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Perturbed Characteristic Functions: Application to the Linearized Gravitational Field

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Abstract

The first-order perturbation induced by the first-order perturbation of a given Lagrangian in its associated characteristic function V is given by a simple integral along the unperturbed extremals. This result is applied to the world characteristic of the linearized gravitational field. Various specialized situations are considered. For example, to obtain $V^{(1)}$ in the quasi-Newtonian approximation one merely needs to evaluate a three-dimensional volume integral over the product of the energy density and a type of two-point Green's function.

1. Introduction

Let $\{q^k(u)\}$ be a set of n arbitrarily selected, continuous, sufficiently often differentiable functions of one independent variable u. A set of values of

$$q^1(u), q^2(u), \ldots, q^n(u), u$$

may be represented by a point P in an $(n+1)$-dimensional representative space R_{n+1}. As u increases continuously from $u = t$ to t' the point P traces out a representative curve C which joins the initial point $A[x^1, \ldots, x^n, t]$ to the final point $A'[x^{1'}, \ldots, x^{n'}, t']$, where $q^k(t) =: x^k$ and $q^k(t') =: x^{k'}$. An arbitrary neighbouring curve C^* through the same endpoints is generated by the set of functions

$$\{q^{*k}(u) := q^k(u) + \varepsilon\, \phi^k(u)\},$$

where ε is a numerical parameter such that $|\varepsilon|$ is sufficiently small and $\{\phi^k\}$ is a set of arbitrary differentiable functions which vanish at the endpoints. If L is a given function of $\{q^k\}$, the first derivatives $\{\dot{q}^k\}$ of these, and of u explicitly, define the functional

$$\tilde{V} := \int_A^{A'} L(\dot{q}^k, q^l, u)\, \mathrm{d}u\,.$$

Then a 'variational problem' is a prescription which selects a particular curve E joining A and A' by the requirement that the value of $\tilde{V}$ evaluated by integration along E is stationary as compared with its value when integrated along an arbitrary neighbouring curve E^* through the same endpoints. It will be taken for granted that there is only one such curve when values of $\{|x^{k'}-x^k|, |t'-t|\}$ are sufficiently small. The curve E is the 'extremal' of the variational problem. The corresponding value V of $\tilde{V}$ depends solely upon the coordinates of A and A' so that the variational problem generates a function $V(x^{1'}, ..., x^{n'}, t', x^1, ..., x^n, t)$ of $2n+2$ variables; and this is the 'characteristic function' of the problem. In physical applications it sometimes goes under other names, e.g. 'point characteristic' in geometrical optics, 'principal function' in analytical dynamics, 'world characteristic' in general relativity theory. At any rate, given the explicit form of V one is in possession of all possible information about the extremals, i.e. of the solution of the variational problem: having chosen any pair of points A, A' the derivatives of V with respect to x^k and $x^{l'}$ provide the directions at these points of the extremal which joins them. Explicitly, let $v_k(u) := \partial L/\partial \dot{q}^k$ and write $p_k := v_k(t)$ and $p_{k'} := v_k(t')$. Then (e.g. Lanczos 1970)

$$\partial V/\partial x^{k'} = p_{k'}, \qquad \partial V/\partial x^k = -p_k. \tag{1a, b}$$

If one regards A and the values of the p_k as given, the set of equations (1b) constitutes the equations of the extremal; and they satisfy the equations which express the vanishing of the functional derivatives of L with respect to $\{q^k\}$. The function V satisfies a pair of first-order partial differential equations but these may be left aside here.

Contemplate now a family of Lagrangians $L(\lambda)$ of the kind

$$L(\lambda) = L^{(0)} + \lambda L^{(1)} + \lambda^2 L^{(2)} + \dots, \tag{2}$$

where the $L^{(s)}$ $(s = 0, 1, ...)$ are given functions and λ is a numerical parameter, sufficiently small in absolute value. Correspondingly,

$$V(\lambda) = V^{(0)} + \lambda V^{(1)} + \lambda^2 V^{(2)} + \dots \tag{3}$$

will be the characteristic function which belongs to $L(\lambda)$. Given $V^{(0)}$ there now arises the problem of determining the 'perturbations' $V^{(1)}, V^{(2)}, ...$ of V. In this paper I consider $V^{(1)}$ alone. That the required result then follows almost trivially (see Section 2*a* below) in no way detracts from its usefulness.

In applications the situation is particularly simple of course when $V^{(0)}$ is known from first principles; for instance when $L^{(0)}$ corresponds to (i) a homogeneous medium in geometrical optics or (ii) flat space in general relativity theory. It is just the second of these examples which is considered here in greater detail. In that case λ may be thought of as Newton's constant whilst one may take

$$V^{(0)} = |\eta_{ij}(x^{i'}-x^i)(x^{j'}-x^j)|^{\frac{1}{2}}, \tag{4}$$

where $\eta_{ij} = \mathrm{diag}(-1, -1, -1, 1)$. The corresponding extremals (i.e. geodesics) are straight lines. Further, the explicit form of $L^{(1)}$ is provided by the solution of the linearized gravitational field equations (Section 2*b*). The function $L^{(1)}$ has to be integrated along the unperturbed geodesics (Section 2*c*) and in the special case when the field is stationary this integration can be carried out once and for all (Section 2*d*).

Then the determination of $V^{(1)}$ only requires the evaluation of three-dimensional (cartesian) volume integrals of the form

$$\int d^3\xi \; S_{ij}(\xi) F(\xi; x', x),$$

where

$$F(\xi; x', x) := \operatorname{artanh}[|x'-x|/(|x'-\xi| + |x-\xi|)]$$

is a kind of two-point Green's function and $S_{ij}(\xi)$ is a given source tensor (x' and x are the cartesian spatial coordinates of the endpoints). One may further specialize to the even simpler quasi-Newtonian approximation (Section 2*e*) which includes the Schwarzschild field remote from the origin (Section 2*f*). In the static case the optical point characteristic (to within terms $O(\lambda^2)$) may be extracted directly from the world function Ω (Section 2*g*).

2. The Perturbation $V^{(1)}$

(*a*) *General Result*

Quite generally

$$V(\lambda) - V^{(0)} = \int_{E(\lambda)} L(\lambda)\, du - \int_{E(0)} L^{(0)}\, du,$$

where $E(\lambda)$ denotes the extremal joining A and A' defined by $L(\lambda)$. Therefore

$$V(\lambda) - V^{(0)} = \int_{E(\lambda)} L^{(0)}\, du + \lambda \int_{E(\lambda)} L^{(1)}\, du - \int_{E(0)} L^{(0)}\, du + O(\lambda^2).$$

Because of the extremality of the curve $E(0)$ the difference between the first and third integrals is $O(\lambda^2)$, whilst

$$\int_{E(\lambda)} L^{(1)}\, du - \int_{E(0)} L^{(1)}\, du = O(\lambda).$$

There follows at once the required result that

$$V^{(1)} = \int_{E(0)} L^{(1)}\, du. \tag{5}$$

In an optical context an analogous result appears in Section 109 of the text by Buchdahl (1970*a*).

(*b*) *Generic Form for Weak Gravitational Fields*

When V belongs to the Lagrangian

$$L = |g_{ij}\dot{q}^i \dot{q}^j|^{\frac{1}{2}}, \tag{6}$$

where g_{ij} is the metric tensor of a four-dimensional normal hyperbolic Riemann space (of signature -2), I call it the world characteristic. Synge's (1960) world function is then given by

$$\Omega := \tfrac{1}{2}\varepsilon V^2, \tag{7}$$

where $\varepsilon = 1, 0, -1$ for time-like, light-like and space-like geodesics respectively.

Confronted with sufficiently weak gravitational fields one may adopt coordinates such that $g_{ij} = \eta_{ij}+\lambda h_{ij}$, where λ may be identified with Newton's constant. For the time being it is convenient to contemplate only time-like geodesics. Then from equation (6)

$$L^{(0)} = (\eta_{ij}\dot{q}^i\dot{q}^j)^{\frac{1}{2}}, \qquad L^{(1)} = \tfrac{1}{2}h_{ij}\dot{q}^i\dot{q}^j/L^{(0)}. \tag{8a, b}$$

According to equation (5) the function $L^{(1)}$ is required only on the curve $E(0)$ which is here a straight line. Further, the special parameter u may be taken to be the geodesic arc length along $E(0)$, varying from 0 to s ($\equiv V^{(0)}$) as one goes from A to A'. Therefore in equation (8b) we have

$$q^i = (x^{i'}-x^i)u/s+x^i, \tag{9}$$

whence

$$\dot{q}^i = (x^{i'}-x^i)/s =: \alpha^i. \tag{10}$$

The denominator of equation (8b) now reduces to unity so that

$$L^{(1)} = \tfrac{1}{2}h_{ij}\alpha^i\alpha^j. \tag{11}$$

Restoring the argument q^k to h_{ij}, one has finally

$$V^{(1)} = \tfrac{1}{2}\alpha^i\alpha^j\int_0^s h_{ij}(\alpha^k u+x^k)\,\mathrm{d}u. \tag{12}$$

(*c*) *Linearized Gravitational Field due to Given Sources*

Let the gravitational field contemplated in the preceding section be due to sources described by the energy–momentum tensor $T_{ij}(q^k)$. Then (e.g. Eddington 1923)

$$h_{ij} = -4\int\frac{S_{ij}(\boldsymbol{\xi}, q^4-|\boldsymbol{q}-\boldsymbol{\xi}|)}{|\boldsymbol{q}-\boldsymbol{\xi}|}\,\mathrm{d}^3\xi, \tag{13}$$

where $S_{ij} := T_{ij}-\tfrac{1}{2}\eta_{ij}T$. (If w^k is any contravariant vector, I write $w^k =: (\boldsymbol{w}, w^4)$ and $w := (-\eta_{ab}w^a w^b)^{\frac{1}{2}} = |\boldsymbol{w}|$. Indices a, b, c go over the range 1, 2, 3 and it should be borne in mind that on the right of equation (13) one is implicitly dealing with the Euclidean metric η_{ij}. The quantity $\boldsymbol{\xi}$ is an auxiliary cartesian three-vector.)

Equation (12) now becomes, using equations (9) and (10),

$$V^{(1)} = -2\alpha^i\alpha^j\int\mathrm{d}^3\xi\int_0^s\frac{S_{ij}(\boldsymbol{\xi}, \alpha^4 u+t-|\boldsymbol{\alpha}u+\boldsymbol{x}-\boldsymbol{\xi}|)}{|\boldsymbol{\alpha}u+\boldsymbol{x}-\boldsymbol{\xi}|}\,\mathrm{d}u. \tag{14}$$

It is of some advantage to choose a new variable of integration $\bar{u} := u-\tfrac{1}{2}s$. At the same time write

$$\boldsymbol{\eta}' := \boldsymbol{x}'-\boldsymbol{\xi}, \qquad \boldsymbol{\eta} = \boldsymbol{x}-\boldsymbol{\xi}. \tag{15}$$

Omitting the bar, one then obtains from equation (14)

$$V^{(1)} = -2\alpha^i\alpha^j\int\mathrm{d}^3\xi\int_{-\frac{1}{2}s}^{\frac{1}{2}s}\frac{S_{ij}\big(\boldsymbol{\xi}, \alpha^4 u+\tfrac{1}{2}(t'+t)-|\boldsymbol{\alpha}u+\tfrac{1}{2}(\boldsymbol{\eta}'+\boldsymbol{\eta})|\big)}{|\boldsymbol{\alpha}u+\tfrac{1}{2}(\boldsymbol{\eta}'+\boldsymbol{\eta})|}\,\mathrm{d}u. \tag{16}$$

This is as far as one can go in the general case.

(d) Stationary Case

Suppose now that the sources are stationary in the sense that the T_{ij} are independent of q^4. Equation (16) then reduces to

$$V^{(1)} = -2\alpha^i\alpha^j \int S_{ij}(\xi)\,\Gamma(\xi; x^{k'}, x^l)\,\mathrm{d}^3\xi\,, \tag{17}$$

where

$$\Gamma(\xi; x^{k'}, x^l) := \int_{-\frac{1}{2}s}^{\frac{1}{2}s} \frac{\mathrm{d}u}{|\boldsymbol{\alpha} u + \frac{1}{2}(\boldsymbol{\eta}' + \boldsymbol{\eta})|}\,. \tag{18}$$

The integral on the right of equation (18) is elementary but its evaluation is tedious. Bearing in mind that

$$s\boldsymbol{\alpha} = \boldsymbol{x}' - \boldsymbol{x} = \boldsymbol{\eta}' - \boldsymbol{\eta}$$

and so

$$|\boldsymbol{x}' - \boldsymbol{x}| =: l = \alpha s\,,$$

one finds that

$$\Gamma = 2sl^{-1}\,\mathrm{artanh}[l/(\eta' + \eta)]\,. \tag{19}$$

Thus, explicitly,

$$V^{(1)} = -4\alpha^i\alpha^j s l^{-1} \int S_{ij}(\xi)\,\mathrm{artanh}\left(\frac{l}{|\boldsymbol{x}' - \boldsymbol{\xi}| + |\boldsymbol{x} - \boldsymbol{\xi}|}\right)\mathrm{d}^3\xi\,. \tag{20}$$

(e) Quasi-Newtonian Approximation

The quasi-Newtonian approximation corresponds to the assumption that all internal motions are sufficiently slow. This is equivalent to the assumption that the component $T_{44} =: \rho$, the energy density, is effectively the sole non-vanishing component of T_{ij}. Then

$$S_{ij} = \tfrac{1}{2}\rho\,\mathrm{diag}(1, 1, 1, 1)\,, \tag{21}$$

and

$$\alpha^i\alpha^j S_{ij} = \tfrac{1}{2}[l^2 + (t' - t)^2]s^{-2}\rho\,.$$

Consequently,

$$V^{(1)} = -\frac{2[l^2 + (t' - t)^2]}{ls} \int \rho(\xi)\,\mathrm{artanh}\left(\frac{l}{|\boldsymbol{x}' - \boldsymbol{\xi}| + |\boldsymbol{x} - \boldsymbol{\xi}|}\right)\mathrm{d}^3\xi \tag{22}$$

and, since $\Omega^{(0)} = \frac{1}{2}s^2$,

$$\Omega^{(1)} = s\,V^{(1)}\,. \tag{23}$$

In the strictly Newtonian approximation one contemplates only geodesics for which α is negligible compared with α^4, that is, l negligible compared with $t' - t$. The factor $l^2 + (t' - t)^2$ on the right of equation (22) is then to be replaced by $(t' - t)^2$.

(f) Schwarzschild Field

Since $\rho(\xi)$ in equation (22) denotes the 'Newtonian density', a point source corresponds to the choice

$$\rho(\xi) = m\,\delta(\xi)/2\pi\xi^2\,, \tag{24}$$

where m is the mass of the source. The required integration is now trivial. Identifying

m with the parameter λ, units having now been chosen as usual so as to give the speed of light and Newton's constant the value unity, one has finally

$$\Omega = \tfrac{1}{2}s^2 - 2ml^{-1}[l^2 + (t'-t)^2]\operatorname{artanh}[l/(|\boldsymbol{x}'| + |\boldsymbol{x}|)] + O(m^2). \tag{25}$$

This result is not so easily obtained otherwise. However, it may be confirmed that it is in harmony with the series found by Buchdahl and Warner (1979). There is, of course, no need to confine oneself to time-like geodesics now, that is, Ω need not be positive.

(g) Optical Point Characteristic $\tilde{V}$

Given a static gravitational field, take adapted coordinates such that $g_{a4} = 0$ and $g_{ij,4} = 0$. The optical point characteristic $\tilde{V}$ may then be obtained from Ω by replacing $t'-t$ by $\tilde{V}$ and solving the equation $\Omega = 0$ for $\tilde{V}$ (cf. Buchdahl 1970*b*). It will suffice to choose the special result (25) for the purpose of illustration. One finds at once that

$$\tilde{V} = l + 4m\operatorname{artanh}[l/(r'+r)] + O(m^2), \tag{26}$$

where $r := |\boldsymbol{x}|$ and $r' := |\boldsymbol{x}'|$.

In an equatorial plane, say the plane $z = 0$, consider the ray which joins the point (x, y) to the point $(x'=x, y'=-y)$. For this ray, from equation (26),

$$-\psi := \partial\tilde{V}/\partial x = 2m|y|/x(x^2+y^2)^{\frac{1}{2}} + O(m^2).$$

Now let $|y| \to \infty$ and reject all terms not linear in m. Then $\psi_{|y|=\infty} = -2m/x$. However, to the required order, x is the distance R of closest approach to the origin and ψ is the angle between the ray and the y axis. By symmetry, the total deflection is $2|\psi| = 4m/R$, which is the usual result. (There is, of course, no meaningful Newtonian approximation in this context, since l is now very nearly equal to $t'-t$.)

3. Concluding Remarks

The question arises as to the possibility of extending the approximation to V to a higher order in λ by extending the procedure adopted above. The situation is that such higher order approximations can indeed be found though the work involved is very cumbersome. It must be borne in mind, however, that experience shows other methods for finding approximations to V also to be very cumbersome. I hope to return to this problem at a later date.

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