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Fluxoid Dynamics in Relativistically Streaming Plasmas

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Abstract

When particle inertia causes the magnetohydrodynamic flux-freezing condition to fail, a flux-vorticity conservation theorem can be valid: for relativistically streaming, multifluid, pressure-free plasmas, each species has a 'fluxoid' that is frozen-in to its motion. Here, fluxoid dynamics is treated with particular reference to the effects of scalar partial pressure gradients corresponding to non-relativistic thermal speeds. Several forms of the theorem are given, including explicitly covariant ones valid in general relativity. The circumstances under which the separate contributions to the fluxoid law have significant effects on fluxoid dynamics are made clear by order-of-magnitude analysis. The case of steadily rotating systems is developed and used to investigate the applicability of the fluxoid theorem to pulsar magnetospheres.

1. Introduction

Ohm's law, with scalar conductivity, is obtained for multifluid plasmas after neglecting pressure gradients and acceleration differences between the species (Burman *et al.* 1976). When the conductivity is effectively infinite, Alfvén's magnetohydrodynamic 'flux-freezing' theorem follows. In magnetohydrodynamics, Ampère's law is used, the displacement current being neglected. However, the magnitude of the electric current density is limited because the particle speeds are bounded by c, the free-space speed of light. If the maximum electric current density that the medium can provide is inadequate to maintain the curl of the magnetic field, then the displacement current cannot be neglected. Furthermore, Ohm's law is then invalid: acceleration of the plasma particles occurs (Swann 1962*a*, 1962*b*; Syrovat-skii 1966 and (as Syrovatsky) 1970) and their inertial effects must be included in the description; flux is not then frozen-in. If the electric current is carried essentially by particles of charge q and number density N, and if the magnetic field $\boldsymbol{B}$ varies on a characteristic length scale L, then Ampère's law fails when $B/L \gtrsim 4\pi N|q|$. It should be noted that Alfvén (1968, 1971) has emphasized that, for various reasons, the flux-freezing theorem is often inapplicable in both laboratory and astrophysical plasmas.

Because of the above considerations, Buckingham *et al.* (1972, 1973) developed a generalization of the flux-freezing theorem that fully incorporates relativistic inertial effects in multifluid plasmas. The component species were represented as pressure-free ideal gases that interact through their mutual electromagnetic field but do not interact mechanically, except that hydrodynamic sinks were incorporated for each fluid to allow for processes such as recombination. For each fluid, a flux-vorticity or 'fluxoid' conservation theorem was obtained that generalizes both the vorticity conservation theorem of hydrodynamics and the flux conservation theorem of magnetohydrodynamics. For each fluid there is a fluxoid that is frozen-in to the motion of that fluid. Both differential and integral forms of the theorem were obtained. Later, explicitly covariant forms were given (Burman 1977).

Wright (1978) has discussed various aspects of magnetic and inertial effects, including the flux-vorticity theorem, in pressure-free charge-separated plasmas, with particular reference to pulsar magnetospheres.

The fluxoid theorem should be useful in the description of plasmas in which particle acceleration is occurring, a common and important phenomenon in the Universe. However, the theorem needs further development in order to establish the significance of physical effects neglected so far. The most significant restriction in previous work is the omission of pressure effects: in many applications, pressure gradients will be important in the plasma dynamics. In this paper, the effects of partial pressure gradients on the flux-vorticity or fluxoid theorem will be studied. The mean thermal speeds are restricted to being nonrelativistic; hence only pressure *gradients*, not the partial pressures themselves, enter the dynamics. This is not a significant restriction for most applications, since only for temperatures greater than about 10^{10} K are electron thermal speeds highly relativistic. It will be shown that the neglect of pressure gradient effects in the fluxoid theorem in its application to pulsar magnetospheres (Wright 1978; Burman and Mestel 1978, 1979; Mestel *et al.* 1979; Mestel 1980; Burman 1980) appears to be valid.

2. The Fluxoid Theorem

Consider relativistic plasmas consisting of an arbitrary number of species, each of which is regarded as a fluid. Viscosity and heat flow will be neglected. Attention will be restricted to fluids that are at nonrelativistic temperatures: the mean thermal speed of the particles of any species is nonrelativistic so that $p \ll \rho_0 c^2$, where p and ρ_0 are the pressure and the proper mass density of that species. Thus, effects of pressure gradients will be included, but the pressure itself will not enter the dynamics.

Consider a representative species in the plasma. Let $\boldsymbol{v}$ denote its fluid 3-velocity and γ the corresponding Lorentz factor $(1-\boldsymbol{v}^2/c^2)^{-\frac{1}{2}}$. The charge and rest mass of its particles are denoted by e and m_0, while n and n_0 represent their number density and proper number density, connected by $n = \gamma n_0$. The equation of motion of this component fluid is

$$(\partial/\partial t + \boldsymbol{v}\,.\,\nabla)\boldsymbol{p} = e(\boldsymbol{E} + c^{-1}\,\boldsymbol{v}\times\boldsymbol{B}) - n^{-1}\nabla p, \tag{1}$$

where t represents time in an inertial frame, $\boldsymbol{p} \equiv \gamma m_0\,\boldsymbol{v}$ and $\boldsymbol{E}$ is the electric field.

Since we have $\boldsymbol{\alpha}\times(\nabla\times\boldsymbol{\alpha}) \equiv \frac{1}{2}\nabla(\boldsymbol{\alpha}^2) - \boldsymbol{\alpha}\,.\,\nabla\boldsymbol{\alpha}$ for any vector $\boldsymbol{\alpha}$, it follows that

$$\boldsymbol{v}\,.\,\nabla\boldsymbol{p} \equiv -\boldsymbol{v}\times(\nabla\times\boldsymbol{p}) + m_0\,c^2\,\nabla\gamma\,; \tag{2}$$

the relationship $\gamma^2\boldsymbol{v}^2/c^2 \equiv \gamma^2 - 1$ has been used to eliminate $\boldsymbol{v}^2$ from the last term. Invoking the two Maxwell equations that do not involve sources, so that $\boldsymbol{E}$ and $\boldsymbol{B}$ may be eliminated in favour of the scalar and vector potentials ϕ and $\boldsymbol{A}$, and using equation (2), we find that the equation of motion (1) becomes

$$\frac{\partial}{\partial t}\left(\boldsymbol{p} + \frac{e\boldsymbol{A}}{c}\right) - \boldsymbol{v}\times\left\{\nabla\times\left(\boldsymbol{p} + \frac{e\boldsymbol{A}}{c}\right)\right\} + \nabla(\gamma m_0\,c^2 + e\phi) = -n^{-1}\nabla p\,. \tag{3}$$

Taking the curl of (3) shows that

$$\partial\boldsymbol{\omega}/\partial t - \nabla\times(\boldsymbol{v}\times\boldsymbol{\omega}) = -\nabla\times\boldsymbol{f}, \tag{4}$$

where

$$-\nabla\times\boldsymbol{f} \equiv (n_0^{-1}\nabla n_0 + \gamma^{-1}\nabla\gamma)\times(n^{-1}\nabla p) \tag{5}$$

and

$$\boldsymbol{\omega} \equiv \nabla\times(\boldsymbol{p} + e\boldsymbol{A}/c) \equiv e\boldsymbol{\beta}/c, \tag{6}$$

with

$$\boldsymbol{\beta} \equiv \boldsymbol{B} + (cm_0/e)\nabla\times(\gamma\boldsymbol{v}). \tag{7}$$

The vector $\boldsymbol{\omega}$ is the generalized vorticity of the species under consideration, while $\boldsymbol{\beta}$ can be regarded as a generalized magnetic field or 'magneto-inertial field' associated with that species. Equation (4) holds for each species and describes the development of the generalized vorticity field or magneto-inertial field of each species with respect to the motion of that species: it expresses the flux-vorticity or fluxoid theorem in a differential form. Equation (4) reduces to the previously known theorem (Buckingham *et al.* 1972, 1973) when the right-hand side is negligible. For an uncharged fluid, equation (4) reduces to a theorem for the relativistic vorticity $\nabla\times(\gamma\boldsymbol{v})$ of that fluid. For a charged fluid with negligible inertial terms, $\boldsymbol{\omega}$ reduces to $e\boldsymbol{B}/c$ and equation (4) describes the development of the magnetic field lines *relative to that fluid.*

Equation (4) has known implications in certain special cases. When $\boldsymbol{\omega}\times(\nabla\times\boldsymbol{f})$ is negligible, this equation shows that

$$\boldsymbol{\omega}\times\{\partial\boldsymbol{\omega}/\partial t - \nabla\times(\boldsymbol{v}\times\boldsymbol{\omega})\} = \boldsymbol{0}, \tag{8}$$

which is the necessary and sufficient condition that the vector lines of the solenoidal field $\boldsymbol{\omega}$ be material lines (Truesdell 1954, Section 28). When $\nabla\times\boldsymbol{f}$ is negligible, equation (4) reduces to

$$\partial\boldsymbol{\omega}/\partial t - \nabla\times(\boldsymbol{v}\times\boldsymbol{\omega}) = \boldsymbol{0}, \tag{9}$$

which is the necessary and sufficient condition that the strengths of all vector tubes of the solenoidal field $\boldsymbol{\omega}$ at a given cross section remain constant as the motion proceeds (Truesdell 1954, Section 28).

For the special case of nonrelativistic motion of barotropic fluids, $\nabla\times\boldsymbol{f}$ vanishes: equation (9) holds and $\boldsymbol{\omega}$ is 'frozen-in' to the motion of that fluid. It should be emphasized that the right-hand side of equation (4) is not usually negligible: the theorem, in general, is not one of freezing-in of $\boldsymbol{\omega}$ or $\boldsymbol{\beta}$ but one that incorporates diffusion of the lines of $\boldsymbol{\omega}$ or $\boldsymbol{\beta}$ of each species with respect to that species.

An integral form of the fluxoid theorem can be obtained as follows. Let C denote a contour that moves at each of its points with the local fluid velocity of a representative species, and let S denote any surface that caps C and is also fixed in the fluid. If $\mathrm{d}/\mathrm{d}t \equiv \partial/\partial t + \boldsymbol{v}.\nabla$ is the convective derivative for that fluid then, for any vector $\boldsymbol{\alpha}$,

$$\int_S \left(\frac{\partial(\nabla\times\boldsymbol{\alpha})}{\partial t} - \nabla\times\{\boldsymbol{v}\times(\nabla\times\boldsymbol{\alpha})\}\right).\mathrm{d}\boldsymbol{S} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_S (\nabla\times\boldsymbol{\alpha}).\mathrm{d}\boldsymbol{S}\right). \tag{10}$$

A proof of equation (10) for the case when $\nabla\times\boldsymbol{\alpha}$ is $\boldsymbol{B}$ has been presented by, for example, Tanenbaum (1967); since $\nabla.\boldsymbol{B} = 0$ is the only property of $\boldsymbol{B}$ used in that proof, equation (10) in fact follows for any vector $\boldsymbol{\alpha}$. On replacing $\boldsymbol{\alpha}$ in equation (10) by $\boldsymbol{p}+e\boldsymbol{A}/c$, it is seen that the relativistic flux-vorticity theorem (4) may be expressed, after using the Stokes theorem, as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\oint_C (\boldsymbol{p}+e\boldsymbol{A}/c).\mathrm{d}\boldsymbol{s}\right) = -\oint_C \boldsymbol{f}.\mathrm{d}\boldsymbol{s}, \tag{11}$$

where $\mathrm{d}\boldsymbol{s}$ denotes an element of the curve C. For a charged component fluid the quantity Φ, where

$$\begin{aligned}\Phi &\equiv \int_S \boldsymbol{B}.\mathrm{d}\boldsymbol{S} + (cm_0/e)\oint_C \gamma\boldsymbol{v}.\mathrm{d}\boldsymbol{s} \\ &= \int_S \boldsymbol{\beta}.\mathrm{d}\boldsymbol{S} = (c/e)\int_S \boldsymbol{\omega}.\mathrm{d}\boldsymbol{S} = (c/e)\oint_C (\boldsymbol{p}+e\boldsymbol{A}/c).\mathrm{d}\boldsymbol{s},\end{aligned} \tag{12}$$

is the fluxoid through C. When $\nabla\times\boldsymbol{f}$ vanishes, equation (11) reduces to the theorem obtained by Buckingham *et al.* (1972, 1973).

The vorticity theorem of nonrelativistic hydrodynamics has been expressed in numerous forms, and the subject has been reviewed in detail by Truesdell (1954). These forms can be adapted so as to apply to the relativistic flux-vorticity theorem. For example, equation (4) can be written in the form

$$\mathrm{d}\boldsymbol{\omega}/\mathrm{d}t = \boldsymbol{\omega}.\nabla\boldsymbol{v} - \boldsymbol{\omega}\nabla.\boldsymbol{v} - \nabla\times\boldsymbol{f}. \tag{13}$$

For a neutral fluid when $\nabla\times\boldsymbol{f} = \boldsymbol{0}$, equation (13) is the d'Alembert–Euler vorticity equation (Truesdell 1954, Section 94) for the fluid.

Let ω_p, v_p and f_p denote cartesian components of $\boldsymbol{\omega}$, $\boldsymbol{v}$ and $\boldsymbol{f}$. Write $v_{pq} \equiv 2\omega_{[p}v_{q]}$ where square brackets around indices denote a skew-symmetric part. The summation convention will be used. The flux-vorticity theorem (4) can be written in cartesian tensor form as

$$\partial\omega_q/\partial t + \partial_r v_{qr} = -\varepsilon_{qrs}\partial_r f_s, \tag{14}$$

where $\partial_r \equiv \partial/\partial x_r$ and ε_{pqr} is the permutation symbol.

I have obtained previously (Burman 1977) two explicitly covariant forms, one differential and one integral, of the pressure-free relativistic fluxoid theorem. These will now be extended to become explicitly covariant forms of the fluxoid theorem of this paper. Take the fourth space–time coordinate to be ct and define two skew-symmetric second-rank 4-tensors $(V_{\mu\nu})$ and $(F_{\mu\nu})$ by the following specifications of their components in an inertial frame: $V^{pq} \equiv 2\omega_{[p}v_{q]}$, $V^{r4} = c\omega_r$, $F^{qr} = -\varepsilon_{qrs}f_s$ and $F^{r4} = 0$. The equation

$$\partial_\nu V^{\mu\nu} = \partial_\nu F^{\mu\nu} \tag{15}$$

reduces to equation (14) for $\mu = q$ and to the identity $\nabla.\boldsymbol{\omega} = 0$ for $\mu = 4$; equation (15) is an explicitly covariant form of the flux-vorticity theorem.

Let (P^μ) denote the canonical fluid 4-momentum associated with each particle of one of the component species: $P^\mu = m_0 U^\mu + eA^\mu/c$ where (U^μ) is the fluid 4-velocity of the species and (A^μ) is the electromagnetic 4-potential. Note that $(P^\mu) = (\boldsymbol{p} + e\boldsymbol{A}/c,\ W/c)$ where W is the total energy (the sum of the rest energy, kinetic energy and electromagnetic potential energy) per particle. Hence, taking L to be some closed space–time path with typical element $(\mathrm{d}x^\mu) = (\mathrm{d}\boldsymbol{s}, c\,\mathrm{d}t)$ and using the signature -2, we have

$$\oint_L P_\mu\,\mathrm{d}x^\mu = -\oint_L (\boldsymbol{p} + e\boldsymbol{A}/c)\boldsymbol{.}\mathrm{d}\boldsymbol{s} + \oint_L W\,\mathrm{d}t\,. \tag{16}$$

Also,

$$\oint_L n^{-1} p_{,\mu}\,\mathrm{d}x^\mu = -\oint_L \boldsymbol{f}\boldsymbol{.}\mathrm{d}\boldsymbol{s} + \oint_L n^{-1}(\partial p/\partial t)\,\mathrm{d}t\,, \tag{17}$$

where the subscript comma denotes partial differentiation. Consider the equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\oint_C P_\mu\,\mathrm{d}x^\mu\right) = -\gamma\oint_C n^{-1} p_{,\mu}\,\mathrm{d}x^\mu, \tag{18}$$

where τ denotes proper time $(\mathrm{d}t/\mathrm{d}\tau = \gamma)$ and C is a closed contour that moves with the species under consideration. Since C must be space-like, a Lorentz frame can always be found in which C lies entirely in a spatial section of space–time, so equations (16) and (17) show that (18) reduces to (11): equation (18) is another explicitly covariant form of the fluxoid theorem.

In their explicitly covariant forms, the fundamental equations of electrodynamics and of fluid dynamics can be extended to apply in general relativity by replacing partial derivatives with covariant derivatives. Hence equation (18) is already in general relativistic form, while equation (15) becomes

$$V^{\mu\nu}{}_{;\nu} = F^{\mu\nu}{}_{;\nu}\,, \tag{19}$$

where the subscript semicolon denotes covariant differentiation. These generally covariant forms of the theorem could be useful in developing the theory of possible magnetospheres around black holes.

3. Analysis of the Fluxoid Theorem

It is necessary to perform an order-of-magnitude analysis of the fluxoid law so as to obtain a general understanding of fluxoid dynamics. This was done by Wright (1978) for the pressure-free case, and the extension to include pressure gradients will be made here. In this way, the circumstances under which the separate contributions to the fluxoid law are likely to be important will be made clear.

Let L denote a length scale over which the macroscopic plasma properties vary significantly. Write ω_B for $|e|B/m_0 c$, the nonrelativistic angular gyrofrequency of the species under consideration; for electrons $\omega_B \approx 2\times10^7 B$ with B in gauss (10^{-4} T). The ratio of the magnitudes of the inertial and magnetic contributions to $\boldsymbol{\omega}$ or $\boldsymbol{\beta}$ is typically, for charged species, of order ε_M where $\varepsilon_M \equiv \gamma v/\omega_B L$. This number was introduced by Wright (1978), who called it the 'magnetic Rossby number' by analogy with the Rossby number $v/\omega L$ used to estimate the relative importance of inertial and Coriolis effects in a nonrelativistic fluid rotating at angular speed ω. The number $2\pi\varepsilon_M$ can be interpreted as the ratio of the distance travelled by the fluid in a relativistic gyro-period (of one of its particles that happens to have negligible random motion) to the scale length L. As expected from qualitative considerations (Buckingham *et al.* 1972, 1973), inspection of the expression for ε_M indicates semi-quantitatively that the inertial contribution is likely to be important in the vicinity of magnetic neutral lines or sheets or in regions where rapid relativistic acceleration is occurring. For example, there may be thin layers in which $|\nabla\gamma|$ is large (Mestel *et al.* 1979): if $|\nabla\gamma| \gg \omega_B/c$, then $\varepsilon_M \gg 1$. When $\varepsilon_M \ll 1$, indicating that magnetic effects predominate over inertial effects, and, in addition, the right-hand side of the fluxoid law (4) is negligible, the magnetic flux is frozen-in to the motion of the species under consideration. For $\varepsilon_M \gtrsim 1$, inertial effects are important and the magnetic flux is not frozen-in.

Let S_i and S_m denote the ratios of the magnitude of the right-hand side $-\nabla\times\boldsymbol{f}$ of the fluxoid theorem (4) to the magnitudes of the inertial and magnetic contributions, respectively, to the term $\nabla\times(\boldsymbol{v}\times\boldsymbol{\omega})$.

Consider first the nonrelativistic and moderately relativistic cases, defined as satisfying $|\nabla\gamma|/\gamma \ll |\nabla n_0|/n_0$. Let Λ denote the angle between ∇p and ∇n_0; for a barotropic fluid, $\Lambda = 0$. Estimating p to be of order $m_0 v_{th}^2 n_0$, where v_{th} denotes the mean thermal speed of the particles of the representative species in its local rest frame, shows that

$$S_i \sim \gamma^{-2}(v_{th}/v)^2 \sin\Lambda\,, \qquad S_m \sim \gamma^{-1}(v/\omega_B L)(v_{th}/v)^2 \sin\Lambda\,. \tag{20a, b}$$

For uncharged species or inertially dominated charged species, S_i is the relevant ratio; note that the length scale does not appear in S_i. For magnetically dominated species, S_m is the relevant ratio, and it contains the small factor $v/\omega_B L$. It is seen that pressure gradients can be important in fluxoid dynamics for non-barotropic species with drift speeds less than about their mean thermal speeds. The electron drift speed in plasmas has been observed to be limited above by the ion thermal speed (Stix 1962, p. 203; Alfvén 1968). Hence, for the electrons in a plasma in which the electron and ion temperatures are of the same order T, we have

$$(v/\omega_B L)(v_{th}/v)^2 \gtrsim T^{\frac{1}{2}}/BL\,, \tag{21}$$

with T in kelvin, B in gauss and L in centimetres. This quantity can exceed 1 in laboratory plasmas, but will typically be extremely small in cosmic plasmas except for highly localized phenomena.

Now consider very relativistic species, defined as satisfying $|\nabla\gamma|/\gamma \gtrsim |\nabla n_0|/n_0$. Let l denote the length scale for variation of γ and L a common length scale for variation of p, n_0 and $\boldsymbol{B}$. In this case, $l \lesssim L$ and $\varepsilon_M \equiv \gamma c/\omega_B l$. (There may be thin regions in which γ varies more rapidly than the other macroscopic plasma properties. In moderately relativistic plasmas, as defined above, we have $l \gg L$, meaning that the Lorentz factor changes only slightly in distances over which the other macroscopic plasma properties vary significantly.) The contribution of the partial pressure gradient to the fluxoid law is not much affected by whether or not conditions are approximately barotropic, because of the presence of the $\nabla\gamma$ term in equation (5) for $\nabla\times\boldsymbol{f}$. The estimate $p \sim m_0 v_{th}^2 n_0$ shows that

$$S_i \sim \gamma^{-2}(v_{th}/c)^2 l/L\,, \qquad S_m \sim \gamma^{-1}(c/\omega_B l)(v_{th}/c)^2\,. \tag{22a, b}$$

The estimates (20) and (22) indicate that partial pressure gradients typically have little effect on fluxoid dynamics in relativistically streaming plasmas that are not relativistically hot—see, however, Section 4 below.

Since attention has been restricted to plasmas that are not relativistically hot, only pressure *gradients*, not the partial pressures themselves, can be significant. However, in determining the importance of the pressure gradients, the length scales have cancelled out of the estimate (20a), entered through small factors in (20b) and (22b) and entered through a factor which is less than about 1 in (22a). Note that the term $\partial\boldsymbol{\omega}/\partial t$ in the fluxoid law can sometimes partially cancel with the term $-\nabla\times(\boldsymbol{v}\times\boldsymbol{\omega})$, thus increasing the relative importance of the pressure gradient contribution; an example of this will be discussed in Section 4 below when considering steadily rotating systems. Also, these order-of-magnitude estimations are, of course, not rigorous and must be checked in any particular application.

It must be emphasized that the above considerations apply only to the significance of pressure gradients in the fluxoid law, and not to their importance elsewhere, such as in a generalized Ohm law for the plasma or in the equation of motion of a species. For example, if $\boldsymbol{E}+c^{-1}\boldsymbol{v}\times\boldsymbol{B}\approx\boldsymbol{0}$ holds for a species, then $-n^{-1}\nabla p$ must balance the inertial force $(\partial/\partial t+\boldsymbol{v}\,.\,\nabla)\boldsymbol{p}$ of that species.

The analysis given in this section is sufficient to indicate that in many circumstances pressure gradient effects on fluxoid dynamics can be safely neglected. The important special case of steadily rotating systems requires a separate analysis, because of the cancellation effect just mentioned.

4. Steadily Rotating Systems

The special case of steady rotation is relevant to a number of areas in which the fluxoid theorem has been, or could be, applied. These include studies of pulsar and possible black hole magnetospheres, and of magnetic field generation in the early Universe (Harrison 1970). The term 'steady rotation' here refers to a system that is steady in a rotating frame: it means that there exists a structure that rotates at a constant rate, but the motions of the individual particles, or the individual plasma species, are not themselves so restricted (Mestel *et al.* 1976). The fluxoid theorem, with pressure gradient effects included, will now be developed for this special case.

Let ϖ, ϕ and z be cylindrical polar coordinates with the z axis as the rotation axis. The system is steady in the rotating frame: the changes in time at points fixed in the inertial frame result only from the steady rotation of the whole structure with constant angular frequency Ω. Hence it follows from the two sourceless members of Maxwell's set of equations, namely Faraday's law and $\nabla\,.\,\boldsymbol{B}=0$, that $\boldsymbol{E}$ and $\boldsymbol{B}$ are connected by (Mestel 1971)

$$\boldsymbol{E}+c^{-1}\Omega\varpi\boldsymbol{t}\times\boldsymbol{B}=-\nabla\Phi,\tag{23}$$

where $\boldsymbol{t}$ is the unit toroidal vector and the gauge-invariant quantity Φ is defined in terms of the familiar scalar and vector potentials by the relation (Endean 1972*a*)

$$\Phi\equiv\phi-(\Omega\varpi/c)A_{\phi}.\tag{24}$$

The steady-rotation condition

$$\partial/\partial t=-\Omega\,\partial/\partial\phi\tag{25}$$

(Mestel 1971; Endean 1972*a*) is valid for, in particular, cylindrical polar components of vectors. Hence the identity (2) for $\boldsymbol{v}.\nabla\boldsymbol{p}$ gives (Burman and Mestel 1978)

$$(\partial/\partial t+\boldsymbol{v}.\nabla)\boldsymbol{p} = -\boldsymbol{u}\times(\nabla\times\boldsymbol{p})+\nabla(\gamma mc^2-\Omega\varpi p_\phi), \tag{26}$$

where $\boldsymbol{u}\equiv\boldsymbol{v}-\Omega\varpi\boldsymbol{t}$. Equation (23) and $\boldsymbol{B}=\nabla\times\boldsymbol{A}$ show that (Burman and Mestel 1978)

$$\boldsymbol{E}+c^{-1}\boldsymbol{v}\times\boldsymbol{B} = c^{-1}\boldsymbol{u}\times(\nabla\times\boldsymbol{A})-\nabla\Phi. \tag{27}$$

From equations (26) and (27), the equation of motion (1) of the representative species takes the simple form

$$\boldsymbol{u}\times\{\nabla\times(\boldsymbol{p}+e\boldsymbol{A}/c)\} = e\nabla\Psi+n^{-1}\nabla p, \tag{28}$$

where

$$\Psi\equiv\Phi+\frac{\gamma m_0 c^2}{e}\left(1-\frac{\Omega\varpi}{c}\frac{v_\phi}{c}\right). \tag{29}$$

The quantity Ψ was obtained as a constant of the motion in the pressure-free case by Endean (1972*a*, 1972*b*). Taking the curl of equation (28) we obtain the fluxoid theorem in the form

$$\nabla\times(\boldsymbol{u}\times\boldsymbol{\omega}) = \nabla\times(n^{-1}\nabla p). \tag{30}$$

Comparison with the more general form (4) shows that, in the case of steadily rotating systems, the time derivative term has cancelled with part of the $-\nabla\times(\boldsymbol{v}\times\boldsymbol{\omega})$ term, leaving $-\nabla\times(\boldsymbol{u}\times\boldsymbol{\omega})$.

Typical magnitudes of the contributions to the theorem may be compared, as before, by estimating p to be of order $m_0 v_{\mathrm{th}}^2 n_0$. For the nonrelativistic and moderately relativistic cases, in which $|\nabla\gamma|/\gamma \ll |\nabla n_0|/n_0$, the estimates (20) are replaced by

$$S_{\mathrm{i}}\sim\gamma^{-2}(v_{\mathrm{th}}^2/uv)\sin\Lambda, \qquad S_{\mathrm{m}}\sim\gamma^{-1}(v/\omega_{\boldsymbol{B}}L)(v_{\mathrm{th}}^2/uv)\sin\Lambda. \tag{31a, b}$$

For very relativistic species, in which $|\nabla\gamma|/\gamma \gtrsim |\nabla n_0|/n_0$, the estimates (22) are replaced by

$$S_{\mathrm{i}}\sim\gamma^{-2}(v_{\mathrm{th}}^2/uc)l/L, \qquad S_{\mathrm{m}}\sim\gamma^{-1}(c/\omega_{\boldsymbol{B}}l)(v_{\mathrm{th}}^2/uc). \tag{32a, b}$$

These numbers suggest that the pressure gradient effect on the flux-vorticity dynamics can be important when u is sufficiently small, meaning that the species concerned is close to being in a state of rigid corotation. In fact, S_{i} and S_{m} diverge for perfectly rigid rotation, corresponding to the vanishing of the left-hand side of the flux-vorticity theorem (30), which thus reduces to the degenerate form

$$\nabla\times(n^{-1}\nabla p) = \boldsymbol{0}. \tag{33}$$

The equation of motion (28) of the species reduces, for rigid corotation, to

$$e\nabla\Psi = -n^{-1}\nabla p, \tag{34}$$

which is, of course, just the integral of the degenerate fluxoid theorem (33).

In pulsar magnetospheres, the magnetic field is so strong that magnetic forces predominate over inertial forces throughout most of the region inside the 'light cylinder', which is defined as the surface $\varpi = c/\Omega$ on which the corotational speed equals c. In some of this region, the magnetic field will enforce approximate corotation of charged particles: there v will be approximately $\Omega\varpi$ and L will typically be of the same order of magnitude as the distance ϖ from the rotation axis. Hence, the magnetic Rossby number is estimated by

$$\varepsilon_M \sim \gamma\Omega/\omega_B, \tag{35}$$

reducing to Ω/ω_B well inside the light cylinder. The estimate (35) shows that the ratio of inertial to magnetic effects is of the order of the ratio of a macroscopic rotational frequency to a microscopic gyrofrequency, as Mestel (1971) showed by considering the corresponding energy densities. When magnetic forces predominate over inertia, the relevant number for estimating the effects of pressure gradients on the fluxoid law is S_m; this is given by equation (31b), which becomes, for $v \sim \Omega\varpi$ and $L \sim \varpi$,

$$S_m \sim \frac{1}{\gamma}\frac{c}{\Omega\varpi}\frac{\Omega}{\omega_B}\frac{v_{th}^2}{uc}; \tag{36}$$

this contains the very small ratio Ω/ω_B and is likely to be very small itself, implying that pressure gradient effects are likely to be negligible in the fluxoid theorem for an approximately corotating zone.

In the vicinity of the light cylinder, corotational speeds approach c and the relativistic inertial effects will cause the particles to break away from the magnetic field lines. From the estimate (32a), we have

$$S_i \lesssim \gamma^{-2}(v_{th}/c)^2, \tag{37}$$

which is very small.

Magnetic field strengths near pulsar surfaces are estimated to be of order 10^{12} G; this is so strong that use of scalar pressures is inappropriate, and a better approximation would be to use a uniaxial pressure tensor, the pressure having two different components, one along the magnetic field and one across it. But for the purpose of making order-of-magnitude estimates of the effects of pressure gradients, formulae based on scalar pressures are adequate.

Mestel *et al.* (1979) have proposed a model of the axisymmetric pulsar magnetosphere which contains a corotating zone and a 'circulation zone' in which electrons emitted from the pulsar near the magnetic poles travel out along magnetic field lines and then back along other field lines, having been caused to drift across the lines by radiation reaction forces occurring in a 'dissipation zone' in the neighbourhood of the light cylinder and beyond. In the circulation zone, relatively little acceleration occurs and the poloidal flow is nonrelativistic. Taking $L \sim \varpi$ we obtain the estimate (36) for S_m in this zone: again pressure gradient effects are likely to be negligible in the fluxoid theorem.

Various workers have proposed that electrons emitted from a pulsar surface rapidly reach highly relativistic energies very near the surface, accelerated by a substantial local electric field component parallel to the magnetic field. If the particles are accelerated to a Lorentz factor γ in a distance of l cm above the surface, the estimate (32b) shows that

$$S_{\mathrm{m}} \sim (10^{-9}/\gamma l)v_{\mathrm{th}}^2/uc. \tag{38}$$

In particular, Michel (1974) proposed that $\gamma \sim 10^4$ is reached about 100 m above the surface; this would imply that $\varepsilon_M \sim 10^{-9}$ and that a very tiny value, of order $10^{-17}(v_{\mathrm{th}}/c)^2$, obtains for S_{m}.

The above estimates suggest that pressure gradient effects will not be important in the fluxoid law for pulsar magnetospheres, at least for the region from the stellar surface to somewhat beyond the light cylinder; thus the neglect of pressure gradient effects in the fluxoid law by Wright (1978), Burman and Mestel (1978, 1979), Mestel *et al.* (1979), Mestel (1980) and Burman (1980) appears to be justified. There are so many effects that might be significant that it is a relief to see that one of them probably is not. But since the flow dynamics theory based on only the Lorentz and relativistic inertial forces leads to singularities in the Lorentz factors (Mestel *et al.* 1979; Burman 1980), some dissipative force must be essential somewhere in the dynamical analysis.

It should be emphasized that pressure gradients may be significant in dynamical equations other than the fluxoid theorem. For example, consider electrons and positive ions in a corotating zone. The small gravitational, centrifugal and pressure gradient forces acting on, for example, the electrons in a negatively charged region will generally have a resultant component parallel to the magnetic field, and this force will be balanced by a small electric force in that direction; the corresponding electric force in that direction acting on the ions will assist the non-electromagnetic forces in draining away the ions, thus increasing the degree of charge separation (Mestel 1980).

Failure to detect continuous X-radiation from the Crab pulsar during lunar occultations has shown that its effective surface temperature must be less than about 5×10^6 K (Wolff *et al.* 1975). Other pulsars, being older, are likely to be cooler. Thus temperatures in the magnetospheric plasmas are very likely to be nonrelativistic unless violent heating occurs. Ardavan (1976) has claimed that the plasma just outside the light cylinder will be relativistically hot, but I have shown (Burman 1980) that his analysis is faulty. A referee of the present paper has said that he would expect the pressure to be relativistic along the magnetic field and negligible across it; if this is so, then the formulae given here are not adequate.

5. Concluding Remarks

In this paper, a generalized flux conservation theorem has been obtained for plasmas in which Alfvén's flux conservation law fails because there are too few current-carrying particles to maintain the electric current density that would be required and relativistic particle inertia becomes important. A multifluid plasma model has been used. For each species, a generalized flux, the 'fluxoid', is defined and the theorem derived here describes its behaviour. When pressure gradient effects are negligible in this law, the fluxoid is frozen-in to the motion of the species concerned. Conditions for this to be so have been considered.

The fluxoid theorem should be useful in the description of plasmas in which acceleration is occurring; for example, pulsar magnetospheres, for which Burman and Mestel (1978, 1979) have used the theorem together with the Endean integral to simplify the equations of motion of the species. Also, Havnes (1971) has discussed evidence that low-energy galactic cosmic rays have been accelerated by electromagnetic forces acting in partially ionized plasmas; this could also be the case for low-energy solar cosmic rays (Lanzerotti *et al.* 1972). Other areas for application of the fluxoid theorem are the study of magnetic field generation in the early Universe (Harrison 1970) and the description of the dynamics of plasmas in the vicinity of neutral lines and sheets of magnetic fields and in possible magnetospheres around black holes.

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