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Mathematical Theory of One-dimensional Isothermal Blast Waves in a Magnetic Field

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Abstract

An investigation is made of the self-similar flow behind a one-dimensional blast wave from a planar explosion (situated on $z = 0$) in a medium whose density and magnetic field vary with distance as $z^{-\omega}$ ahead of the blast front, with the assumption that the flow is isothermal. It is found that: if $\omega < 0$ a continuous, single-valued solution does not exist; if $\omega = 0$ the solution is singular and piecewise continuous with an inner region where no fluid flow occurs and an outer region where the fluid flow gradually increases; if $\frac{1}{2} > \omega > 0$ the governing equation possesses a set of movable critical points. For a weak, but nonzero, magnetic field it is shown that the value of the smallest critical point does not lie in the physical domain $z > 0$. The post-shock fluid flow then cannot intersect the critical point, and is smoothly continuous. It is shown that to be physically acceptable, the fluid flow speed must pass through the origin. It is also shown that ω must be less than $\frac{1}{2}$ for the magnetic energy swept up by the blast wave to remain finite. The overall conclusion from the investigation is that the behaviour of isothermal blast waves in the presence of an ambient magnetic field differs substantially from the behaviour calculated for no magnetic field. These results point to the inadequacy of previous attempts to apply the theory of self-similar flows to evolving supernova remnants without making any allowance for the dynamical influence of magnetic field pressure.

1. Introduction

The theory of self-similar flows behind blast waves (see e.g. the exposition by Sedov 1959) has been extensively applied in the analysis and interpretation of observations of supernova remnants (SNRs) (see e.g. Woltjer 1972; Gorenstein *et al.* 1974; Rappaport *et al.* 1974; and references therein). The large temperature gradients predicted by adiabatic models, however, may be inconsistent with the assumption that heat flux can be neglected. This difficulty was pointed out by Sedov (1959) and Parker (1963), both of whom suggested that it might be more appropriate to adopt an isothermal rather than an adiabatic treatment. Recently Solinger *et al.* (1975) demonstrated quantitatively the internal inconsistency of adiabatic blast wave models for SNRs and advocated the use of isothermal models instead. They used Korobeinikov's (1956) solution to reinterpret the properties of several observed SNRs. More recently still, Lerche and Vasyliunas (1976) showed that isothermal blast wave models themselves suffer from both global and local instabilities. Initial deviations do not then decay, and the system does not tend toward a self-similar form.

Now all the above authors assume either that any magnetic field is zero, so that it cannot influence the dynamical evolution of the blast wave, or that it is so 'weak' that the magnetic field evolves kinematically (i.e. the fluid equations are solved ignoring the field, and the field structure and evolution are then determined from Lenz's law). However, a recent investigation (Caswell and Lerche 1979*a*, 1979*b*) of the radio brightness variations across 33 SNRs has demonstrated that the galactic magnetic field plays a dominant role in their evolution. It is apparent therefore that proper consideration must be given to the effects of magnetic fields, and in particular that their influence on the dynamical evolution of blast waves must be properly analysed.

The mathematical development of solutions for isothermal self-similar flows in the presence of magnetic fields has been rather neglected to date. No investigation seems to have been made of the mathematical properties of flows under any conditions when magnetic field pressure plays a dynamical role. The topics of interest include the topology of the solutions, the influence of boundary conditions and the nature and effects of singularities. In order to emphasize the important role played by a magnetic field in the evolution of a blast wave we consider here the simplest possible case of a plane one-dimensional isothermal blast wave. We recognize, of course, that the temporal behaviour of SNRs is, presumably, more accurately described by a spherical blast wave. However, as has been emphasized by Cox (1972) and McCray *et al.* (1975), a simplified one-dimensional treatment (ignoring curvature of the shock front) is sufficient to bring out the underlying physics very succinctly. While two- and three-dimensional effects (such as the bending of blast waves around density fluctuations and oblique magnetic fields) will no doubt modify the results obtained, the basic behaviour is nevertheless adequately described by a one-dimensional treatment.

Thus there are strong arguments, both mathematical and physical, for developing a one-dimensional theory of self-similar isothermal flow including magnetic field effects.

2. Properties of One-dimensional Magnetoactive Isothermal Self-similar Blast Waves

(*a*) *Formulation of the Problem*

Since the general method of constructing self-similar blast waves is described in standard texts (e.g. Landau and Lifschitz 1959; Sedov 1959; Parker 1963) this section is brief and serves chiefly to introduce notation. Assume that the density of the cold ambient medium varies with distance z from the plane of the explosion as $\rho(z) = \rho_0(a/z)^\omega$, where ρ_0 is the density at the reference distance a (only values of $\omega < 1$ are of physical interest; $\omega \geqslant 1$ would imply an infinite total mass contained within the blast wave). Assume that the magnetic field embedded in the ambient medium points in the x direction and varies with distance z from the plane of the explosion as $B_x(z) = B_0(a/z)^\Lambda$, where B_0 is the magnetic field strength at the reference distance a. (Only values of $\Lambda < \frac{1}{2}$ are of physical interest; $\Lambda \geqslant \frac{1}{2}$ would imply an infinite total magnetic energy contained within the blast wave.)

Let a blast wave move out from the plane $z = 0$ at $t = 0$ so that at time t the blast front is at position $z_s(t)$. The assumption of self-similarity implies that, within the blast wave, the density $\rho(z,t)$, the z-directed flow speed $V_z(z,t)$, the temperature $T(z,t)$ and the magnetic field $B_x(z,t)$ are to be written in the forms

$$\rho = \eta\,\rho_0(a/z_s)^\omega\,R(\lambda), \tag{1}$$

$$V_z = (\eta-1)\eta^{-1}V_s V(\lambda), \tag{2}$$

$$kT/m = (\eta-1)\eta^{-2}V_s^2\,\Theta(\lambda)-(B_0^2/8\pi\rho_0)(\eta^2-1)\eta^{-1}(a/z_s)^{2\Lambda-\omega}, \tag{3}$$

$$B_x = \eta\, B_0(a/z_s)^{\Lambda}\, B(\lambda). \tag{4}$$

In these equations R, V, Θ and B are dimensionless functions of the argument $\lambda = z/z_s$, and $V_s = \mathrm{d}z_s/\mathrm{d}t$. If the constant η is chosen to be the density magnification factor across the shock wave then the equations of mass, momentum and flux conservation across the shock wave are satisfied with $R(1) = V(1) = B(1) = \Theta(1) = 1$. The assumption of isothermal flow corresponds to setting $\Theta(\lambda) = 1$. When this is done the parameter η is determined by the solution to the flow equations and cannot be set to the customary value 4, which is appropriate to adiabatic post-shock flow for a constant speed shock.

Now the equations of continuity, momentum and magnetic induction are respectively

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial z}\Big(\rho V_z\Big) = 0, \tag{5}$$

$$\rho\left(\frac{\partial V_z}{\partial t}+V_z\frac{\partial V_z}{\partial z}\right) = -\frac{\partial}{\partial z}\left(p+\frac{B_x^2}{8\pi}\right), \tag{6}$$

$$\frac{\partial B_x}{\partial t}+\frac{\partial}{\partial z}\Big(B_x V_z\Big) = 0, \tag{7}$$

where p is the gas pressure: $p = \rho kT/m$.

Insertion of equations (1)–(4) into equations (5)–(7) yields three equations for the three functions $R(\lambda)$, $V(\lambda)$ and $B(\lambda)$:

$$\frac{\mathrm{d}R}{\mathrm{d}\lambda}\Big((\eta-1)\eta^{-1}V-\lambda\Big)+R\left((\eta-1)\eta^{-1}\frac{\mathrm{d}V}{\mathrm{d}\lambda}-\omega\right) = 0, \tag{8}$$

$$\frac{\mathrm{d}B}{\mathrm{d}\lambda}\Big((\eta-1)\eta^{-1}V-\lambda\Big)+B\left((\eta-1)\eta^{-1}\frac{\mathrm{d}V}{\mathrm{d}\lambda}-\Lambda\right) = 0, \tag{9}$$

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}\lambda}\Big((\eta-1)\eta^{-1}V-\lambda\Big)&+Vz_s\dot{V}_s V_s^{-2}\\ &= -\eta\Big(mV_s^2(\eta-1)\Big)^{-1}kT\left(R^{-1}\frac{\mathrm{d}R}{\mathrm{d}\lambda}-\eta^2(\eta^2-1)^{-1}R^{-1}\frac{\mathrm{d}B^2}{\mathrm{d}\lambda}\right)\\ &\quad-\eta(\eta^2-1)^{-1}R^{-1}\frac{\mathrm{d}B^2}{\mathrm{d}\lambda}.\end{aligned} \tag{10}$$

But the self-similar assumption demands that R, V and B be functions only of λ. Equation (10) is therefore valid only when (i) the temperature T is proportional to V_s^2, and (ii) $z_s\dot{V}_s/V_s^2$ = const. However, if $B_0 \neq 0$ it follows from equation (3) that T is proportional to V_s^2 only when $V_s^2 \propto z_s^{-2\Lambda+\omega}$, that is, when $z_s \propto t^{2/(2-\omega+2\Lambda)}$. In this case we have $z_s\dot{V}_s/V_s^2 = -(\Lambda-\frac{1}{2}\omega)$. Inspection of equations (8) and (9) reveals that to avoid a singularity in *either R or B* as $V(\lambda)$ passes* through $\lambda\eta/(\eta-1)$ it is *necessary* that $\omega = \Lambda < \frac{1}{2}$; therefore $\omega < \frac{1}{2}$ since $\Lambda < \frac{1}{2}$.

Under the conditions noted above, we have $z_s \propto t^{2/(2+\omega)}$ and $\omega < \frac{1}{2}$. Equations (8), (9) and (10) then yield

$$B(\lambda) = R(\lambda), \tag{11}$$

$$\frac{dR}{d\lambda}\left((\eta-1)\eta^{-1}V-\lambda\right)+R\left((\eta-1)\eta^{-1}\frac{dV}{d\lambda}-\omega\right) = 0, \tag{12}$$

$$\frac{dV}{d\lambda}\left((\eta-1)\eta^{-1}V-\lambda\right)-\tfrac{1}{2}\omega V = -Y(\eta R)^{-1}\frac{dR}{d\lambda}-2\eta(1-Y)(\eta^2-1)^{-1}\frac{dR}{d\lambda}, \tag{13}$$

where

$$Y = \eta^2 kT/(\eta-1)mV_s^2 = \text{const.} < 1\,.$$

Equation (3) implies $Y \leqslant 1$ with equality when $B_0 = 0$.

It is convenient to define the new variables

$$x = \lambda\eta Y^{-\frac{1}{2}}(\eta-1)^{-\frac{1}{2}}, \quad u = VY^{-\frac{1}{2}}(\eta-1)^{\frac{1}{2}}, \quad r = 2\eta^2(1-Y)(\eta^2-1)^{-1}Y^{-1}R, \tag{14}$$

in terms of which equations (13) and (12) respectively become

$$\frac{du}{dx}\left(u-x\right)-\tfrac{1}{2}\omega u = -\left(1+r^{-1}\right)\frac{dr}{dx}, \tag{15}$$

$$\frac{dr}{dx}\left(u-x\right)+r\left(\frac{du}{dx}-\omega\right) = 0. \tag{16}$$

Note that the parameter η no longer appears explicitly. Here we explore analytically the nature of the solutions to the $u(x)$ and $r(x)$ equations, in the physical domain $x \geqslant 0$.

The physical requirements that $V(\lambda) = R(\lambda) = 1$ on $\lambda = 1$ (the shock front) yield the requirements that

$$x_s = \eta Y^{-\frac{1}{2}}(\eta-1)^{-\frac{1}{2}}, \quad u_s = Y^{-\frac{1}{2}}(\eta-1)^{\frac{1}{2}}, \quad r_s = 2\eta^2(1-Y)Y^{-1}(\eta^2-1)^{-1}. \tag{17}$$

Elimination of η from equations (17) gives the shock curve equations

$$x_s = u_s+(Yu_s)^{-1}, \qquad r_s = 2(1-Y)Y^{-2}u_s^{-2}(1+Yu_s^2)^2(2+Yu_s^2)^{-1}, \tag{18a, b}$$

with

$$\eta = 1+Yu_s^2 \geqslant 1\,. \tag{19}$$

* It should be noted that if we take $\omega = \Lambda$ *before* we manipulate equations (8), (9) and (10) we find, as will be seen in Section 3 below, that in fact $V(\lambda)$ is everywhere *less* than $\lambda\eta/(\eta-1)$. Thus neither R nor B has a singularity. Whether the same is true when $\omega \neq \Lambda$ is unknown. The point is that the structure of the equations determining the post-shock flow properties depends on the parameters ω and Λ. For $\omega \neq \Lambda$, elimination of, say, R and B in favour of V leads to a third-order ordinary nonlinear differential equation. The topological nature of the flow pattern is determined by such an equation. However, for $\omega = \Lambda$ the governing equation, while nonlinear, is only second order. This is such a tremendous simplification that the present investigation has been restricted to precisely the $\omega = \Lambda$ case. The author would, of course, be most interested to see the results of calculations bearing on the more general problem.

Note that the minimum value of x_s is $2Y^{-\frac{1}{2}}$ occurring when $u_s = Y^{-\frac{1}{2}}$, and that at this value of u_s we have $r_s = \frac{8}{3}(1-Y)Y^{-1}$ and $\eta = 2$.

Both the flow equations (15) and (16) *and* the values of their solutions (18) on the shock are then no longer dependent on η explicitly. Hence the topology of solutions $u(x)$ and $r(x)$ can be discussed independently of the value of η.

Equations (15) and (16) are two first-order ordinary differential equations. They require specification of two boundary conditions. Physically, an obvious requirement is that the flow speed $u(x)$ vanish at the origin; thus an appropriate boundary condition is $u(0) = 0$ (more precisely $u \to 0$ as $x \to 0$). The second physical boundary condition is that the density $r(x)$ be finite at the origin; $r(x = 0) = r_0$, say, with $r_0 > 0$.

Equations (15) and (16) can be combined into a single second-order ordinary differential equation. An appropriate dependent variable is

$$M(x) = \int_0^x r(x')\,\mathrm{d}x', \tag{20}$$

with $M(x)$ obeying the equation

$$\frac{\mathrm{d}^2M}{\mathrm{d}x^2}\left\{1+\frac{\mathrm{d}M}{\mathrm{d}x}-(1-\omega)^2M^2\left(\frac{\mathrm{d}M}{\mathrm{d}x}\right)^{-2}\right\} = +\tfrac{1}{2}\omega\left\{x\frac{\mathrm{d}M}{\mathrm{d}x}+(1-\omega)M\right\}, \tag{21}$$

where

$$u(x) = x-(1-\omega)M(\mathrm{d}M/\mathrm{d}x)^{-1}. \tag{22}$$

The boundary conditions on $M(x)$ are

$$\mathrm{d}M/\mathrm{d}x\,|_{x=0} = r_0 > 0, \qquad M \to r_0 x \quad \text{as} \quad x \to 0. \tag{23}$$

Equation (22) automatically yields $u(x) \to \omega x$ as $x \to 0$. Note that, since $r(x)$ is proportional to the gas density, we require $M(x) \geqslant 0$ and $\mathrm{d}M/\mathrm{d}x \geqslant 0$ for all $x \geqslant 0$.

Before considering the topological structure of solutions to equation (21) for arbitrary ω ($<\frac{1}{2}$), we first examine in detail the case $\omega = 0$ for which a piecewise analytic solution is obtainable.

(b) Solution for $\omega = 0$

For $\omega = 0$ equation (21) reduces to

$$\frac{\mathrm{d}^2M}{\mathrm{d}x^2}\left\{1+\frac{\mathrm{d}M}{\mathrm{d}x}-M^2\left(\frac{\mathrm{d}M}{\mathrm{d}x}\right)^{-2}\right\} = 0, \tag{24}$$

so that *either*

(i) $$\mathrm{d}^2M/\mathrm{d}x^2 = 0, \qquad \text{implying} \quad M = a+bx; \tag{25}$$

or

(ii) $$(\mathrm{d}M/\mathrm{d}x)^3+(\mathrm{d}M/\mathrm{d}x)^2-M^2 = 0. \tag{26}$$

Consider equation (26). On introduction of the parametric variable P, through

$$\mathrm{d}M/\mathrm{d}x = PM, \tag{27}$$

equation (26) yields

$$M = (1-P^2)P^{-3}. \tag{28}$$

Using $\mathrm{d}M/\mathrm{d}x = (\mathrm{d}M/\mathrm{d}P)(\mathrm{d}P/\mathrm{d}x)$, from the relations (27) and (28) we then obtain the equation

$$\mathrm{d}P/\mathrm{d}x = -P^2(1-P^2)(3-P^2)^{-1}$$

whose general solution is

$$(1+P)(1-P)^{-1}\exp(-3P^{-1}) = A\exp(-x), \tag{29}$$

where, at the moment, A is an arbitrary, but positive, constant of integration. Note that the density and flow velocity are given parametrically through the relations

$$r(x) = \frac{\mathrm{d}M}{\mathrm{d}x} = (1-P^2)P^{-2}, \qquad \frac{\mathrm{d}r}{\mathrm{d}x} = 2(1-P^2)P^{-1}(3-P^2)^{-1} \geqslant 0, \tag{30a}$$

$$u(x) = x-P^{-1}, \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2(3-P^2)^{-1} \geqslant 0. \tag{30b}$$

Now the requirement $M \geqslant 0$ demands either $1 \geqslant P \geqslant 0$ or $-\infty \leqslant P \leqslant -1$; the requirement $\mathrm{d}M/\mathrm{d}x \geqslant 0$ then restricts the range of P to $1 \geqslant P \geqslant 0$.

Equation (29) shows, however, that $x(P)$ is a monotonically decreasing function of increasing P in the range $1 \geqslant P \geqslant 0$ with $x(P = 0) = +\infty$ and $x(P = 1) = -\infty$. But $M = 0$ on $x = 0$. Inspection of equation (28) reveals that M is zero only on $P = 1$ in the range $1 \geqslant P \geqslant 0$. Hence it must be concluded that the solution branch following from equation (26) cannot extend to the origin $x = 0$. On the other hand, the solution branch (25), if valid at the origin $x = 0$, implies $M = r_0 x$, $r(x) = r_0$ *and* $u(x) = 0$. Hence this solution branch cannot extend out to the shock front. Thus: the only possibility is that the solution branch (25) extends out from $x = 0$ to some point $x = x_*$; for $x \geqslant x_*$ the solution branch (26) must take over and carry on out to the shock. Consider then the matching conditions at $x = x_*$. We require $u(x_*) = 0$ and $M(x_*) = r_0 x_*$. From equation (30b) we see that $u(x_*) = 0$ implies an associated P_* with

$$P_* = x_*^{-1}. \tag{31}$$

Using equations (31) and (29) we now determine A as a function of P_*:

$$A = (1+P_*)(1-P_*)^{-1}\exp(-2P_*^{-1}). \tag{32}$$

From equation (28) we also have

$$r_0 x_* = (1-P_*^2)P_*^{-3}. \tag{33}$$

Combination of equations (31) and (33) then yields the value for x_*:

$$x_* = (1+r_0)^{\frac{1}{2}}, \qquad P_* = (1+r_0)^{-\frac{1}{2}}. \tag{34}$$

Note that we have $P_* < 1$ as required. The slopes $\mathrm{d}u/\mathrm{d}x$ and $\mathrm{d}r/\mathrm{d}x$ of the solution branch (26) on $x = x_*$ are

$$\mathrm{d}u/\mathrm{d}x = 2(1+r_0)(2+3r_0)^{-1}, \qquad \mathrm{d}r/\mathrm{d}x = 2r_0(1+r_0)^{+\frac{1}{2}}(2+3r_0)^{-1}, \tag{35}$$

so that the solution branch valid for $0 \leqslant x \leqslant x_*$ matches (with discontinuous derivatives) to the solution branch valid for $x > x_*$.

Consider now the shock conditions (18a, b) and (19). From equations (30b) we see that on $x = x_s$, where $u = u_s$, we require $P_s = Yu_s$. But then, from equations (18b) and (30a), we have

$$r_s = (1-P_s^2)P_s^{-2} = 2Y^{-2}(1-Y)u_s^{-2}(1+Yu_s^2)^2(2+Yu_s^2)^{-1}. \tag{36}$$

Equation (36) yields a positive value for u_s solely in terms of Y:

$$u_s = (2Y)^{-\frac{1}{2}}(2-Y)^{-\frac{1}{2}}\{(9+4Y-4Y^2)^{\frac{1}{2}}-(3-2Y)\}^{\frac{1}{2}}, \tag{37}$$

with

$$x_s(Y) = u_s(Y)+\{Yu_s(Y)\}^{-1}, \qquad r_s(Y) = (Yu_s)^{-2}\{1-(Yu_s)^2\} > 0.$$

(Note that we have $P_s \leqslant 1$ $(u_s \leqslant Y^{-1})$.) But, from equation (29) evaluated at $x = x_s$ and $u = u_s$ with equation (32), we obtain

$$(1+P_s)(1-P_s)^{-1}\exp\{-2P_s^{-1}+P_s Y^{-1}\}$$
$$= \{(1+r_0)^{\frac{1}{2}}+1\}\{(1+r_0)^{\frac{1}{2}}-1\}^{-1}\exp\{-2(1+r_0)^{\frac{1}{2}}\}, \tag{38}$$

so that equation (38) determines r_0 as a function solely of Y.

The slopes of the solution branch (26) on the shock curve specified by equations (18) are

$$\frac{du}{dx} = 2\{3-(Yu_s)^2\}^{-1} > 0, \quad \frac{dr}{dx} = 2(Yu_s)^{-1}\{1-(Yu_s)^2\}\{3-(Yu_s)^2\} > 0. \tag{39}$$

Note from equations (30a) and (30b) that, in the range $x_s \geqslant x \geqslant x_*$, $r(x)$ and $u(x)$ are monotonically increasing functions of increasing x, and that $Yu_s < P < (1+r_0)^{-\frac{1}{2}}$. The value of η is then determined as a function of Y alone through

$$\eta = 1+Yu_s^2 = \{1+(9+4Y-4Y^2)^{\frac{1}{2}}\}(4-2Y)^{-1}. \tag{40}$$

Thus we have that:

$$\text{as} \quad Y\to 0, \qquad Yu_s^2 \to 0 \qquad \text{so that} \qquad \eta \to 1; \tag{41a}$$

and

$$\text{as} \quad Y\to 1, \qquad Yu_s^2 \to 1 \qquad \text{so that} \qquad \eta \to 2. \tag{41b}$$

This is most unlike the situation in the complete absence of an external magnetic field (Lerche and Vasyliunas 1976). The difference arises from the magnetic pressure term in the equation of motion. In the complete absence of a magnetic field both the continuity equation *and* the equation of motion contain r only in the form $r^{-1}\,dr/dx$, so that no fundamental scale of density amplitude is present in either equation. On the other hand, inclusion of a finite-strength magnetic field completely destroys this property, since a term directly proportional to dr/dx appears in the equation of motion. Solutions in the absence of a magnetic field are therefore topologically different from those in the presence of a magnetic field (a first-order differential equation for the fluid flow obtains instead of a second-order differential equation). More simply, if $Y = 1$ then $u_s(Y) = 1$ and $r_s = 0$, that is, the scaling transformation (14) fails since division by zero is implied.

Thus in order to obtain a self-similar isothermal blast wave in the case $\omega = 0$ and for a given value of $Y\,(<1)$, it is necessary to match the two solution branches (25) and (26) at a precise value of x. The matching has to be done with discontinuous slopes for $\mathrm{d}u/\mathrm{d}x$ and $\mathrm{d}r/\mathrm{d}x$. All parameters of the solution (namely u_s, η, r_0, r_s, x_* and x_s) are then uniquely determined as specified functions of Y by the requirement that the solution branch (26) must pass through the shock. There is no other self-similar solution with continuous post-shock velocity and density.

3. Topology of Solutions for $\omega \neq 0$

For $\omega \neq 0$, the behaviour of solutions to equation (21) is more difficult to analyse. This is, essentially, a consequence of the fact that when equation (21) is written in the form

$$\frac{\mathrm{d}^2M}{\mathrm{d}x^2} = \tfrac{1}{2}\omega\left(\frac{\mathrm{d}M}{\mathrm{d}x}\right)^2\left\{(1-\omega)M + x\frac{\mathrm{d}M}{\mathrm{d}x}\right\}\left\{\left(\frac{\mathrm{d}M}{\mathrm{d}x}\right)^3 + \left(\frac{\mathrm{d}M}{\mathrm{d}x}\right)^2 - (1-\omega)^2M^2\right\}^{-1} \tag{42}$$

it clearly has movable critical points (Ince 1956). Since the behaviour of solutions is dependent on the structure of the equation at the critical points, and since the structure of movable critical points depends on the initial ($x = 0$) values of M and $\mathrm{d}M/\mathrm{d}x$, an analysis of the topological behaviour of solutions to equation (42) is an extremely difficult problem.†

(a) Behaviour in the Vicinity of the Origin $x = 0$

For $x \approx 0$, the solution to equation (42) with $M(x) = r_0 x + O(x^3) + \ldots$ is

$$M(x) \approx r_0 x + \tfrac{1}{12} r_0\,\omega(2-\omega)(1+r_0)^{-1}x^3 + O(x^5), \tag{43}$$

so that

$$r(x) \approx r_0 + \tfrac{1}{4} r_0\,\omega(2-\omega)(1+r_0)^{-1}x^2 + O(x^4), \tag{44a}$$

$$u(x) \approx \omega x + \tfrac{1}{6}\omega(2-\omega)(1+r_0)^{-1}(1-\omega)x^3 + O(x^5). \tag{44b}$$

From equation (43) it follows that, at small x,

$$D \equiv \left(\frac{\mathrm{d}M}{\mathrm{d}x}\right)^3 + \left(\frac{\mathrm{d}M}{\mathrm{d}x}\right)^2 - (1-\omega)^2M^2$$

$$\approx r_0^2(1+r_0) + \frac{r_0 x^2}{(1+r_0)}\left(\tfrac{5}{4}\omega(2-\omega) - r_0(1+r_0)(1-\omega)^2\right) + O(x^3) \tag{45a}$$

and

$$N \equiv (1-\omega)M + x\,\mathrm{d}M/\mathrm{d}x$$

$$\approx r_0 x(2-\omega)\left\{1 + \tfrac{1}{12}\omega(4-\omega)x^2(1+r_0)^{-1}\right\} + O(x^4). \tag{45b}$$

† For a second-order differential equation *not* to have movable critical points it is necessary that it should be of the form (Ince 1956)

$$\mathrm{d}^2y/\mathrm{d}x^2 = A(x,y)(\mathrm{d}y/\mathrm{d}x)^2 + B(x,y)\,\mathrm{d}y/\mathrm{d}x + C(x,y).$$

This is clearly not the case with equation (42). Hence it has movable critical points.

Consider firstly the behaviour for $\omega < 0$. In this case $\mathrm{d}u/\mathrm{d}x|_{x=0} = \omega < 0$, so that u is negative for small x. Therefore if the fluid flow is ever to cross the shock curve $x_s = u_s + (Yu_s)^{-1}$ in the region $x_s > 0$, it follows that u must eventually become *positive* as x increases. Hence, somewhere between $x = 0$ and the crossing point where $u = 0$ again, u must take on its most negative value where $\mathrm{d}u/\mathrm{d}x = 0$. Let this be at $x = x_*$, with $u(x_*) = -u_0$ (<0); $r(x_*) = r_* > 0$. From equation (15) we then obtain

$$\mathrm{d}r/\mathrm{d}x|_{x=x_*} = -\tfrac{1}{2}\omega u_0 r_*(1+r_*)^{-1} > 0 . \tag{46a}$$

But since we have $\mathrm{d}r/\mathrm{d}x|_{x\to 0} < 0$ it follows that r goes through a minimum in the region $x < x_*$. At the minimum of r, we have $\mathrm{d}^2M/\mathrm{d}x^2 = 0$. However, from equation (21), $\mathrm{d}^2M/\mathrm{d}x^2 = 0$ at $x = x_\dagger$ say only when

$$r(x_\dagger) \equiv \mathrm{d}M/\mathrm{d}x|_{x=x_\dagger} = -(1-\omega)M/x_\dagger < 0 . \tag{46b}$$

But the density must be positive. Hence for $\omega < 0$ it must be concluded that *there is no self-similar solution, with continuous post-shock velocity starting at the origin* $x = 0$, *which eventually crosses the shock*. Hereinafter we therefore restrict our attention to the regime $0 < \omega < \tfrac{1}{2}$.

(b) Miscellaneous Solution Properties for $0 < \omega < \tfrac{1}{2}$

Since we have $u = x-(1-\omega)M/M_x$ and since M and M_x are required to be positive, it follows that u lies in the range $0 \leqslant u < x$. Hence u never crosses the line $u = x$. Thus the assumption made in Section 2 that Λ must be precisely equal to ω in order to avoid a singularity in either ρ or B as u crossed x is not necessary; Λ and ω can take values independently of each other. Nevertheless the situations which correspond to $\Lambda = \omega$ *are* physically permitted. Throughout the remainder of this paper we shall consider only the $\Lambda = \omega$ situations for the reasons given in the footnote following equation (10). Consideration of the more general case $\Lambda \neq \omega$ (which is much more difficult to investigate) is deferred to a later paper.

Equation (42) has a critical point at $x = x_c$, say, where $M = M_c \geqslant 0$, $u = u_c$ and $\mathrm{d}M/\mathrm{d}x \equiv M_c' \geqslant 0$ with

$$(1-\omega)M_c = -x_c M_c', \qquad M_c'^3 + M_c'^2 - (1-\omega)^2 M_c^2 = 0 . \tag{47}$$

Equations (47) yield

$$M_c = -(1-\omega)^{-1}x_c\{(x_c)^2-1\}, \qquad M_c' = (x_c)^2-1 . \tag{48}$$

Thus a critical point does not exist in the physical domain $x > 0$ since the twin requirements $M_c > 0$, $M_c' > 0$ cannot both be met by equations (48). Consider then the behaviour of M in equation (42) as x increases from zero with (43) as the value of M at small x. Three possibilities exist: (i) M varies in such a way that the numerator $N \equiv (1-\omega)M + x\,\mathrm{d}M/\mathrm{d}x$ goes to zero *before* D goes to zero; (ii) M varies in such a way that D goes to zero *before* N goes to zero; (iii) neither D nor N goes to zero and M steadily increases, eventually crossing the shock when

$$M_s = 2(1-Y)Y^{-3}u_s^{-3}(1+Yu_s^2)^2(2+Yu_s^2)^{-1} , \tag{49a}$$

and

$$\mathrm{d}M/\mathrm{d}x|_{x=x_s} = 2(1-Y)u_s^{-2}(1+Yu_s^2)^2 Y^{-2}(2+Yu_s^2)^{-1} . \tag{49b}$$

Consider now the first two possibilities in turn.

(i) If N goes to zero at some x, say $x_\dagger$, then

$$\mathrm{d}^2M/\mathrm{d}x^2\,|_{x=x\dagger} \equiv \mathrm{d}r/\mathrm{d}x\,|_{x=x\dagger} = 0.$$

But from equations (15) and (16) we have

$$\frac{\mathrm{d}r}{\mathrm{d}x} = -r(1+r)^{-1}\left\{\frac{\mathrm{d}u}{\mathrm{d}x}\left(u-x\right)-\tfrac{1}{2}\omega u\right\}, \tag{50a}$$

$$\frac{\mathrm{d}r}{\mathrm{d}x} = -r\left(\frac{\mathrm{d}u}{\mathrm{d}x}-\omega\right)\left(u-x\right)^{-1}. \tag{50b}$$

At $\mathrm{d}r/\mathrm{d}x = 0$ equation (50a) yields

$$\mathrm{d}u/\mathrm{d}x\,|_{x=x\dagger} = -\tfrac{1}{2}\omega u_\dagger(x_\dagger-u_\dagger)^{-1}, \tag{51a}$$

while equation (50b) yields

$$\mathrm{d}u/\mathrm{d}x\,|_{x=x\dagger} = \omega. \tag{51b}$$

Equations (51) are satisfied only when $u_\dagger = 2x_\dagger$. However, if $u(x)$ ever crosses the line $u = 2x$ then, since $\mathrm{d}u/\mathrm{d}x\,|_{x=0} = \omega < 2$, it must do so with a slope $\mathrm{d}u/\mathrm{d}x > 2$. But this contradicts equation (51b), and hence it must be inferred that the assumed crossing does not take place, and that N cannot go to zero before D. We therefore conclude that the first possibility does not obtain.

(ii) Consider the second possibility that as x increases from zero the variation of M is such that D goes to zero before N. In this case we have $\mathrm{d}^2M/\mathrm{d}x^2\,|_{D=0} \to \pm\infty$. Let D go to zero at $x = x_0$ where $M = M_0$ and $\mathrm{d}M/\mathrm{d}x = M'_0$. Then

$$M_0'^2(1+M'_0) = (1-\omega)^2M_0^2,$$

and also, since we have $N > 0$,

$$-x_0 < (1-\omega)M_0/M'_0.$$

Consider the behaviour of M in the neighbourhood of $D = 0$. Write $\zeta = x-x_0$ and $M = M_0+M'_0\zeta+\delta m$ with $\delta m(\zeta{=}0) = 0$ and $\mathrm{d}(\delta m)/\mathrm{d}\zeta\,|_{\zeta=0} = 0$. Then to lowest order in δm from equation (42) we have

$$\delta m_{\zeta\zeta} = \tfrac{1}{2}\omega M_0'^2\{(1-\omega)M_0+x_0M'_0\}\{\delta m_\zeta M'_0(2+3M'_0)-2M_0M'_0(1-\omega)^2\zeta\}^{-1}, \tag{52}$$

where a subscript ζ denotes $\mathrm{d}/\mathrm{d}\zeta$. Equation (52) can be integrated once to give

$$\tfrac{1}{2}\delta m_\zeta^2 M'_0(2+3M'_0)-2M_0M'_0(1-\omega)^2\zeta\,\delta m_\zeta+\delta m\,2M_0M'_0(1-\omega)^2 = \tfrac{1}{2}\omega M_0'^2\{(1-\omega)M_0+x_0M'_0\}\zeta, \tag{53}$$

where we have used the fact that δm_ζ vanishes on $\zeta = 0$ to determine the arbitrary constant of integration.

On the requirement that $\delta m(\zeta=0) = 0 = \delta m_\zeta(\zeta=0)$ the appropriate behaviour of equation (53) in the vicinity of $\zeta = 0$ is

$$\delta m_\zeta = \pm[\omega M'_0(2+3M'_0)^{-1}\{(1-\omega)M_0 + x_0 M'_0\}]^{\frac{1}{2}}\zeta^{\frac{1}{2}}. \tag{54}$$

By hypothesis the coefficient of $\zeta^{\frac{1}{2}}$ in equation (54) is real, since N has not yet reached zero and, as $x \to 0$, N is positive. However, then for $x < x_0$, $\zeta^{\frac{1}{2}}$ is imaginary. Hence as $x \to x_0$ from below δm is pure imaginary. But M is required to be real. Therefore it must be inferred that D does *not* tend to zero while N is still positive. We thus conclude that the second possibility also does not obtain.

Overall it must be concluded that for $1 > Y > 0$, $\omega > 0$ the solutions to equation (42) starting with $M = 0$ at $x = 0$ and $\mathrm{d}M/\mathrm{d}x > 0$ at $x = 0$ lie in the range $0 \leqslant u < 2x$ for all values of $x(>0)$ until the shock curve $x = u+(Yu)^{-1}$ is reached. For $\omega < 0$ no single-valued solution starting at the origin with $u = 0$ and $\mathrm{d}M/\mathrm{d}x > 0$ at $x = 0$ exists which crosses the shock. For $\omega = 0$ we have a singular solution which matches piecewise (with discontinuous derivatives at the matching point) at a given point and crosses the shock curve.

4. Discussion and Conclusions

We have analysed the equilibrium properties of isothermal self-similar blast waves in one dimension propagating away from a plane source explosion into a surrounding medium whose density and magnetic field both vary as $z^{-\omega}$ with $\omega < \frac{1}{2}$. Our main results are the following.

(1) For $\omega < 0$ there do not exist physically acceptable self-similar solutions.

(2) For $\omega = 0$ a singular solution exists which is piecewise continuous.

(3) For $\frac{1}{2} > \omega > 0$, the second-order differential equation describing the fluid flow behind the blast wave has movable critical points. Since the line of movable critical points does not exist in the physical domain $x > 0$ we have been able to show that the physical solution curve is smoothly continuous out to the shock.

To the extent that a three-dimensional blast wave can be regarded as planar (Cox 1972; McCray *et al.* 1975) the results reported here accurately portray the evolution of such a blast wave into a magnetized surrounding medium. (The case $\omega \approx 0$ is often regarded as appropriate in describing the evolution of a supernova remnant.)

The fluid flow behaviour uncovered by the analysis here raises several questions, and suggests further lines of investigation to improve our understanding of blast wave expansion into media in the presence of magnetic fields.

The first question is: what is the qualitative and quantitative modification to the present results when allowance is made for the three-dimensional nature of the explosion? Can the shock curvature really be neglected?

Secondly: what modification results to the flow behaviour, even within the one-dimensional framework, when the variation of density and magnetic field ahead of the blast front are allowed to vary differently, that is, $\rho \propto z^{-\omega}$ and $B \propto z^{-\Lambda}$ $(\Lambda \neq \omega)$?

Thirdly: is the magnetic field a stabilizing or a destabilizing influence on the blast waves? The point here is that it is known that in the absence of an external magnetic field the three-dimensional self-similar isothermal blast waves are linearly and nonlinearly unstable (Lerche and Vasyliunas 1976; Bernstein and Book 1978).

Clearly, since our theoretical understanding of the dynamical evolution of supernova remnants is closely tied to our knowledge of the properties of blast waves, it is of some importance to ascertain the temporal behaviour of perturbations introduced into a spherical blast wave expanding into a magnetoactive medium. In the next paper in this series we shall consider the behaviour of a self-similar isothermal spherical blast wave expanding into a surrounding medium which contains a magnetic field. An investigation of the behaviour of perturbations to the self-similar flow will be deferred to the third paper in the series.

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References

Bernstein, I. B., and Book, D. L. (1978). *Astrophys. J.* **225**, 633.
Caswell, J. L., and Lerche, I. (1979*a*). *Mon. Not. R. Astron. Soc.* **187**, 201.
Caswell, J. L., and Lerche, I. (1979*b*). *Aust. J. Phys.* **32**, 79.
Cox, D. P. (1972). *Astrophys. J.* **178**, 143.
Gorenstein, P., Harnden, F. R., Jr, and Tucker, W. H. (1974). *Astrophys. J.* **192**, 661.
Ince, E. L. (1956). 'Ordinary Differential Equations' (Dover: New York).
Korobeinikov, B. P. (1956). *J. Acad. Sci. USSR* **109**, 271.
Landau, L., and Lifschitz, E. M. (1959). 'Fluid Mechanics' (Addison–Wesley: Reading, Mass.).
Lerche, I., and Vasyliunas, V. M. (1976). *Astrophys. J.* **210**, 85.
McCray, R., Stein, R. F., and Kafatos, M. (1975). *Astrophys. J.* **196**, 565.
Parker, E. N. (1963). 'Interplanetary Dynamical Processes' (Interscience: New York).
Rappaport, S., Doxsey, R., Solinger, A., and Borken, R. (1974). *Astrophys. J.* **194**, 329.
Sedov, L. (1959). 'Similarity and Dimensional Methods in Mechanics' (Academic: New York).
Solinger, A., Rappaport, S., and Buff, J. (1975). *Astrophys. J.* **201**, 381.
Woltjer, L. (1972). *Annu. Rev. Astron. Astrophys.* **10**, 129.

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