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The Inverse Scattering Problem for Static Meson Fields and Nonlocal Potentials

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Abstract

It is shown that unlike the problem of determining interaction potentials from phase shifts (a problem which has no unique solution), the meson source density of a nucleon can be determined completely from a knowledge of the reaction matrix for the scattering process. Two main classes of nonlocal interactions are discussed and the conditions for a unique solution to exist are obtained.

Introduction

The problem of determining interaction potentials given cross section data has been examined by many authors, but the method of using reaction matrix parameters was first put forward by Cook (1972). The notation of that paper is used here.

The equation satisfied by the meson interacting with the meson cloud of a nucleon is as given by Drell *et al.* (1956) and also Edwards and Matthews (1957). The Schrödinger equation for the problem is

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{l(l+1)}{r^2}+q^2\right)U_{TJ}(r) = -\frac{C_{TJ}\,V(r)}{\omega}\int_0^\infty V(r')\,U_{TJ}(r')\,\mathrm{d}r' = \Theta_{TJ}(r)\,, \tag{1}$$

where $q^2 = \omega^2-m^2$, with q the meson momentum in the laboratory frame and m its mass, U_{TJ} is the meson wavefunction, with T the total isotopic spin of the state and J the total angular momentum, C_{TJ} is a constant whose value depends upon T and J, and $V(r) = r\,\mathrm{d}\rho/\mathrm{d}r$, with ρ the meson source density.

The subscript TJ is now dropped for convenience, and the wavefunction defined in equation (1) is expanded into the orthogonal set of eigenfunctions

$$U(r) = \sum_\lambda A_\lambda(q^2)\,U_\lambda(r)\,, \tag{2}$$

where

$$A_\lambda(q^2) = \frac{U_\lambda(a)}{q_\lambda^2-q^2}\left(\frac{\mathrm{d}U}{\mathrm{d}r}\right)_a(1-B\mathscr{R})\,,\qquad \mathscr{R} = R/(1+BR)\,,\qquad R = a^{-1}\sum_\lambda U_\lambda^2/(q_\lambda^2-q^2)\,, \tag{3a, b, c}$$

while the interaction radius a and the boundary condition parameter B are selected so that

$$(\mathrm{d}U_\lambda/\mathrm{d}r)_{r=a} = (B/A)\,U_\lambda(a)\,. \tag{4}$$

The corresponding free meson wave equation is

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{l(l+1)}{r^2}+Q^2\right)W(r)=0, \tag{5}$$

where Q is the free particle momentum, and we use the boundary condition (4) on the wavefunction $W(r)$ to define an orthogonal set of wavefunctions $W_\mu(r)$. A matrix $B_{\lambda\mu}$ is defined by the relationship

$$U_\lambda(r)=\sum_\mu B_{\lambda\mu}W_\mu(r) \tag{6}$$

and $\mathbf{B}$ satisfies

$$\mathbf{B}\mathbf{B}^{\mathrm{T}}=\mathbf{I}. \tag{7}$$

Owing to the form of equation (1), an interaction matrix can be defined as

$$V_{\lambda\mu}=(q_\lambda^2-Q_\mu^2)B_{\lambda\mu}, \tag{8}$$

which satisfies

$$\Theta=\sum_\lambda A_\lambda(q^2)\sum_\mu V_{\lambda\mu}W_\mu(r). \tag{9}$$

We now investigate the nature of the interaction term Θ. It is assumed throughout this paper that all potentials have a finite range a, so that integration to infinity over r may be replaced by integration out to the range a.

Interaction Term

By substituting the interaction term (9) into equation (1), and using equations (6) and (8), we find that the interaction matrix is a matrix of rank 1 given by

$$V_{\lambda\mu}=c\theta_\lambda\xi_\mu/\omega_\lambda, \tag{10}$$

where

$$\theta_\lambda=\int_0^a V(r)\,U_\lambda(r)\,\mathrm{d}r \qquad\text{and}\qquad \xi_\mu=\int_0^a V(r)\,W_\mu(r)\,\mathrm{d}r. \tag{11a, b}$$

We also have the discrete energies

$$\omega_\lambda=(q_\lambda^2+m^2)^{\frac{1}{2}}.$$

Eliminating $V_{\lambda\mu}$ from equations (8) and (10), we see that

$$B_{\lambda\mu}=c\theta_\lambda\xi_\mu/\omega_\lambda(q_\lambda^2-Q_\mu^2). \tag{12}$$

However, we also have by way of equation (6) the conditions

$$\theta_\lambda=\sum_\mu B_{\lambda\mu}\xi_\mu \qquad\text{and}\qquad \xi_\mu=\sum_\lambda B_{\lambda\mu}\theta_\lambda. \tag{13a, b}$$

Substituting equation (12) into (13a) and (13b), we get the solutions

$$c\sum_\lambda \theta_\lambda^2/\omega_\lambda(q_\lambda^2-Q_\mu^2)=\mathscr{I}_\mu, \qquad\text{where}\qquad \mathscr{I}_\mu\equiv(1,1,1,\dots,1),$$

and

$$\sum_\mu \xi_\mu^2/(q_\lambda^2-Q_\mu^2)=\omega_\lambda/c.$$

These equations yield

$$\theta_\lambda^2 = \sum_\mu [q_\lambda^2 - Q_\mu^2]^{-1} \mathscr{I}_\mu \omega_\lambda / c \qquad \text{and} \qquad \xi_\mu^2 = \sum_\lambda [q_\lambda^2 - Q_\mu^2]^{-1} \omega_\lambda / c, \tag{14a, b}$$

where the quantity in square brackets is to be treated as a matrix. Thus exact solutions in terms of the momentum eigenvalues alone exist for θ_λ and ξ_μ up to a sign on each. Since, however, the matrix (12) must also satisfy the equations

$$U_\lambda(a) = \sum_\mu B_{\lambda\mu} W_\mu(a) \qquad \text{and} \qquad W_\mu(a) = \sum_\lambda B_{\lambda\mu} U_\lambda(a), \tag{15}$$

and the $U_\lambda(a)$ and $W_\mu(a)$ are known quantities, the matrix $\mathbf{B}$ is overdetermined by $2N$ degrees of freedom, where $\mathbf{B}$ is an $N \times N$ matrix. This can be overcome by supposing that only part of the spectrum of eigenvalues of q_λ^2 has been determined. It can be taken into account by leaving an additional row and column of $\mathbf{B}$ undetermined, thereby introducing a further $2N-1$ unknowns found by solving equation (2) and leaving the constant c to be determined. Thus a knowledge of the phase shifts permits a complete solution for the meson source density

$$\rho(r) = \int_r^a \{V(r')/r'\}\, \mathrm{d}r', \tag{16}$$

where

$$V(r) = \sum_\mu \xi_\mu W_\mu(r). \tag{17}$$

Equation (1) can then be used to find Θ.

It may be true that as $N \to \infty$, equations (13) and (15) are not inconsistent and the matrix $\mathbf{B}$ is not overdetermined in this limit. From a practical point of view, one can usually determine only a few of the components of $U_\lambda(a)$ from the fitted reaction matrix poles, while the infinity of undetermined poles contributes a roughly constant background contribution to each element of the matrix. The above device of including one extra row and column of $B_{\lambda\mu}$ and one extra U_λ, such that

$$\sum_{\lambda=1}^{N+1} U_\lambda^2(a) = \sum_{\mu=1}^{N+1} W_\mu^2(a),$$

in effect approximates this infinity of unknown residues, i.e. we then have

$$U_{N+1}(a) \approx \sum_{\lambda=N+1}^{\infty} B_{\lambda\mu}^{(\infty)} W_\mu(a) \qquad \text{and} \qquad W_{N+1}(a) \approx \sum_{\mu=N+1}^{\infty} B_{\lambda\mu}^{(\infty)} U_\lambda(a),$$

where $B_{\lambda\mu}^{(\infty)}$ is the exact transformation matrix. Such a device ensures that the approximate finite $B_{\lambda\mu}$ is an orthogonal matrix.

Nonlocal Potentials

The static meson-field-theory interaction described by equation (1) represents the extreme case of a separable nonlocal interaction of the form

$$\theta = \int_0^a V(r, r')\, U(r)\, \mathrm{d}r', \tag{18}$$

where $V(r, r')$ is the nonlocal potential. Recently, Bagchi and Mulligan (1974) have discussed various types of nonlocal interactions, including the separable form of type (a).

Separable Potentials of Type (a)

Separable potentials of type (a) have the form

$$V(r, r') = \sum_{\alpha=1}^{M} V_\alpha(r)\, V_\alpha(r'). \tag{19}$$

From equations (10) and (11) we see that such a potential leads to an interaction matrix of the form

$$\begin{aligned} V_{\lambda\mu} &= \sum_{\alpha=1}^{M} \int_0^a V_\alpha(r)\, W_\mu(r)\, \mathrm{d}r \int_0^a V_\alpha(r')\, U_\lambda(r')\, \mathrm{d}r' \\ &= \sum_\alpha \xi_{\alpha\mu}\, \theta_{\alpha\lambda}. \end{aligned} \tag{20}$$

Thus $V_{\lambda\mu}$ is a matrix of rank M, and there are $2NM$ variables to be determined. Equations (13) and (15) still hold, however, and so remove $4N$ degrees of freedom, leaving $2N(M-2)$ unknowns. Evidently, for $M = 2$ an exact solution exists but for $M > 2$ no unique solution can be found.

Separable Potentials of Type (b)

The nonlocal potentials of type (b) are not symmetric under interchange of r and r'. A special class of this type has the form

$$V(r, r') = V(r')\, Z(r-r'). \tag{21}$$

Using a double orthonormal expansion of $Z(r-r')$, we obtain

$$V(r, r') = V(r') \sum_{\nu\rho} a_{\nu\rho}\, U_\nu(r')\, U_\rho(r), \tag{22}$$

where the $a_{\nu\rho}$ are a set of constants to be determined. Substituting equation (22) into the expression for $V_{\lambda\mu}$, we obtain

$$\begin{aligned} V_{\lambda\mu} &= \sum_{\nu\rho} \int_0^a V(r')\, W_\mu(r')\, U_\nu(r')\, \mathrm{d}r' \int_0^a a_{\nu\rho}\, U_\rho(r)\, U_\lambda(r)\, \mathrm{d}r \\ &= \sum_\nu a_{\nu\lambda} \int_0^a V(r')\, W_\mu(r')\, U_\nu(r')\, \mathrm{d}r', \end{aligned} \tag{23}$$

that is, we have

$$\mathbf{V} = \mathbf{a}\mathbf{V}', \qquad \text{where} \qquad \mathbf{V}' \equiv \int_0^a V(r')\, W_\mu(r')\, U_\nu(r')\, \mathrm{d}r'.$$

There are thus N^2 unknown components of $\mathbf{a}$ and N^2 unknown components of $\mathbf{V}'$, while there are $2N$ constraints. However, we can absorb the constants $\mathbf{a}$ into $\mathbf{V}'$ and this leaves us with N^2-2N spare degrees of freedom. The problem can therefore be solved only if $N \leqslant 2$, that is, for potentials of the type

$$V(r,r') = a_1\, U_\rho(r)\, V(r')\, U_\nu(r') + a_2\, U_\sigma(r)\, V(r')\, U_\zeta(r'), \tag{24}$$

for any ρ, ν, σ and ζ.

The local limit is obtained from equation (22) by the conditions $\mathbf{a} \to \mathbf{I}$, the unit matrix, as $N \to \infty$, in which case there is an infinity of components of $\mathbf{V}$, and the closure theorem

$$\sum_{\nu=1}^{\infty} U_\nu(r)\, U_\nu(r') \to \delta(r-r') \tag{25}$$

may be applied to give

$$V(r,r') \to V(r')\,\delta(r-r'). \tag{26}$$

The local limit therefore represents a case not attainable by a finite sum of the form (22). Even if we approximate the summation (25) by a finite sum to N terms, there are N^2-2N spare degrees of freedom and no unique solution exists for $N > 2$.

The curious feature about the local form (26) is that if we approximate the local potential by a series of step functions, as done by Cook (1973), then we obtain

$$V(r,r') = \sum_{\sigma=1}^{P} V_\sigma\, \theta(r-r_\sigma)\, \theta(r_{\sigma+1}-r_\sigma)\, \delta(r-r'). \tag{27}$$

In this case a unique solution exists for the components of the vector V_σ, provided that we have $P \leqslant N$. For $P > N$, no unique solution exists, which is consistent with the above findings. It should be noted that it is nonsymmetric separable potentials of the type (21) which can produce a local limit, and not symmetric potentials of the type (19).

Conclusions

Unlike the problem of determining local interaction potentials, the static meson source density can be determined exactly from a knowledge of the reaction matrix. The resulting equations are linear and can be solved using modern electronic computers to yield a unique value for the meson source density. More general classes of nonlocal potentials have been considered above and conditions for the existence of unique solutions have been given. The local limit proves to be a case in which no unique solution exists.

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