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Nonstatic Spherically Symmetric Isotropic Solutions for a Perfect Fluid in General Relativity

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Abstract

The present author (Wyman 1946) showed that all perfect fluids which can be represented by nonstatic, spherically symmetric, isotropic solutions of the Einstein field equations can be found by solving a nonlinear total differential equation of the second order involving an arbitrary function $\Psi(r)$. Since then several particular solutions of this equation have been found. Although the four solutions given recently by Chakravarty *et al.* (1976) involve particular choices of $\Psi(r)$, none of these is the general solution of the equation that results from the specific choice of $\Psi(r)$ that was made. The present paper shows how these four general solutions are obtained.

In a recent paper, Chakravarty *et al.* (1976) considered the problem of determining nonstatic, spherically symmetric, isotropic solutions of the Einstein field equations. Although the specialized technique they used did yield specialized solutions, their method does not, in any instance, find the general solution of the differential equations they seek to solve. The purpose of the present note is to obtain all of these general solutions, and to show that the particular solutions listed by Chakravarty *et al.* flow naturally from these general solutions.

If the line element is assumed to have the spherically symmetric isotropic form

$$\mathrm{d}s^2 = \exp(\nu)\,\mathrm{d}t^2 - \exp(\mu)\,(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2 \sin^2\theta\,\mathrm{d}\phi^2) \tag{1}$$

then, as proved by the present author (Wyman 1946), the perfect fluid solutions of the Einstein field equations can be found by solving

$$\exp(\tfrac{1}{2}\mu)\left\{\frac{\partial^2 \mu}{\partial r^2} - \tfrac{1}{2}\left(\frac{\partial \mu}{\partial r}\right)^2 - \frac{1}{r}\frac{\partial \mu}{\partial r}\right\} = \Psi(r), \tag{2}$$

where $\Psi(r)$ is an arbitrary function of its argument. When $\mu = \mu(r,t)$ is known, the second gravitational potential is given by

$$\nu = 2\ln(\partial\mu/\partial t) + \Phi(t), \tag{3}$$

where $\Phi(t)$ is also an arbitrary function of its argument. The substitution $x = r^2$ and $R = \exp(-\tfrac{1}{2}\mu)$ reduces equation (2) to the form

$$R'' = \Gamma(x)\,R^2, \tag{4}$$

where $\Gamma(x)$ is an arbitrary function of x, and primes are used to denote differentiation with respect to x.

Differential equations of the type (4) were studied in considerable detail by Ince (1927). He introduced three functions of x, namely λ, γ and τ which satisfied the following conditions

$$\lambda = (6/\Gamma)^{1/5}, \qquad \gamma' = \lambda^{-2}, \qquad \tau = \lambda''/2\lambda\Gamma. \tag{5a–5c}$$

The substitution

$$R = \lambda y + \tau, \qquad z = \gamma(x) \tag{6}$$

places equation (4) into the form

$$\partial^2 y/\partial z^2 = 6y^2 + S, \qquad \text{where} \qquad S = (\Gamma\tau^2 - \tau'')/(\gamma')^{3/2}. \tag{7}$$

Hence, if S is a constant, say $S = -\frac{1}{2}g_2$, then

$$\partial^2 y/\partial z^2 = 6y^2 - \tfrac{1}{2}g_2, \tag{8}$$

and we therefore obtain

$$(\partial y/\partial z)^2 = 4y^3 - g_2\, y - g_3, \tag{9}$$

where g_3 is an arbitrary function of t. The general solution of this equation is well known to be

$$y = \wp(z+\alpha;\, g_2, g_3), \tag{10}$$

where $\wp$ is the Weierstrass elliptic function and α is a second arbitrary function of t. In this instance, the success of the Ince method depends on the finding of a $\Gamma(x)$ for which S is a constant. The particular choice:

$$\Gamma = 6(ax^2 + 2bx + c)^{-5/2}, \tag{11}$$

where a, b and c are arbitrary constants, leads to

$$\lambda = (ax^2 + 2bx + c)^{\frac{1}{2}}, \qquad \gamma = \int \frac{\mathrm{d}x}{ax^2 + 2bx + c}, \qquad \tau = h(ax^2 + 2bx + c)^{\frac{1}{2}}, \tag{12}$$

with

$$h = (ac - b^2)/12, \qquad S = -6h^2. \tag{13}$$

Equation (4) then becomes

$$R'' = 6(ax^2 + 2bx + c)^{-5/2} R^2. \tag{14}$$

Hence, the general solution of equation (14) is

$$R = (ax^2 + 2bx + c)^{\frac{1}{2}} \{ \wp(z+\alpha;\, 12h^2, g_3) + h \}, \tag{15}$$

where

$$z = \int \frac{\mathrm{d}x}{ax^2 + 2bx + c}. \tag{16}$$

The elementary integration of equation (16) leads to the four following cases

Case A. $\quad a = b = 0; \qquad z = x/c. \qquad (17)$

Case B. $\quad a = 0,\ b \neq 0; \qquad z = \tfrac{1}{2}b^{-1}\ln(2bx+c). \qquad (18)$

Case C. $\quad a \neq 0,\ 12h = ac-b^2 = 0; \qquad z = -(ax+b)^{-1}. \qquad (19)$

Case D. $\quad a \neq 0,\ h \neq 0; \qquad z = (12h)^{-\frac{1}{2}}\arctan\big((12h)^{-\frac{1}{2}}(ax+b)\big). \qquad (20)$

For case D, it has not been assumed that $h > 0$; if $h < 0$, the resulting expression for z is still real. Since α is an arbitrary function of t, a constant of integration was not included in the four integrations of equation (16). Coupled with equation (14), the above cases yield four different differential equations, the solutions of which (in each case) will be determined by the Ince method.

It is well known that the Weierstrass elliptic function will degenerate into an elementary function if and only if

$$g_3 = \pm 8h^3. \tag{21}$$

Without specifying the sign of h, the elementary solutions are known and are given by

$$R = (ax^2+2bx+c)^{\frac{1}{2}}\big\{\pm 3h\cot^2\big((\pm 3h)^{\frac{1}{2}}(z+\beta)\big) + h \pm 2h\big\}, \tag{22}$$

where the same sign must be chosen in each term. The special case $h = 0$ can be obtained by a limiting procedure as

$$R = (z+\beta)^{-2}. \tag{23}$$

Now, the recent solutions found by Chakravarty *et al.* (1976) correspond to the two solutions given in equation (22) for the conditions given in (20). When the plus signs are chosen in equation (22) we obtain

$$R = 3h(ax^2+2bx+c)^{\frac{1}{2}}\operatorname{cosec}^2\big\{\tfrac{1}{2}\arctan\big((12h)^{-\frac{1}{2}}(ax+b)\big) + \beta\big\}, \tag{24}$$

where $(3h)^{\frac{1}{2}}\beta$ has been replaced by β in (22). Hence we have

$$\begin{aligned} R &= \frac{6h(ax^2+2bx+c)^{\frac{1}{2}}}{1-\cos\big\{\arctan\big((12h)^{-\frac{1}{2}}(ax+b)\big)+\beta\big\}} \\ &= \frac{6h(ax^2+2bx+c)}{(ax^2+2bx+c)^{\frac{1}{2}} - (12h/a)^{\frac{1}{2}}\cos\beta + (ax+b)(a)^{-\frac{1}{2}}\sin\beta}. \end{aligned} \tag{25}$$

Regardless of the sign of h or a, equation (25) can always be put in a form in which R is real. For example, if a is negative then h/a is positive, where we have taken $c > 0$ so that $(ax^2+2bx+c)^{\frac{1}{2}}$ is real at $r = x = 0$. Replacing β by $(-1)^{\frac{1}{2}}\beta$, makes equation (25) take the form

$$R = \frac{6h(ax^2+2bx+c)}{(ax^2+2bx+c)^{\frac{1}{2}} - (12h/a)^{\frac{1}{2}}\cosh\beta + (ax+b)(-a)^{-\frac{1}{2}}\sinh\beta}. \tag{26}$$

In any event, the gravitational potentials μ and ν are given by

$$\exp\mu = R^{-2}$$

$$= \frac{[(ar^4+2br^2+c)^{\frac{1}{2}} - (12h/a)^{\frac{1}{2}}\cos\beta + (ar^2+b)a^{-\frac{1}{2}}\sin\beta]^2}{36h^2(ar^4+2br^2+c)}, \tag{27a}$$

$$\exp\nu = K(t)\left(\frac{(12h/a)^{\frac{1}{2}}\sin\beta + (ar^2+b)a^{-\frac{1}{2}}\cos\beta}{(ar^4+2br^2+c)^{\frac{1}{2}} - (12h/a)^{\frac{1}{2}}\cos\beta + (ar^2+b)a^{-\frac{1}{2}}\sin\beta}\right)^2, \tag{27b}$$

where K and β are arbitrary functions of t. Returning to equation (22), the choice of minus signs yields

$$R = -h(ax^2+2bx+c)[3\cot^2\{\tfrac{1}{2}\operatorname{arctanh}((-12h)^{\frac{1}{2}}(ax+b)) + \beta\} + 1], \tag{28}$$

and this seems to be the simplest form in which this solution can be placed.

Cases A, B and C above yield three differential equations whose general solutions are now easily determined. These equations have particular solutions which can be expressed in terms of elementary functions. By using equations (22) and (23) it is not difficult to show that these elementary solutions correspond to the so-called known solutions listed by Chakravarty *et al.* (1976).

The Ince method of finding solutions of equation (4) depends for its success on finding the functions $\Gamma(x)$ for which

$$S = (\Gamma\tau^2 - \tau'')/(\gamma')^{3/2} = \text{const.} \tag{29}$$

It is not too difficult to prove that equation (29) implies that λ must be a solution of

$$2(\lambda^4\lambda'')'' - \lambda^3(\lambda'')^2 = \text{const.}\,\lambda^{-3}. \tag{30}$$

Once λ is known, Γ is determined by equation (5a).

Although equation (22) does give a necessary and sufficient condition for the Ince procedure to reduce (4) to the Weierstrass elliptic differential equation, solutions of (22) have not proven easy for the present author to find. In a recent paper (Wyman 1976) he used a slightly modified Ince procedure to yield extremely complicated solutions of the problem.

References

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