delta_{i2})\}. \quad (53)$$

This ensures that the first two components of ${}_5\mathscr{O}^{\theta\psi}_{-\phi}$ vanish when ψ equals 3.

Since the ${}_5\mathcal{O}^{\theta\psi}_{-\phi}$ form a closed renormalization set, decoupling q_{+3} in $(3,2,1)$ must give

$$ {}_6\mathcal{O}^{\theta\psi}_{-\phi} = \boldsymbol{B}^{\theta\psi}_{5,-\phi}(y_6, g_6)\, {}_5\mathcal{O}^{\theta\psi}_{-\phi} + O(1/m_6^{(6)2}), \tag{54} $$

where we introduce

$$ y_k(\mu) = m_k^{(k)}(\mu)/\mu . \tag{55} $$

Applying $\mu(\mathrm{d}/\mathrm{d}\mu)\mathbf{1}$ to both sides of (54), we obtain the RG equation satisfied by $\boldsymbol{B}^{\theta\psi}_{5,-\phi}$:

$$ \big([y_6\{\gamma_m^{(6)}(g_6)-1\}(\partial/\partial y_6)+\beta_6(g_6)(\partial/\partial g_6)]\boldsymbol{1}+\boldsymbol{\gamma}^{(6)}(g_6)\big)\boldsymbol{B}^{\theta\psi}_{5,-\phi} - \boldsymbol{B}^{\theta\psi}_{5,-\phi}\boldsymbol{\gamma}^{(6)}(g_5) = 0, \tag{56} $$

where

$$ \gamma_m^{(k)}(g_k) = (\mu/m_k^{(i)})(\mathrm{d}m_k^{(i)}/\mathrm{d}\mu) \tag{57} $$

and is flavour (i) independent.

The matrix diagonalizing the one loop approximation to $\boldsymbol{\gamma}^{(k)}$ is subtraction point independent and therefore commutes with $\mu(\mathrm{d}/\mathrm{d}\mu)\mathbf{1}$. For this reason (56) may be reduced to a set of decoupled scalar RG equations for which we know the solutions. Proceeding in this manner one finds the scaling solution of (56) to be

$$ \boldsymbol{B}^{\theta\psi}_{5,-\phi}(y_6, g_6) = \exp\left(\int_{g_6}^{\tilde{g}_6(Q)} \frac{\boldsymbol{\gamma}^{(6)}(x)}{\beta_6(x)}\,\mathrm{d}x\right)\boldsymbol{B}^{\theta\psi}_{5,-\phi}(m_6^{(6)}(\mu)/Q, \tilde{g}_6(Q)) $$

$$ \times \exp\left(-\int_{g_6}^{\tilde{g}_6(Q)} \frac{\boldsymbol{\gamma}^{(5)}(g_5(x))}{\beta_6(x)}\,\mathrm{d}x\right), \tag{58} $$

where $\tilde{g}_6(Q)$ is a non-standard running coupling constant, defined by

$$ \ln(Q/\mu) = \int_{g_k}^{\tilde{g}_k(Q)} \frac{1-\gamma_m^{(k)}(x)}{\beta_k(x)}\,\mathrm{d}x . \tag{59} $$

To apply an LLA we scale to $Q = m_6^{(6)}$. Assuming $\tilde{g}_6(Q)$ is sufficiently small at mass scales μ and $m_6^{(6)}$, so as to permit the use of the LLA and also a leading asymptotic expansion in (58), we obtain

$$ \boldsymbol{B}^{\theta\psi}_{5,-\phi}(y_6, g_6) = \exp\{-\tilde{s}_6(m_6^{(6)}, \mu)\tilde{\boldsymbol{\gamma}}_0^{(6)}\}\boldsymbol{B}^{\theta\psi}_{5,-\phi}(1,0)\exp\{\tilde{s}_6(m_6^{(6)}, \mu)\tilde{\boldsymbol{\gamma}}_0^{(5)}\}, \tag{60} $$

where we make use of the fact that $g_5 = g_6 + O(g_6^3)$ (see Miller and McKellar 1981*a*). The $\tilde{\boldsymbol{\gamma}}^{(k)}$ are defined in (44) and (47), while the scalars

$$ \tilde{s}_k(P, Q) = (1/16\pi^2 b_0^{(k)})\ln(\tilde{g}_k(p)^2/\tilde{g}_k(Q)^2) \tag{61} $$

are defined by analogy with $s_k(P, Q)$ in equation (41).

Using (60) in (54) and the result in (51) gives us the structure of the effective Hamiltonian at this stage:

$$ \mathscr{H}^{\Delta}_{\mathrm{eff}}(-\phi, \mathrm{PG}) = \sqrt{\tfrac{1}{2}}4G \sum_{\theta=1}^{\phi-1}\sum_{\psi=1}^{3} [\boldsymbol{A}^{\theta\psi}_{-\phi}(1,0)]^{\mathrm{T}}\exp[\{s_6(m_{\mathrm{W}}, \mu) - \tilde{s}_6(m_6^{(6)}, \mu)\}\tilde{\boldsymbol{\gamma}}_0^{(6)}] $$

$$ \times \boldsymbol{B}^{\theta\psi}_{5,-\phi}(1,0)\exp\{s_6(m_6^{(6)}, \mu)\tilde{\boldsymbol{\gamma}}_0^{(5)}\}\, {}_5\mathcal{O}^{\theta\psi}_{-\phi} . \tag{62} $$

The matrix exponentials combine since $\tilde{\boldsymbol{\gamma}}_0^{(6)}$ commutes with itself. For $\psi = 3$ the first two components of ${}_5\mathcal{O}^{\theta\psi}_{-\phi}$ are *not* defined. It is quite convenient to maintain this

notation, nonetheless. Since the free field coefficients for these first two components vanish (see equation 53) and since the other components (which do exist, i.e. $[{}_5\mathcal{O}^{\theta\psi}_{-\phi}]_i$, $i = 3, \ldots, 6$) cannot generate them upon renormalization (as can be seen in equation 47), it follows that they never enter the final expressions. This is why it makes no difference to 'retain' them. The premultiplying matrices in (62) will annihilate those components which apparently contain explicit reference to heavy fields which have actually been decoupled. This trick allows us to maintain the 6×6 structure of the problem and greatly simplifies both the formal structure and the numerical evaluation of our final results.

Proceeding in the same manner we next decouple q_{-3} from the operators ${}_5\mathcal{O}^{\theta\psi}_{-\phi}$, giving the general result

$$\begin{aligned}\mathscr{H}^{\Delta}_{\text{eff}}(-\phi, \text{PG}) = \sqrt{\tfrac{1}{2}}4G \sum_{\theta=1}^{\phi-1} \sum_{\psi=1}^{3} [\boldsymbol{A}^{\theta\psi}_{-\phi}(1,0)]^{\text{T}} \exp[\{s_6(m_{\text{W}}, \mu) - \tilde{s}_6\}\tilde{\gamma}^{(6)}_0] \\ \times \prod_{l=1}^{k-1} [\boldsymbol{B}^{\theta\psi}_{6-l,-\phi}(1,0) \exp\{(\tilde{s}_{6-l+1} - \tilde{s}_{6-l})\tilde{\gamma}^{(6-l)}_0\}] \\ \times \boldsymbol{B}^{\theta\psi}_{6-k,-\phi}(1,0) \exp(\tilde{s}_{6-k+1}\, \tilde{\gamma}^{(6-k)}_0)\, {}_{6-k}\mathcal{O}^{\theta\psi}_{-\phi}, \end{aligned} \tag{63}$$

where the free field coefficients are given by

$$\boldsymbol{B}^{\theta\psi}_{6-l,-\phi}(1,0)_{ij} = \delta_{ij}\{1 - \delta_{\psi,3-\frac{1}{2}(l-1)}(\delta_{i1} + \delta_{i2})\}, \qquad l \text{ odd}, \tag{64a}$$

$$= \delta_{ij}, \qquad l \text{ even}, \tag{64b}$$

and $\tilde{s}_k$ denotes $\tilde{s}_k(m^{(k)}_k, \mu)$.

We have previously mentioned that when ψ corresponds to a heavy quark q_+ index the first two components of ${}_{6-k}\mathcal{O}^{\theta\psi}_{-\phi}$ are not defined. Because of the structure of the $\boldsymbol{B}$ and γ matrices, which is preserved upon exponentiation, it is easy to show that these components are not actually present at all. Our choice of the l-odd free field coefficients in (64a) makes use of the fact that once these components are removed when first discovered to contain a heavy quark during sequential decoupling, then at subsequent heavy quark decouplings they do not have to be removed a second time. This is why at subsequent decouplings it appears we have assigned them non-vanishing free field coefficients. This notation simply avoids an unnecessary complication.

In practice k will never exceed 3, so one may ask why we derive such a general equation as (63). The answer is that in a single expression (63) describes the penguin generating sector of both b and s decays and allows for flexibility in defining which quarks are to be considered heavy. For example, some may wish to consider t heavy relative to b and others may not, while others may wish to consider t, b and c or t and b or just t as heavy relative to s. In the next subsection we shall see that (63) can also be used to describe the penguin generating sector of both c and t decays. In short it is not being extravagant but economical to write down such a general result.

The Case $\mathscr{H}^{\Delta}_{\text{eff}}(+, \text{PG})$ *and Summary.* We wish to perform the same calculation for $\mathscr{H}^{\Delta}_{\text{eff}}(+\phi, \text{PG})$ as was done above for $\mathscr{H}^{\Delta}_{\text{eff}}(-\phi, \text{PG})$. The relevant starting point is the free field result in (24). Our final result may be obtained by observing the symmetry between (24) and (26), i.e.

$$\mathscr{H}^{\Delta}_{\text{eff}}(+\phi, \text{PG}) = \mathscr{H}^{\Delta}_{\text{eff}}(-\phi, \text{PG}) \begin{pmatrix} (+) \leftrightarrow (-) \\ V \leftrightarrow V^+ \end{pmatrix}. \tag{65}$$

With this one can quote almost immediately what the final result would be, the only subtlety being in correctly converting the free field coefficients in (64).

Using this symmetry or going to the trouble of performing the detailed calculation one attains the final theoretical result: Upon decoupling in (3, 2, 1) all heavy quarks, k of which are lighter than W, and W itself in a sequential fashion one finds

$$\begin{aligned}\mathscr{H}^{\Delta}_{\text{eff}}(\pm\phi, \text{PG}) = \sqrt{\tfrac{1}{2}}4G \sum_{\theta=1}^{\phi-1}\sum_{\psi=1}^{3} [\boldsymbol{A}^{\theta\psi}_{\pm\phi}(1,0)]^{\text{T}} \exp[\{s_6(m_{\text{W}},\mu)-\tilde{s}_6\}\tilde{\boldsymbol{\gamma}}^{(6)}_0] \\ \times \prod_{l=1}^{k-1} [\boldsymbol{B}^{\theta\psi}_{6-l,\pm\phi}(1,0)\exp\{(\tilde{s}_{6-l+1}-\tilde{s}_{6-l})\tilde{\boldsymbol{\gamma}}^{(6-l)}_0\}] \\ \times \boldsymbol{B}^{\theta\psi}_{6-k,\pm\phi}(1,0)\exp\{\tilde{s}_{6-k+1}\tilde{\boldsymbol{\gamma}}^{(6-k)}_0\}_{6-k}\mathscr{O}^{\theta\psi}_{\pm\phi}, \end{aligned} \tag{66}$$

where $\tilde{s}_k$ denotes $\tilde{s}_k(m_k^{(k)}, \mu)$. The free field coefficients are

$$[\boldsymbol{A}^{\theta\psi}_{-\phi}(1,0)]^{\text{T}} = (0, [V^{+}]_{\theta\psi} V_{\psi\phi}, 0, 0, 0, 0)\,, \tag{67a}$$

$$[\boldsymbol{A}^{\theta\psi}_{+\phi}(1,0)]^{\text{T}} = (0, V_{\theta\psi}[V^{+}]_{\psi\phi}, 0, 0, 0, 0)\,; \tag{67b}$$

$$[\boldsymbol{B}^{\theta\psi}_{6-l,-\phi}(1,0)]_{ij} = \delta_{ij}\{1-\delta_{\psi,3-\frac{1}{2}(l-1)}(\delta_{i1}+\delta_{i2})\}\,, \quad l \text{ odd}\,, \tag{67c}$$

$$= \delta_{ij}\,, \quad l \text{ even}; \tag{67d}$$

$$[\boldsymbol{B}^{\theta\psi}_{6-l,+\phi}(1,0)]_{ij} = \delta_{ij}\,, \quad l \text{ odd}\,, \tag{67e}$$

$$= \delta_{ij}\{1-\delta_{\psi,3-\frac{1}{2}(l-2)}(\delta_{i1}+\delta_{i2})\}\,, \quad l \text{ even}; \tag{67f}$$

and the ${}_k\mathscr{O}^{\theta\psi}_{\pm\phi}$ are given in (46).

At this stage all the necessary theoretical work has been completed. Reducing these results to their most useful form for phenomenological application is a problem dealt with in the next section.

4. Numerical Aspects

The purpose of this section is to reduce $\mathscr{H}^{\Delta}_{\text{eff}}(\text{PF})$ and $\mathscr{H}^{\Delta}_{\text{eff}}(\pm, \text{PG})$ to their most useful numerical forms.

From equations (45) and (66) the problem reduces to evaluating products of exponentiated matrices. The matrices depend on the scalars $s_k(P, Q)$ and $\tilde{s}_k(P, Q)$ defined in (41) and (61). Numerical evaluation of these is straightforward when we know the running coupling constants, both standard and non-standard, and numerical estimates for the mass scales of the problem, P and Q. We cover the evaluation of these parameters in the following subsection.

(a) *Preliminary Considerations*

Running Coupling Constants, Quark Mass and RG Invariants. Solving an RG equation in the Landau gauge when $m_k^{(i)}(\mu)$ and $g_k(\mu)$ are treated as independent functions of μ leads to the standard definition of the running coupling constant, a definition of the running fermion mass and the associated RG invariants. These are given by

$$\ln\left(\frac{Q}{\Lambda'_k}\right) = \int^{\bar{g}_k(Q)} \frac{\mathrm{d}x}{\beta_k(x)}, \qquad \overline{m}_k^{(i)}(Q) = \tilde{\Lambda}_k^{(i)} \exp\left(\int^{\bar{g}_k(Q)} \frac{\gamma_m^{(k)}(x)}{\beta_k(x)}\,\mathrm{d}x\right), \tag{68a, b}$$

where $\gamma_m^{(k)}(x) = (\mu/m_k^{(i)})\mathrm{d}m_k^{(i)}/\mathrm{d}\mu$ is the anomalous mass dimension. (Here Λ' is not the standard definition of the RG invariant Λ (i.e. the one for which experimental numbers are quoted—the Bardeen *et al.* 1978 definition), although it is often mistakenly defined as such. The relation between Λ and Λ' is given shortly.) The second definition depends upon the auxiliary equations chosen in solving the RG equation; we find this the most natural choice. Eliminating Λ_k' from (68a) gives the earlier definition of $\bar{g}_k(Q)$ in (39). The Λ_k' and $\tilde{\Lambda}_k^{(i)}$ are RG invariants to be fixed by experiment. This aspect is discussed shortly. These invariants are both scheme dependent *and* effective theory (k) dependent, features which we take into account in our numerical work.

The quantities we have considered (i.e. 'operator coefficients') can be expressed in terms of two independent functions of μ:

$$y_k = m_k^{(k)}(\mu)/\mu \quad \text{and} \quad g_k(\mu).$$

Solving RG equations using these variables gives rise to the non-standard definition of the running coupling constant $\tilde{g}_k(Q)$. In terms of the RG invariants introduced above we have

$$\ln\left(\frac{Q}{\Lambda_k'}\frac{\tilde{\Lambda}_k^{(k)}}{m_k^{(k)}}\right) = \int^{\tilde{g}_k(Q)} \frac{1-\gamma_m^{(k)}(x)}{\beta_k(x)}\,\mathrm{d}x. \tag{69}$$

The ratio $\tilde{\Lambda}_k^{(k)}/m_k^{(k)}$ is in fact flavour independent as follows from (68b) ($\gamma_m^{(k)}(x)$ is flavour independent in the $\overline{\mathrm{ms}}$ scheme) and thus $\tilde{g}_k(Q)$ is not flavour dependent. The RG invariants may be eliminated from (69) to give the, possibly, more familiar definition in equation (59) (Gilman and Wise 1979). Unlike $\bar{g}_k(Q)$, $\tilde{g}_k(Q)$ is *also* subtraction point dependent, as is clear from (69), since $m_k^{(k)} \equiv m_k^{(k)}(\mu)$ appears on the LHS.

Clearly it should not matter how we solve the RG equation, as we must get the same results by either method. This suggests the two running coupling constants ought to be related in some way. It is not difficult to show that this is indeed the case and we find

$$\tilde{g}_k(Q' = \{m_k^{(i)}/\overline{m}_k^{(i)}(Q)\}\,Q) = \bar{g}_k(Q). \tag{70}$$

In other words, the non-standard running coupling constant is in reality only the standard running coupling constant with a differently defined variable.

When Q is sufficiently large, equations (68) may be expanded asymptotically. Retaining *all* integrand pole terms we find

$$1/\bar{g}_k(Q)^2 = R_k \ln(Q^2/\Lambda_k'^2) + T_k \ln\{1/\bar{g}_k(Q)^2\}, \tag{71a}$$

$$\overline{m}_k^{(i)}(Q) = \tilde{\Lambda}_k^{(i)}\{1/\bar{g}_k(Q)^2\}^{-4/16\pi^2 R_k}. \tag{71b}$$

The standard definition of Λ (Bardeen *et al.* 1978) has for a trivial reason come to be accepted as

$$1/\bar{g}_k(Q)^2 = R_k \ln(Q^2/\Lambda_k^2) + T_k \ln\{1/R_k\,\bar{g}_k(Q)^2\}, \tag{71c}$$

where

$$R_k = b_0^{(k)} = (1/16\pi^2)(11-\tfrac{2}{3}k), \tag{72a}$$

$$T_k = -b_1^{(k)}/b_0^{(k)} = (1/16\pi^2)(102-\tfrac{38}{3}k)/(11-\tfrac{2}{3}k). \tag{72b}$$

(Hence one sees that $\Lambda_k'^2 = \Lambda_k^2 R_k^{T_k/R_k}$, for example $\Lambda_4'^2 = 0{\cdot}1137\,\Lambda_4^2$.)

In this paper we shall use the second order asymptotic expansion for $\bar{g}_k(Q)^2$ given in (71c). The reason for this is as follows. From the point of view of *consistency* we should expand *all* our integrands up to and including the $1/x$ pole terms in their Laurent expansions. It is essential that the integrands of all scaling coefficients (see equation 40 for example) be expanded to $1/x$ in their Laurent expansions, otherwise they would all be approximated by one. Thus to be consistent all other integrands should be expanded to exactly the same order, i.e. up to and including $1/x$ pole terms in their Laurent expansions.

The running coupling constant is thus expanded to second order. This in turn implies the need (Baće 1978) to incorporate scheme dependence in RG invariants. This dependence is fairly strong as shown by the Celmaster–Gonsalves (1979) relation. Another consequence is that we must now observe the μ dependence of the quark mass, and so we *must* use a running quark mass in our work. A second point which is relevant is that in non-leptonic weak decays of s and c we would like to choose relatively 'moderate' subtraction points. We then find that the second order terms in (71c) can be fairly significant at such moderate Q^2 values *even* when third order terms are small. Roberts (1981) argues that the experimental situation is in general agreement with $\Lambda_{5(\overline{\mathrm{ms}})} \approx 0{\cdot}4$ GeV. However, two recent determinations from e^+e^- annihilation (Ali 1981) and the gluonic width of the upsilon (MacKenzie and Lepage 1981) suggest that $\Lambda_{5(\overline{\mathrm{ms}})}$ could be significantly smaller, i.e. $\approx 0{\cdot}1$ GeV. We shall compromise and use as our input

$$\Lambda_{5(\overline{\mathrm{ms}})} = 0{\cdot}250 \text{ GeV}. \tag{73}$$

(Note that all the above authors are talking of the same standard Λ as defined through equation 71c.)

Table 1. Renormalization group invariants

The Λ_k are calculated in the $\overline{\mathrm{ms}}$ scheme, with Λ_5 as input, as are the $\tilde{\Lambda}_k^{(k)}$, where we have used quark threshold mass estimates $m^{(4)} = 1{\cdot}6$ GeV, $m^{(5)} = 4{\cdot}7$ GeV and $m^{(6)} = 35$ GeV. The $Q_4^2(k)$ represent minimum Q^2 values above which (i) third order contributions to $\bar{g}_k(Q)^2$ are less than 5% and (ii) the method of second iteration gives good approximations. Finally, the μ_k represent conventional subtraction point choices for s, c, b and t decay, corresponding respectively to $k = 3$, 4, 5 and 6

k	Λ_k (GeV)	$\tilde{\Lambda}_k^{(k)}$ (GeV)	$Q_5^2(k)$ (GeV^2)	μ_k (GeV)
3	0·420	—	6·0	$\sqrt{1{\cdot}3}$
4	0·354	0·898	3·0	1·6
5	0·250	2·99	1·3	4·7
6	0·126	26·7	1·3	35·0

We have recently shown (Miller and McKellar 1981*b*) how one may calculate other effective RG invariants Λ_k^2 and $\tilde{\Lambda}_k^{(i)2}$, given a value for Λ_l for a specific l. For the purpose of this paper we need $\Lambda^2_{k(\overline{\mathrm{ms}})}$ and $\tilde{\Lambda}^{(k)2}_{k(\overline{\mathrm{ms}})}$ for $k = 3$, 4, 5 and 6, the results of the calculation of these being given in Table 1. The calculations were performed using the Ovrut–Schnitzer (1981*a*, 1981*b*) formulation of the decoupling process.

Of course (71c) only defines $\bar{g}_k(Q)^2$ by a transcendental equation. But (71c) itself is an approximate equation and we require only an approximate solution.

The common method adopted in the literature is to approximate the solution by second iteration of (71c). We have investigated the validity of this method (Miller and McKellar 1981*b*). Basing the numerical work on (73), we find that the third order terms in (71c) are uniformly less than 5% significant (for $k = 3, \ldots, 8$) if we work above $Q^2 = 1{\cdot}3\ \mathrm{GeV}^2$. We establish a minimum Q^2, denoted by $Q_5^2(k)$, above which it is true that the second iteration is a good approximation to the exact solution of (71c) (i.e. within the order of the third order term of the exact solution which we restrict to be no larger than 5%). These are also given in Table 1. The third iteration is always a good approximation and we prefer to always employ it.

We note that the second iteration of (71c) is given by

$$1/\bar{g}_k(Q)^2 = R_k \ln(Q^2/\Lambda_k^2) + T_k \ln\{\ln(Q^2/\Lambda_k^2)\}. \tag{74}$$

The appearance of an (ln ln) term rather than an ($\ln R_k$ ln) term in (74) was originally the sole motivating factor (Bardeen *et al.* 1978) in the redefinition of Λ when proceeding from (69) to (71c). We find it easiest to evaluate $\tilde{g}_k(Q)$ by using third iteration and (70), for in this way we can always keep track on the validity of the iterative method. To do this we need to know $\tilde{\Lambda}_k^{(k)}$ for $k = 4$, 5 and 6, and as mentioned these are also tabulated in Table 1. It is also helpful to keep in mind that (from equation 70)

$$\tilde{g}_k(\mu) = \bar{g}_k(\mu) = g_k. \tag{75}$$

Subtraction Point Choice and m_W. To completely specify $s_k(P, Q)$ and $\tilde{s}_k(P, Q)$ we need to know m_W and to choose μ.

First we consider the subtraction point μ. We must choose μ so that we can make reliable estimates of $\bar{g}_k(\mu)$. These can be obtained using third iteration of (71c) if we choose $\mu^2 \geqslant 1{\cdot}3\ \mathrm{GeV}^2$, as we saw in the last subsection. Second, μ must be chosen in a manner consistent with the process to be considered. By this we mean μ^2 should be chosen to $O(q^2)$, where q^2 is a typical external momentum for that process. Such a choice attempts to minimize higher order effects in the physical matrix elements of the four quark operators. As the principal application of our results will be to flavour changing non-leptonic decays then the above requirements reduce to

$$\mu_k^2 = \max(1{\cdot}3\ \mathrm{GeV}^2, m^{(k)2}), \tag{76}$$

where $m^{(k)2}$ is the threshold mass of the decaying quark. Our choices of μ for s, c, b and t decays, determined in this way, are given in Table 1.

We point out that this mode of μ choice (which we refer to as the conventional μ choice) is consistent with the $\overline{\mathrm{ms}}$ scheme, but not the ms scheme. A discussion of this point is given in Appendix 3, the conclusions being that an appropriate μ choice for the ms scheme displays physical results (and operator coefficients) which are independent of whether we use ms or $\overline{\mathrm{ms}}$.

In the Introduction we mentioned that the GLAM–GW method ignores the subtraction point dependence of m_W. For m_W we use (as did Gilman and Wise 1979) the threshold mass obtained from essentially a zero loop calculation of G in μ decay and measurements of the Weinberg angle θ_W in ν_μe scattering. Recent estimates (see Tran 1981) seem to centre on a value

$$m_W = 80\ \mathrm{GeV}, \tag{77}$$

which is the one we shall employ.

(b) Numerical Results for $\mathscr{H}^{\Delta}_{\text{eff}}(\text{PF})$

The theoretical result is expressed in equation (45). The numerical evaluation of it is quite simple.

We begin by applying to (45) the summation decomposition in (29), and hence we write

$$\mathscr{H}^{\Delta}_{\text{eff}}(\text{PF}) = \sum_{\phi=2}^{3} \mathscr{H}^{\Delta}_{\text{eff}}(+\phi, \text{PF}) + \sum_{\chi=2}^{3} \mathscr{H}^{\Delta}_{\text{eff}}(-\chi, \text{PF}) + \text{h.c.}, \tag{78}$$

where

$$\mathscr{H}^{\Delta}_{\text{eff}}(+\phi, \text{PF}) = \sqrt{\tfrac{1}{2}}4G \sum_{\theta=1}^{\phi-1} \sum_{\substack{\psi=\chi \\ \psi,\chi=1}}^{\phi} [A^{\theta\phi\psi\chi}(1,0)]^{\text{T}} \exp\{s_6(m_{\text{W}}, \mu)\tilde{\gamma}_0^{(6)}\}\, {}_6\mathscr{O}^{\theta\phi\psi\chi}, \tag{79a}$$

$$\mathscr{H}^{\Delta}_{\text{eff}}(-\chi, \text{PF}) = \sqrt{\tfrac{1}{2}}4G \sum_{\psi=1}^{\chi-1} \sum_{\substack{\theta,\phi=1 \\ \theta\neq\phi}}^{\chi-1} [A^{\theta\phi\psi\chi}(1,0)]^{\text{T}} \exp\{s_6(m_{\text{W}}, \mu)\tilde{\gamma}_0^{(6)}\}\, {}_6\mathscr{O}^{\theta\phi\psi\chi}. \tag{79b}$$

The first four terms of (78) describe PF non-leptonic weak decays of c, t and b respectively. They vanish when $-\chi = -2$, there being no PF Hamiltonian for s decays. When $\mathscr{H}^{\Delta}_{\text{eff}}(\text{PF})$ is applied to describe PF non-leptonic weak decays of c, b and t, subtraction points are chosen appropriately, as given in Table 1 where

$$\mu = \mu_{2\phi}, \quad \text{for } +\phi; \qquad \mu_{2\chi-1}, \quad \text{for } -\chi. \tag{80}$$

Table 2. Scalars $s_6(m_{\text{W}}, \mu_k)$

These scalars needed for $\mathscr{H}^{\Delta}_{\text{eff}}(\text{PF})$ are calculated using third iteration and lead to the coefficients a_k and b_k appearing in (82). The a_k and b_k have been changed from 1 and 0, their respective values in the absence of QCD effects

k	$s_6(m_{\text{W}}, \mu_k)$	a_k	b_k
3	−0·1422	—	—
4	−0·1232	1·209	−0·428
5	−0·0767	1·108	−0·251
6	−0·0183	1·020	−0·056

The scalars to be calculated are $s_6(m_{\text{W}}, \mu_k)$ for $k = 3$, 4, 5 and 6 ($k = 3$ will be needed later). We use third iteration, ensuring the validity of our calculations. The results are given in Table 2.

The final part of the problem is to evaluate the matrices $\exp\{s_6(m_{\text{W}}, \mu_k)\tilde{\gamma}_0^{(6)}\}$. This is most easily accomplished with an eigenvector/eigenvalue analysis of $\tilde{\gamma}^{(6)}$. Doing so (using equations 31 and 44), we find that

$$\exp(c\tilde{\gamma}_0^{(6)}) = \tfrac{1}{2}\begin{pmatrix} e^{2c}+e^{-4c} & e^{2c}-e^{-4c} \\ e^{2c}-e^{-4c} & e^{2c}+e^{-4c} \end{pmatrix}. \tag{81}$$

Setting

$$\exp\{s_6(m_{\text{W}}, \mu_k)\tilde{\gamma}_0^{(6)}\} = \begin{pmatrix} a_k & b_k \\ b_k & a_k \end{pmatrix}, \tag{82}$$

we get the values of a_k and b_k given in Table 2. It follows that (using equations 30 and 34)

$$[A^{\theta\phi\psi\chi}(1,0)]^{\mathrm{T}}\exp\{s_6(m_{\mathrm{W}},\mu_k)\tilde{\gamma}_0^{(6)}\}\,{}_6\mathcal{O}^{\theta\phi\psi\chi} = [V^+]_{\psi\phi}\,V_{\theta\chi}(b_i\,{}_6\mathcal{O}_1^{\theta\phi\psi\chi} + a_i\,{}_6\mathcal{O}_2^{\theta\phi\psi\chi}). \quad (83)$$

From equations (79) we thus conclude that

$$\mathcal{H}^{\Delta}_{\mathrm{eff}}(\mathrm{s},\mathrm{PF}) = 0 \quad (\text{no PF } \mathcal{H}_{\mathrm{eff}} \text{ for s decays}), \quad (84a)$$

$$\mathcal{H}^{\Delta}_{\mathrm{eff}}(\mathrm{b},\mathrm{PF}) = \sqrt{\tfrac{1}{2}}4G\sum_{\psi=1}^{2}\sum_{\substack{\theta,\phi=1\\ \theta\neq\phi}}^{2}[V^+]_{\psi\phi}\,V_{\theta 3}(b_5\,{}_6\mathcal{O}_1^{\theta\phi\psi 3} + a_5\,{}_6\mathcal{O}_2^{\theta\phi\psi 3}), \quad (84b)$$

$$\mathcal{H}^{\Delta}_{\mathrm{eff}}(\mathrm{c},\mathrm{PF}) = \sqrt{\tfrac{1}{2}}4G\sum_{\substack{\psi,\chi=1\\ \psi\neq\chi}}^{2}[V^+]_{\psi 2}\,V_{1\chi}(b_4\,{}_6\mathcal{O}_1^{12\psi\chi} + a_4\,{}_6\mathcal{O}_2^{12\psi\chi}), \quad (84c)$$

$$\mathcal{H}^{\Delta}_{\mathrm{eff}}(\mathrm{t},\mathrm{PF}) = \sqrt{\tfrac{1}{2}}4G\sum_{\theta=1}^{2}\sum_{\substack{\psi,\chi=1\\ \psi\neq\chi}}^{3}[V^+]_{\psi 3}\,V_{\theta\chi}(b_6\,{}_6\mathcal{O}_1^{\theta 3\psi\chi} + a_6\,{}_6\mathcal{O}_2^{\theta 3\psi\chi}); \quad (84d)$$

and for convenience the four quark operators are

$${}_k\mathcal{O}^{\theta\phi\psi\chi} = \begin{pmatrix}(\bar{q}^i_{+\theta}\,q^i_{+\phi})_{\mathrm{L}} & (\bar{q}^j_{-\psi}\,q^j_{-\chi})_{\mathrm{L}}\\ (\bar{q}^i_{+\theta}\,q^j_{+\phi})_{\mathrm{L}} & (\bar{q}^j_{-\psi}\,q^i_{-\chi})_{\mathrm{L}}\end{pmatrix}. \quad (85)$$

It is unnecessary to expand equations (84) in any greater detail. The relevant coefficients for a particular decay channel can be read off directly.

(*c*) *Numerical Results for* $\mathcal{H}^{\Delta}_{\mathrm{eff}}(\pm,\mathrm{PG})$

In evaluating $\mathcal{H}^{\Delta}_{\mathrm{eff}}(\pm,\mathrm{PG})$ the matrix notation that we have employed becomes especially convenient.

To summarize, the calculations with which we deal are:

(1) $\mathcal{H}^{\Delta}_{\mathrm{eff}}(-2,\mathrm{PG},3)$, penguin generating s decays where t, b and c are all regarded as heavy quarks. (Hence effectively three flavours remain.)
(2) $\mathcal{H}^{\Delta}_{\mathrm{eff}}(-2,\mathrm{PG},4)$, where only t and b are regarded as heavy quarks. (Effectively four flavours remain.)
(3) $\mathcal{H}^{\Delta}_{\mathrm{eff}}(+2,\mathrm{PG})$, penguin generating c decays with both b and t as heavy quarks.
(4) $\mathcal{H}^{\Delta}_{\mathrm{eff}}(-3,\mathrm{PG})$, penguin generating b decays, where t is regarded as a heavy quark.
(5) $\mathcal{H}^{\Delta}_{\mathrm{eff}}(+3,\mathrm{PG})$, penguin generating t decays.

The c quark is a borderline case when classifying quarks as either heavy or light. We calculate both cases (1) and (2) so that our results are useful whatever one's personal preference on the matter.

Equation (66) is the central theoretical result with which we deal. It can be written in the form

$$\mathcal{H}^{\Delta}_{\mathrm{eff}}(\pm\phi,\mathrm{PG}) = \sqrt{\tfrac{1}{2}}4G\sum_{\theta=1}^{\phi-1}\sum_{\psi=1}^{3}[A^{\theta\psi}_{\pm\phi}(1,0)]^{\mathrm{T}}E^{\theta\psi}_{\pm\phi}\,{}_{6-k}\mathcal{O}^{\theta\psi}_{\pm\phi}, \quad (86)$$

where $E^{\theta\psi}_{\pm\phi}$ represents the intervening 6×6 matrix products in (66); the numerical problem is to evaluate these matrices.

Preliminary to this is the evaluation of the necessary scalars $\tilde{s}_k$ and s_k. According to (66) we require (see Table 2)

$$s_6(m_W, \mu_k), \qquad k = 3, \ldots, 6. \tag{87}$$

Further, from (66) we need

$$\tilde{s}_l(m_l^{(l)}, \mu_k); \qquad l = 6, \ldots, 6-k+1, \quad k = 3, \ldots, 6, \tag{88}$$

which in turn implies (see equation 61) that we need $\tilde{g}_l(m_l^{(l)})$ and $\tilde{g}_l(\mu)$, $\mu = \mu_3, \ldots, \mu_6$. The latter may be calculated using (75) and (71c), while the former derive from (71c) and (70), where one observes that

$$\tilde{g}_l(m_l^{(l)}) = \bar{g}_l(Q_l), \qquad Q_l = \overline{m}_l^{(l)}(Q_l), \tag{89a, b}$$

and where Q_l may be calculated using $\overline{m}(Q)$ in (71a). In both cases we use at least third iteration of (71c). The $\tilde{s}_l$ of (88) so calculated are given in Table 3.

Table 3. Scalars $\tilde{s}_l(P, Q)$ required for $\mathscr{H}^{\Delta}_{\text{eff}}(\pm\phi, \text{PG})$

		μ_3			μ_4	μ_5
	$l = 6$	5	4	6	5	6
$\tilde{s}_l(m_l^{(l)}, \mu_k)$	−0·1252	−0·0814	−0·0366	−0·1064	−0·0573	−0·0598

From equation (66) the essential structure (from the viewpoint of programming) of $E^{\theta\psi}_{\pm\phi}$ is

$$E^{\theta\psi}_{\pm\phi} = \exp(s_0 G_6) \times B_1 \exp(s_1 G_5) \times \quad \ldots \quad \times B_k \exp(s_k G_{6-k}), \tag{90}$$

where G and B are matrices and s are scalars. For a given flavour decay, $\pm\phi$ is fixed. Equations (66) and (67) reveal that $E^{\theta\psi}_{\pm\phi}$ have *no* θ dependence, whereas they do depend upon ψ in general. So for each $\pm\phi$ there are three matrices to calculate corresponding to $\psi = 1$, 2 and 3.

The matrix products in (90) are easily evaluated by computer. We point out that we found it more convenient to exponentiate matrices by a simple Taylor series subroutine; this is both quick and allows one to control the size of errors. This procedure compares more favourably with the conventional method in the literature where, for each exponentiation, an eigenvalue/eigenvector analysis is performed. The latter method is in practice more cumbersome and the size of errors depends upon the stability of the problem.

Observe that the ψ dependence in (90) lies in the B_k matrices only (see equation 66). The B_k matrices take on only two forms (see equations 67c–67f):

$$B_k = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}. \tag{91}$$

The scalars in (90) have just been calculated. The matrix G_k is just $\tilde{\gamma}_0^{(k)}$ which may be calculated using (47). It is useful to compare the ease and flexibility of this matrix calculation with the rather formidable specific methods employed by Gilman and Wise (1979) and Wise (1980).

Only the second rows of $E^{\theta\psi}_{\pm\phi}$ are required (equations 67a and 67b). Calculated as described above, these second rows are presented in Table 4.

Our intention is to now reduce $\mathscr{H}^{\Delta}_{\text{eff}}(\pm\phi, \text{PG})$ to their most practically useful form. Using (67a) and (67b), equation (86) may be written as

$$\mathscr{H}^{\Delta}_{\text{eff}}(\pm\phi, \text{PG}) = \sqrt{\tfrac{1}{2}}4G \sum_{\theta=1}^{\phi-1} \sum_{\psi=1}^{3} C^{\theta\psi}_{\pm\phi,i}[{}_{6-k}\mathscr{O}^{\theta\psi}_{\pm\phi}]_i, \tag{92}$$

where we introduce the shorthand notation

$$C^{\theta\psi}_{+\phi,i} = V_{\theta\psi} V^{+}_{\psi\phi}(E^{\theta\psi}_{+\phi})_{2i}, \qquad C^{\theta\psi}_{-\phi,i} = V^{+}_{\theta\psi} V_{\psi\phi}(E^{\theta\psi}_{-\phi})_{2i}. \tag{93a, b}$$

Observe that when ψ in (92) corresponds to a heavy quark index, the $C^{\theta\psi}_{\pm\phi,1;2}$ vanish (see Table 4). This is exactly as promised earlier following equations (62) and (64).

Table 4. Program output for second rows of matrices $E^{\theta\psi}_{\pm\phi}$

				$(E^{\theta\psi}_{\phi\pm})_{2i}$			
$\phi\pm$	ψ	$i=1$	2	3	4	5	6
2−[A]	1,2	−0·50671	1·25922	0·02617	−0·05069	0·01316	−0·07584
	3	0	0	0·00363	−0·00515	0·00092	−0·01127
2−[B]	1	−0·50671	1·25922	0·02666	−0·05198	0·01356	−0·07725
	2	0	0	0·02099	−0·03712	0·00888	−0·06092
	3	0	0	0·00370	−0·00534	0·00098	−0·01147
2+	1,2	−0·42764	1·20925	0·02128	−0·04319	0·01164	−0·06090
	3	0	0	0·01244	−0·02154	0·00506	−0·03574
3−	1,2	−0·25071	1·10847	0·01118	−0·02570	0·00754	−0·03164
	3	0	0	0·00274	−0·00524	0·00135	−0·00765
3+	1,2,3	−0·05591	1·01999	0·00217	−0·00606	0·00196	−0·00635

[A] c is regarded as a light quark in s decay.
[B] c is regarded as a heavy quark in s decay.

Next we observe that $[{}_{6-k}\mathscr{O}^{\theta\psi}_{\pm\phi}]_i$, $i = 3, \ldots, 6$, are independent of ψ, and so we write them as $[{}_{6-k}\mathscr{O}^{\theta 1}_{\pm\phi}]_i$. Combining both these features we may write (92) in the form

$$\mathscr{H}^{\Delta}_{\text{eff}}(\pm\phi, \text{PG}) = \sqrt{\tfrac{1}{2}}4G \sum_{\theta=1}^{\phi-1} \left(\sum_{i=1}^{2} \sum_{\psi=1}^{m} C^{\theta\psi}_{\pm\phi,i}[{}_{6-k}\mathscr{O}^{\theta\psi}_{\pm\phi}]_i + \sum_{i=3}^{6} D^{\theta}_{\pm\phi,i}[{}_{6-k}\mathscr{O}^{\theta 1}_{\pm\phi}]_i \right), \tag{94}$$

where

$$D^{\theta}_{\pm\phi,i} = \sum_{\psi=1}^{3} C^{\theta\psi}_{\pm\phi,i},$$

and

$$\begin{aligned} m &= 1 \text{ for s decay with c heavy;} \quad 2 \text{ for s decay with c light;} \\ &= 2 \text{ for c decay;} \quad 2 \text{ for b decay;} \quad 3 \text{ for t decay.} \end{aligned} \tag{95}$$

We must now consider whether or not the operators remaining in (94) are linearly independent. Referring to (46) and (94) it is clear that $\{[{}_{k}\mathscr{O}^{\theta\psi}_{-\phi}]_i : i = 1,2; \psi = 1, \ldots, m; \theta = 1, \ldots, \phi-1\}$ are linearly independent. Secondly, it is clear that $[{}_{k}\mathscr{O}^{\theta 1}_{-\phi}]_i$ for $i = 5, 6$ may be added to this set and linear independence will be preserved; this follows from the L–R structure of these operators, compared with the initial set

being pure L–L. (Further, one cannot express an operator $[\mathcal{O}]_5$ in terms of an $[\mathcal{O}]_6$ and L–L operators by applying a Fierz transformation.) Thirdly, $[{}_k\mathcal{O}^{\theta 1}_{-\phi}]_3$ *or* $[{}_k\mathcal{O}^{\theta 1}_{-\phi}]_4$ may be added to the augmented set and linear independence will be preserved. This follows because of the appearance of $(\bar{q}_- q_-)_L(\bar{q}_- q_-)_L$ operators within them. So let us add $[{}_k\mathcal{O}^{\theta 1}_{-\phi}]_3$ to our set. From (46) it follows that (using the Fierz transformation)

$$[{}_k\mathcal{O}^{\theta\psi}_{-\phi}]_4 = [{}_k\mathcal{O}^{\theta\psi}_{-\phi}]_3 - (\bar{q}^i_{-\theta} q^i_{-\phi})_L \sum_{\substack{l=1 \\ l\neq 2\phi-1 \\ \neq 2\theta-1}}^{k} (\bar{q}^j_l q^j_l)_L + (\bar{q}^i_{-\theta} q^j_{-\phi})_L \sum_{\substack{l=1 \\ l\neq 2\phi-1 \\ \neq 2\theta-1}}^{k} (\bar{q}^j_l q^i_l)_L . \quad (96)$$

From this one can deduce that $[{}_k\mathcal{O}^{\theta\psi}_{-\phi}]_4$ may in general be added to our set and linear independence will be preserved, *except* in the case of s-decay where

(1) $k = 3$,

$$[{}_3\mathcal{O}^{1\psi}_{-2}]_4 = [{}_3\mathcal{O}^{1\psi}_{-2}]_3 - [{}_3\mathcal{O}^{11}_{-2}]_1 + [{}_3\mathcal{O}^{11}_{-2}]_2 ; \quad (97a)$$

(2) $k = 4$,

$$[{}_4\mathcal{O}^{1\psi}_{-2}]_4 = [{}_4\mathcal{O}^{1\psi}_{-2}]_3 - [{}_4\mathcal{O}^{11}_{-2}]_1 - [{}_4\mathcal{O}^{12}_{-2}]_1 + [{}_4\mathcal{O}^{11}_{-2}]_2 + [{}_4\mathcal{O}^{12}_{-2}]_2 . \quad (97b)$$

The operator basis just discussed was for q_- decay. For q_+ decay one finds the analogous result that the operators in (94) form a linearly independent set *except* in the case of c decay where

(1) $k = 4$,

$$[{}_4\mathcal{O}^{1\psi}_{+2}]_4 = [{}_4\mathcal{O}^{1\psi}_{+2}]_3 - [{}_4\mathcal{O}^{11}_{+2}]_1 - [{}_4\mathcal{O}^{12}_{+2}]_1 + [{}_4\mathcal{O}^{11}_{+2}]_2 + [{}_4\mathcal{O}^{12}_{+2}]_2 ; \quad (97c)$$

(2) $k = 5$,

$$[{}_5\mathcal{O}^{1\psi}_{+2}]_4 = [{}_5\mathcal{O}^{1\psi}_{+2}]_3 - [{}_5\mathcal{O}^{11}_{+2}[_1 - [{}_5\mathcal{O}^{12}_{+2}]_1 - [{}_5\mathcal{O}^{13}_{+2}]_1 + [{}_5\mathcal{O}^{11}_{+2}]_2 + [{}_5\mathcal{O}^{12}_{+2}]_2 + [{}_5\mathcal{O}^{13}_{+2}]_2 . \quad (97d)$$

Equation (97a) was first derived by Gilman and Wise (1979) and provoked the above generalizations.

We can now proceed to write down the final expressions for $\mathcal{H}^{\Delta}_{\text{eff}}(\pm\phi, \text{PG})$. For clarity we take the liberty of substituting s, c, b and t for $\pm\phi = -2, +2, -3, +3$ in the coefficients and in the arguments of $\mathcal{H}^{\Delta}_{\text{eff}}$. From (94) we have:

s decay, $-\phi = -2$

$$\mathcal{H}^{\Delta}_{\text{eff}}(\text{s}, \text{PG}, k) = \sqrt{\tfrac{1}{2}}\, 4G \left(\sum_{i=1}^{2} \sum_{\psi=1}^{k-2} C^{1\psi}_{\text{s}(k),i} [{}_k\mathcal{O}^{1\psi}_{-2}]_i + \sum_{i=3}^{6} D^{1}_{\text{s}(k),i} [{}_k\mathcal{O}^{11}_{-2}]_i \right), \quad (98a)$$

for $k = 3$ or 4, covering the cases where c is considered heavy or light.

c decay, $+\phi = +2$

$$\mathcal{H}^{\Delta}_{\text{eff}}(\text{c}, \text{PG}) = \sqrt{\tfrac{1}{2}}\, 4G \left(\sum_{i=1}^{2} \sum_{\psi=1}^{2} C^{1\psi}_{\text{c},i} [{}_4\mathcal{O}^{1\psi}_{+2}]_i + \sum_{i=3}^{6} D^{1}_{\text{c},i} [{}_4\mathcal{O}^{11}_{+2}]_i \right). \quad (98b)$$

b decay, $-\phi = -3$

$$\mathcal{H}^{\Delta}_{\text{eff}}(\text{b}, \text{PG}) = \sqrt{\tfrac{1}{2}}\, 4G \sum_{\theta=1}^{2} \left(\sum_{i=1}^{2} \sum_{\psi=1}^{2} C^{\theta\psi}_{\text{b},i} [{}_5\mathcal{O}^{\theta\psi}_{-3}]_i + \sum_{i=3}^{6} D^{\theta}_{\text{b},i} [{}_5\mathcal{O}^{\theta 1}_{-3}]_i \right). \quad (98c)$$

t decay, $+\phi = +3$

$$\mathscr{H}^{\Delta}_{\text{eff}}(\text{t, PG}) = \sqrt{\tfrac{1}{2}}4G \sum_{\theta=1}^{2}\left(\sum_{i=1}^{2}\sum_{\psi=1}^{3} C^{\theta\psi}_{\text{t},i}[{}_6\mathscr{O}^{\theta\psi}_{+3}]_i + \sum_{i=3}^{6} D^{\theta}_{\text{t},i}[{}_6\mathscr{O}^{\theta 1}_{+3}]_i\right). \tag{98d}$$

Numerical values of the coefficients are given in Tables 5–8.

There is no point in expanding equations (98) in detail as we may readily extract appropriate coefficients for a given process by inspection using (46). It is important to draw the reader's attention to the fact that V in the tables refers to the $n \times n$ mixing matrix of the original complete theory.

Table 5. Coefficients of operator expansion in (98a) for penguin generating Hamiltonian for s decays

i	ψ	Coefficient
		(*a*) *Heavy* c *quark decoupled from theory*
		$C^{1\psi}_{\text{s}(3),i}$
1	1	$-0{\cdot}4547[V^+]_{11}V_{12} + 0{\cdot}0371[V^+]_{12}V_{22} + 0{\cdot}0053[V^+]_{13}V_{32}$
2	1	$1{\cdot}2072[V^+]_{11}V_{12} - 0{\cdot}0371[V^+]_{12}V_{22} - 0{\cdot}0053[V^+]_{13}V_{32}$
		$D^{1}_{\text{s}(3),i}$
3		$-0{\cdot}0253[V^+]_{11}V_{12} - 0{\cdot}0161[V^+]_{12}V_{22} - 0{\cdot}0016[V^+]_{13}V_{32}$
4		0
5		$0{\cdot}0136[V^+]_{11}V_{12} + 0{\cdot}0089[V^+]_{12}V_{22} + 0{\cdot}0010[V^+]_{13}V_{32}$
6		$-0{\cdot}0773[V^+]_{11}V_{12} - 0{\cdot}0609[V^+]_{12}V_{22} - 0{\cdot}0115[V^+]_{13}V_{32}$
		(*b*) *Light* c *quark not decoupled from theory*
		$C^{1\psi}_{\text{s}(4),i}$
1	1	$-0{\cdot}4560[V^+]_{11}V_{12} + 0{\cdot}0507[V^+]_{12}V_{22} + 0{\cdot}0052[V^+]_{13}V_{32}$
	2	$0{\cdot}0507[V^+]_{11}V_{12} - 0{\cdot}4560[V^+]_{12}V_{22} + 0{\cdot}0052[V^+]_{13}V_{32}$
2	1	$1{\cdot}2085[V^+]_{11}V_{12} - 0{\cdot}0507[V^+]_{12}V_{22} - 0{\cdot}0052[V^+]_{13}V_{32}$
	2	$-0{\cdot}0507[V^+]_{11}V_{12} + 1{\cdot}2085[V^+]_{12}V_{22} - 0{\cdot}0052[V^+]_{13}V_{32}$
		$D^{1}_{\text{s}(4),i}$
3		$-0{\cdot}0245[V^+]_{11}V_{12} - 0{\cdot}0245[V^+]_{12}V_{22} - 0{\cdot}0015[V^+]_{13}V_{32}$
4		0
5		$0{\cdot}0132[V^+]_{11}V_{12} + 0{\cdot}0132[V^+]_{12}V_{22} + 0{\cdot}0009[V^+]_{13}V_{32}$
6		$-0{\cdot}0758[V^+]_{11}V_{12} - 0{\cdot}0758[V^+]_{12}V_{22} - 0{\cdot}0113[V^+]_{12}V_{32}$

(*d*) *Two Comparisons with Other Work*

We now give two illustrative comparisons, one in the penguin free sector and the other in the penguin generating sector of the theory. The comparisons given are those presently considered the more important.

Over the past few years the effective weak non-leptonic Hamiltonian describing 'Cabibbo favoured' charm decay has been intensively studied phenomenologically. It belongs to the penguin free sector of the theory. We note that equation (84c) consists of two parts: $(\psi, \chi) = (2, 1)$ which corresponds to 'Cabibbo favoured' (CF), and $(\psi, \chi) = (1, 2)$ which corresponds to 'Cabibbo double suppressed' charm decay. Thus, in detail we have

$$\mathscr{H}^{\Delta c=-1}_{\text{eff CF}} = \sqrt{\tfrac{1}{2}}4G[V^+]_{22}V_{11}\{b_4(\bar{\text{u}}^i\text{c}^i)_{\text{L}}(\bar{\text{s}}^j\text{d}^j)_{\text{L}} + a_4(\bar{\text{u}}^i\text{c}^j)_{\text{L}}(\bar{\text{s}}^j\text{d}^i)_{\text{L}}\}. \tag{99}$$

The common notation has been to define $f_\pm$ coefficients (Cabibbo and Maiani 1978), which relate to a_4 and b_4 via

$$f_\pm = a_4 \pm b_4 . \tag{100}$$

Cabibbo and Maiani estimate that

$$f_+ = 0{\cdot}68, \qquad f_- = 2{\cdot}15, \tag{101}$$

based upon a six quark model and a c quark mass subtraction point. So a comparison with our results does make sense.

We find (from Table 2)

$$f_+ = 0{\cdot}78, \qquad f_- = 1{\cdot}64. \tag{102}$$

Table 6. Coefficients of operator expansion in (98b) for penguin generating Hamiltonian for c decays

i	ψ	Coefficient
		$C^{1\psi}_{c,i}$
1	1	$-0{\cdot}3845V_{11}[V^+]_{12} + 0{\cdot}0432V_{12}[V^+]_{22} + 0{\cdot}0215V_{13}[V^+]_{32}$
	2	$0{\cdot}0432V_{11}[V^+]_{12} - 0{\cdot}3845V_{12}[V^+]_{22} + 0{\cdot}0215V_{13}[V^+]_{32}$
2	1	$1{\cdot}1661V_{11}[V^+]_{12} - 0{\cdot}0432V_{12}[V^+]_{22} - 0{\cdot}0215V_{13}[V^+]_{32}$
	2	$-0{\cdot}0432V_{11}[V^+]_{12} + 1{\cdot}1661V_{12}[V^+]_{22} - 0{\cdot}0215V_{13}[V^+]_{32}$
		$D^1_{c,i}$
3		$-0{\cdot}0219V_{11}[V^+]_{12} - 0{\cdot}0219V_{12}[V^+]_{22} - 0{\cdot}0091V_{13}[V^+]_{32}$
4		0
5		$0{\cdot}0116V_{11}[V^+]_{12} + 0{\cdot}0116V_{12}[V^+]_{22} + 0{\cdot}0051V_{13}[V^+]_{32}$
6		$-0{\cdot}0609V_{11}[V^+]_{12} - 0{\cdot}0609V_{12}[V^+]_{22} - 0{\cdot}0357V_{13}[V^+]_{32}$

Table 7. Coefficients of operator expansion in (98c) for penguin generating Hamiltonian for b decays

Here $\theta = 1$ or 2

i	ψ	Coefficient
		$C^{\theta\psi}_{b,i}$
1	1	$-0{\cdot}2507[V^+]_{\theta 1}V_{13}$
	2	$-0{\cdot}2507[V^+]_{\theta 2}V_{23}$
2	1	$1{\cdot}1085[V^+]_{\theta 1}V_{13}$
	2	$1{\cdot}1085[V^+]_{\theta 2}V_{23}$
		$D^\theta_{b,i}$
3		$0{\cdot}0112[V^+]_{\theta 1}V_{13} + 0{\cdot}0112[V^+]_{\theta 2}V_{23} + 0{\cdot}0027[V^+]_{\theta 3}V_{33}$
4		$-0{\cdot}0257[V^+]_{\theta 1}V_{13} - 0{\cdot}0257[V^+]_{\theta 2}V_{23} - 0{\cdot}0052[V^+]_{\theta 3}V_{33}$
5		$0{\cdot}0075[V^+]_{\theta 1}V_{13} + 0{\cdot}0075[V^+]_{\theta 2}V_{23} + 0{\cdot}0014[V^+]_{\theta 3}V_{33}$
6		$-0{\cdot}0316[V^+]_{\theta 1}V_{13} - 0{\cdot}0316[V^+]_{\theta 2}V_{23} - 0{\cdot}0077[V^+]_{\theta 3}V_{33}$

The difference is due to our numerical approach (see Section 4*a*) and reflects a decrease in the QCD effect which introduces deviations from $f_\pm = 1$. Roughly speaking this can be traced to the smallness of $\Lambda^2_{5(\overline{\text{ms}})}$ employed in this paper. (Cabibbo

and Maiani in fact did not parametrize $\alpha(\mu)$, but simply used an (over)estimate of $\alpha(m_c)$—this in fact corresponds to using an unacceptably large value of $\Lambda^2_{5(\overline{\text{ms}})}$.)

In the penguin generating sector of the theory we compare our results with the work of Gilman and Wise (1979) for strange decays. To make the comparison we must convert our notation to theirs.

Table 8. Coefficients of operator expansion of (98d) for penguin generating Hamiltonian for t decays

Here $\psi = 1$, 2 or 3 and $\theta = 1$ or 2. Note that in a six quark model the coefficients $D^{\theta}_{t,i}$ vanish for the relevant $\theta = 1$ and 2 values by unitarity of V

i	Coefficient
	$C^{\theta\psi}_{t,i}$
1	$-0{\cdot}0559 V_{\theta\psi}[V^+]_{\psi 3}$
2	$1{\cdot}0200 V_{\theta\psi}[V^+]_{\psi 3}$
	$D^{\theta}_{t,i}$
3	$0{\cdot}0022(V_{\theta 1}[V^+]_{13} + V_{\theta 2}[V^+]_{23} + V_{\theta 3}[V^+]_{33})$
4	$-0{\cdot}0061(V_{\theta 1}[V^+]_{13} + V_{\theta 2}[V^+]_{23} + V_{\theta 3}[V^+]_{33})$
5	$0{\cdot}0020(V_{\theta 1}[V^+]_{13} + V_{\theta 2}[V^+]_{23} + V_{\theta 3}[V^+]_{33})$
6	$-0{\cdot}0064(V_{\theta 1}[V^+]_{13} + V_{\theta 2}[V^+]_{23} + V_{\theta 3}[V^+]_{33})$

Table 9. Comparison of C_i coefficients for penguin generating s decays

i	C_i Present work[A]	C_i Gilman and Wise (1979)
1	$-0{\cdot}492+0{\cdot}032\tau$	$-0{\cdot}760+0{\cdot}026\tau$
2	$1{\cdot}244-0{\cdot}032\tau$	$1{\cdot}420-0{\cdot}026\tau$
3	$-0{\cdot}009-0{\cdot}015\tau$	$-0{\cdot}025-0{\cdot}008\tau$
5	$0{\cdot}005+0{\cdot}008\tau$	$0{\cdot}013+0{\cdot}005\tau$
6	$-0{\cdot}016-0{\cdot}050\tau$	$-0{\cdot}065-0{\cdot}060\tau$

[A] Results obtained from equation (105).

Gilman and Wise work in a six quark model with a Kobayashi–Maskawa (1973) mixing matrix parametrization; they regard c as a heavy quark. They define operators

$$Q_i = 4[{}_3\mathcal{O}^{11}_{-2}]_i, \qquad i = 1,2,3,5,6, \tag{103}$$

and a ratio

$$\tau = -[V^+]_{13} V_{32}/[V^+]_{11} V_{12} = s_2^2 + s_2 c_2 s_3 e^{i\delta}/c_1 c_3, \tag{104}$$

where $[V^+]_{11} V_{12} = -c_1 s_1 c_3$ in the Kobayashi–Maskawa parametrization. Equation (98a) can then be written as (for c heavy)

$$\mathscr{H}^{\Delta}_{\text{eff}}(s, \text{PG}, 3) = -\sqrt{\tfrac{1}{2}}\, G s_1 c_1 c_3 \sum_{\substack{i=1 \\ \neq 4}}^{6} C_i Q_i. \tag{105}$$

Gilman and Wise allow for broad variations in $\alpha_3(\mu)$, Λ^2 and m_t. The most 'sensible' comparison with our work is made with their C_i coefficients for $\alpha_3(\mu) = 0{\cdot}75$,

$\Lambda^2 = 0{\cdot}01\ \mathrm{GeV}^2$ and $m_t = 30\ \mathrm{GeV}$. The comparisons are given in Table 9. One will note the overall agreement in sign and order of magnitudes, but as the differences are not negligible they will obviously have physical implications (see Miller and McKellar 1981*c*).

5. Conclusions

In this paper we have presented a complete analysis of flavour changing effective weak non-leptonic Hamiltonians for processes below the W threshold. Our approach stresses the unity of these processes.

We have continually referred to the GLAM–GW method as a way of decoupling in the weak non-leptonic sector of the (3, 2, 1) model for we feel this is how one should view these calculations, setting them in perspective with the recent rapid advances in our understanding of effective field theory.

The main features of our work are

(1) a generalization beyond a six quark model;
(2) a complete matrix formalism of the theory;
(3) a general and uniform theoretical treatment applicable to s, c, b and t quark weak non-leptonic processes;
(4) a detailed numerical analysis combining scheme dependence in parametrizing running coupling constants and fermion mass;
(5) an inclusion of a full effective theory dependence.

The next step will be an analogous investigation of the flavour conserving weak non-leptonic processes, which we hope to present in a subsequent paper.

From here one might consider an extension to the two loop level. Concrete progress has been made in this direction by Altarelli *et al.* (1981*a*, 1981*b*) in the penguin free sector of the weak non-leptonic theory.

Undoubtedly the greatest difficulties which remain in this area concern the evaluation of four quark operator matrix elements between bound state systems.

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Appendix 1. Calculating Anomalous Dimensions of Four Quark Operators

In this appendix we indicate how the anomalous dimension matrices of four quark operators may be derived at the one loop level. (A full and general account covering the more subtle details may be found in Miller and McKellar 1981*d*.)

Dimensional regularization coupled with ms form a natural choice when calculating RG functions, for in this scheme they are mass independent and we may use massless QCD (except of course when we wish to calculate Z_m!). The RG functions of $\overline{\text{ms}}$ and ms are of course the same. The following work is presented for the Landau ($\alpha = 0$) gauge.

We denote by $\mathcal{O}_1$ the four quark operator

$$\mathcal{O}_1 = (\bar{\psi}^i_{A'}\psi^j_{B'})_L(\bar{\psi}^j_{C'}\psi^j_{D'})_L\,, \tag{A1}$$

using our standard notation (see equation 2), and where A', B', C' and D' are unspecified flavour indices. The anomalous dimension calculation commences by inserting $\mathcal{O}_1^0$ into $G_0^{(4\psi)}$ (the unrenormalized amputated Green function with four external quark legs), as illustrated in Fig. 2. The insertion is denoted by $G_0^{(4\psi)}(\mathcal{O}_1^0)$.

Now an operator need not renormalize multiplicatively under QCD renormalization. We can anticipate this by writing $\mathcal{O}_1$ as the first component of a vector $\mathcal{O}$ and thus defining the matrix renormalization constant by

$$\mathcal{O}^0 = \boldsymbol{Z}\mathcal{O}\,. \tag{A2}$$

Of course this includes the possibility that $\boldsymbol{Z}$ is 1×1: we have *not* assumed anything at this stage, the arguments are strictly deductive.

It follows that

$$G_0^{(4\psi)}(\mathcal{O}^0) = Z_{2\psi}^{-2}\,\boldsymbol{Z}G^{(4\psi)}(\mathcal{O})\,, \tag{A3}$$

where $Z_{2\psi}^{\frac{1}{2}}$ is the quark wavefunction renormalization constant of QCD. We define $\boldsymbol{Z}^{(2)}$ by

$$\boldsymbol{Z} = \mathbf{1} + \boldsymbol{Z}^{(2)} + O(g^4)\,. \tag{A4}$$

Then for ms the following is true:

$$G^{(4\psi)}_{0(1)}(\mathcal{O}^0_i)_{\mathrm{p}} = Z^{(2)}_{ij}\, G^{(4\psi)}_{(0)}(\mathcal{O}_j) + O(g^4)\,. \tag{A5}$$

The LHS is the $1/\varepsilon$ pole part of the one loop contribution to $G^{(4\psi)}_0(\mathcal{O}^0_i)$ (and hence the suffixes p and (1)). The $G^{(4\psi)}_{(0)}(\mathcal{O}_j)$ are renormalized zero loop terms and, for $\mathcal{O}_j$ a four quark operator, they are simply its vertex factor.

One major advantage of ms is that one loop pole terms are easily evaluated, and may be done using *Dirac algebra in four dimensions*. To see this, suppose we are working in $n = 4+\varepsilon$ dimensions. We are only interested (at the one loop level) in the coefficient of the $1/\varepsilon$ pole. So once the $\Gamma(\frac{1}{2}\varepsilon)$ factor is isolated from the divergent integrals we may evaluate what remains in four dimensions. This trick was independently conjured by Altarelli *et al.* (1981*a*).

The term $G^{(4\psi)}_{0(1)}(\mathcal{O}^0_1)_{\mathrm{p}}$ when evaluated is substituted into (A5) and one reads off directly the matrix elements $Z^{(2)}_{ij}$ and the vertex factors of newly generated operators (if any). The operators responsible for these vertex factors are easily reconstructed. One may of course lump all vertex factors together at this stage to define one new operator $\mathcal{O}_2$, however, since $\mathcal{O}_2$ itself will have to be inserted in turn into $G^{(4\psi)}_0$, it is best to define a minimum number of new operators each with distinct chiral and colour structures.

The process is repeated with $\mathcal{O}_2$ and then again with all newly generated operators, the renormalization set and $\boldsymbol{Z}^{(2)}$ matrix elements being revealed deductively.

It is sufficient to point out at this stage that four quark operators only generate four quark operators (see Miller and McKellar 1981*d* and Dawson *et al.* 1981 for a discussion on this point).

The renormalization set is guaranteed to close at any finite loop order (a combinatorial problem determines its largest possible size). The anomalous dimension is defined by

$$\boldsymbol{\gamma} = \mu \boldsymbol{Z}^{-1}\, \mathrm{d}\boldsymbol{Z}/\mathrm{d}\mu\,. \tag{A6}$$

In ms and with dimensional regularization in $n = 4+\varepsilon$ dimensions, this implies

$$\boldsymbol{\gamma} = \varepsilon \boldsymbol{Z}^{(2)} + O(g^4)\,. \tag{A7}$$

Two cases relevant to the present paper are now presented as though calculated with an effective theory containing k flavours.

(1) *Penguin free four quark operators*; $(A' \neq B') \neq (C' \neq D')$

(The Gaillard–Lee (1974) and Altarelli–Maiani (1974) calculation.)

The penguin terms of Fig. 2 vanish here because of a flavour conserving $\delta_{FF'}$ factor in the quark–gluon vertex factor. One finds

$${}_k\mathcal{O} = \begin{pmatrix} (\bar{\psi}^i_{A'}\psi^i_{B'})_{\mathrm{L}} & (\bar{\psi}^j_{C'}\psi^j_{D'})_{\mathrm{L}} \\ (\bar{\psi}^i_{A'}\psi^j_{B'})_{\mathrm{L}} & (\bar{\psi}^j_{C'}\psi^i_{D'})_{\mathrm{L}} \end{pmatrix} \tag{A8}$$

forms a closed renormalization set at the one loop level, with

$$\boldsymbol{\gamma}^{(k)} = \frac{g_k^2}{8\pi^2}\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix} + O(g_k^4)\,. \tag{A9}$$

(This we confirm is a gauge independent result.) Altarelli *et al.* (1981*a*, 1981*b*) have recently performed the two loop calculation for this case.

(2) *Penguin generating four quark operators*; $(A' \neq B') \neq (C' = D')$

(The Gilman and Wise (1979) and Wise (1980) calculation.)

Here the penguin diagrams do contribute. One finds that the closed renormalization set is given by

$$ {}_k\mathcal{O} = \begin{bmatrix} (\bar{\psi}^i_{A'}\psi^i_{B'})_L & (\bar{\psi}^j_{C'}\psi^j_{C'})_L \\ (\bar{\psi}^i_{A'}\psi^j_{B'})_L & (\bar{\psi}^j_{C'}\psi^i_{C'})_L \\ (\bar{\psi}^i_{A'}\psi^i_{B'})_L & \sum_{l=1}^{k} (\bar{q}^j_l q^j_l)_L \\ (\bar{\psi}^i_{A'}\psi^j_{B'})_L & \sum_{l=1}^{k} (\bar{q}^j_l q^i_l)_L \\ (\bar{\psi}^i_{A'}\psi^i_{B'})_L & \sum_{l=1}^{k} (\bar{q}^j_l q^j_l)_R \\ (\bar{\psi}^i_{A'}\psi^j_{B'})_L & \sum_{l=1}^{k} (\bar{q}^j_l q^i_l)_R \end{bmatrix} . \tag{A10} $$

Here q_l represents the components of $\boldsymbol{q}$ (where $\boldsymbol{q}^T$ is given by equation 7 with $2n$ replaced by k). For $\boldsymbol{\gamma}^{(k)}$ we obtain the expression (47). Note that although in certain cases the operators ${}_k\mathcal{O}_1, \ldots, {}_k\mathcal{O}_6$ are not linearly independent (see equations 97) *this in no way affects the validity of* (47).

Appendix 2. Remarks on Order of Decoupling Heavy Fields

We consider the process of starting with some complete theory and proceeding to decouple the defined heavy fields to obtain the effective light field theory. From the path integral formalism of decoupling (see Ovrut and Schnitzer 1981*a*) it is not difficult to see that the heavy fields may be integrated out of the theory in any order (including simultaneous decoupling of a number or all of them). At the perturbative level using the GLAM–GW method of decoupling this does not appear to be true. Here we illustrate this by considering the effects of decoupling one heavy quark and the W boson in both possible orders, using the GLAM–GW method.

In Section 2*a* decoupling of heavy quarks before the decoupling of W is discussed, whereas in Section 3 we discussed the decoupling of heavy quarks after decoupling W. So we are familiar with what happens in both cases. To simplify matters we consider the same calculations as in the text but with a model where only q_{2n} and W are the heavy fields of the problem.

Restricting attention to the penguin generating $-\phi$ sector of the theory we find that when the W and then the heavy quark are decoupled

$$ \mathcal{H}^{\Delta}_{\text{eff}}(-\phi, \text{PG}) = \sqrt{\tfrac{1}{2}}4G \sum_{\theta=1}^{\phi-1} \sum_{\psi=1}^{n} [\boldsymbol{A}^{\theta\psi}_{-\phi}(1,0)]^T \exp[\{s_{2n}(m_W, \mu) - \tilde{s}_{2n}(m^{(2n)}_{2n}, \mu)\}\tilde{\boldsymbol{\gamma}}^{(2n)}_0] $$
$$ \times \boldsymbol{B}^{\theta\psi}_{2n-1,-\phi}(1,0) \exp\{\tilde{s}_{2n}(m^{(2n)}_{2n}, \mu)\tilde{\boldsymbol{\gamma}}^{(2n-1)}_0\}_{2n-1} \mathcal{O}^{\theta\psi}_{-\phi} . \tag{A11} $$

All definitions can be found in Section 3*b*.

Decoupling the heavy particles in reverse order leads to the result

$$\mathscr{H}^{\Delta}_{\text{eff}}(-\phi, \text{PG}) = \sqrt{\tfrac{1}{2}}4G \sum_{\theta=1}^{\phi-1} \sum_{\psi=1}^{n-1} [A^{\theta\psi}_{-\phi}(1,0)]^{\text{T}} \exp\{s_{2n-1}(m_{\text{W}}, \mu)\tilde{\gamma}_0^{(2n-1)}\}_{2n-1} \mathscr{O}^{\theta\psi}_{-\phi}. \tag{A12}$$

It is clear that (A11) and (A12) are not the same result. For this reason, in the text we decouple heavy fields in a strictly mass ordered fashion, this being the more logical approach.

Appendix 3. Minimal or Modified Minimal Subtraction as a Renormalization Scheme

The conventional subtraction point choice where μ^2 is chosen to $O(q^2)$ with q^2 a typical external momentum has long been known to be more appropriate for both mom subtraction and $\overline{\text{ms}}$ than for ms.

Let us consider a conventional μ choice reasonably appropriate to $\overline{\text{ms}}$. The purpose of such a choice is to reduce the importance of higher order terms which appear in operator matrix elements roughly in the form $\{\ln(\mu^2/q^2)\}^n$. However, we point out that, with ms, μ^2 dependence appears in the form $\{\ln(\hat{\mu}^2/q^2)\}^n$, where

$$\hat{\mu}^2 = 4\pi \exp(-\gamma_{\text{E}})\mu^2 \equiv E\mu^2, \tag{A13}$$

γ_{E} being Euler's constant.

Thus we argue that if a conventional μ choice is appropriate to $\overline{\text{ms}}$ then to be *consistent* a μ choice for ms must be made such that $\hat{\mu}^2 = O(q^2)$. We therefore suggest:

(1) $\overline{\text{ms}}$; a conventional μ choice is appropriate:

$$\mu^2 = q^2. \tag{A14}$$

(2) ms; an appropriate μ choice is given by

$$\hat{\mu}^2 = q^2. \tag{A15}$$

The factor difference of $E = 7{\cdot}092$ is sizable, despite the vagueness in what we take for q (which is reasonably well defined in decay processes).

With an appropriate μ choice as in (A14) and (A15), we can now show that perturbation theory for physical processes (and operator coefficients!) yields identical results for either ms or $\overline{\text{ms}}$ renormalization schemes.

Let $\Gamma_{\text{ms}}(\mu^2, \bar{g}_{\text{ms}}(\mu^2), \bar{m}_{\text{ms}}(\mu^2), q^2)$ be an ms operator matrix element evaluated perturbatively, where q^2 is a typical external momentum, and $\bar{m}$, $\bar{g}$ and μ indicate explicit running mass, gauge coupling and μ dependence (we assume $\alpha = 0$). Here μ is chosen in accordance with (A15). Thus, our physical matrix element is

$$\Gamma_{\text{ms}}(q^2/E, \bar{g}_{\text{ms}}(q^2/E), \bar{m}_{\text{ms}}(q^2/E), q^2). \tag{A16}$$

However, we observe that (see 71c)

$$\frac{1}{\bar{g}^2_{\text{ms}}(q^2/E)} = R \ln\left(\frac{q^2}{E\Lambda^2_{\text{ms}}}\right) + T \ln\left(\frac{1}{R\bar{g}^2_{\text{ms}}(q^2/E)}\right),$$

yet $\Lambda^2_{\overline{\rm ms}} = E\Lambda^2_{\rm ms}$ and β coefficients are common to both ms and $\overline{\rm ms}$, and thus

$$\bar{g}^2_{\rm ms}(q^2/E) = \bar{g}^2_{\overline{\rm ms}}(q^2). \tag{A17}$$

Since

$$\overline{m}_{\rm ms}(q^2/E) \equiv \overline{m}_{\rm ms}(\bar{g}_{\rm ms}(q^2/E)) = \overline{m}_{\rm ms}(\bar{g}_{\overline{\rm ms}}(q^2)) \equiv \overline{m}_{\overline{\rm ms}}(q^2) \tag{A18}$$

(γ_m and β coefficients are common to both ms and $\overline{\rm ms}$), the matrix element (A16) becomes

$$\Gamma_{\rm ms}(q^2/E, \bar{g}_{\overline{\rm ms}}(q^2), \overline{m}_{\overline{\rm ms}}(q^2), q^2) \equiv \Gamma_{\overline{\rm ms}}(q^2, \bar{g}_{\overline{\rm ms}}(q^2), \overline{m}_{\overline{\rm ms}}(q^2), q^2),$$

because the only functional difference between $\Gamma_{\rm ms}$ and $\Gamma_{\overline{\rm ms}}$ is in the log terms and the E factor compensates for this. Therefore we have

$$\Gamma_{\rm ms}(\mu^2, \bar{g}_{\rm ms}(\mu^2), \overline{m}_{\rm ms}(\mu^2), q^2)\big|_{\hat{\mu}2=q2} = \Gamma_{\overline{\rm ms}}(\mu^2, \bar{g}_{\overline{\rm ms}}(\mu^2), \overline{m}_{\overline{\rm ms}}(\mu^2), q^2)\big|_{\mu^2=q^2}, \tag{A19}$$

and thus an appropriate μ choice for ms or $\overline{\rm ms}$ schemes yields the same physical results. The above argument is equally valid for operator coefficients (which have no explicit dependence on μ or q).

In our recent Letter (Miller and McKellar 1982) a conventional μ choice was made for ms in line with Gilman and Wise (1979). However, in the present paper we have resorted to an appropriate μ choice as dictated by (A14) and (A15). It is partially for this reason that the tables in the Letter may now be reinterpreted as being calculated for $\overline{\rm ms}$ rather than ms (with $\Lambda_{5(\overline{\rm ms})} = 0{\cdot}35$ GeV).

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