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Generation of Solutions of the Brans–Dicke Equations

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Abstract

A method given by Geroch (1971) for generating new vacuum solutions from given solutions of Einstein's field equations is extended to the generation of vacuum Brans–Dicke solutions from both vacuum Einstein and vacuum Brans–Dicke solutions. Two theorems by Buchdahl (1972, 1973) on generating new vacuum Brans–Dicke solutions are discussed in relation to the theory. As an example of the theory, the Brans–Dicke version of the NUT solution is found.

Introduction

In order to understand more fully the equations of Einstein's theory of gravitation it is useful to have a knowledge of some exact solutions of these equations. A number of authors have discussed methods of generating new solutions of these equations from known solutions. In particular, methods of generating vacuum solutions from known vacuum solutions are discussed by Buchdahl (1954), Ehlers (1962), Harrison (1968) and Geroch (1971, 1972). It is known that if these methods are applied to the Schwarzschild solution (which describes a non-rotating black hole) a new solution called the NUT metric is generated. Although this metric is not asymptotically flat, and is not regarded as physical, it has been useful in developing and understanding certain areas of general relativity (see e.g. Misner 1967). Harrison (1968), Kramer *et al.* (1972) and Kinnersley (1973) have discussed the generation of solutions of the Einstein–Maxwell equations (i.e. gravitational fields with electromagnetic fields present) from both vacuum Einstein solutions and other Einstein–Maxwell solutions.

In the Brans–Dicke theory, Einstein's field equations are modified by the presence of a scalar field. Buchdahl (1972) has given a method for generating exact vacuum solutions of the Brans–Dicke equations from known solutions of Einstein's vacuum equations and has also given a method (1973) for generating a new solution from any given Brans–Dicke solution in the case where the trace of the energy–momentum tensor is zero.

In this paper the generation of new vacuum solutions of the Brans–Dicke field equations from both vacuum Einstein and vacuum Brans–Dicke solutions is discussed, and the work of Buchdahl (1972) is extended. The method is used to generate as an example a metric which is the Brans–Dicke analogue of the NUT metric.

Generation of New Vacuum Solutions from Old

Geroch (1971, 1972) discussed methods of generating vacuum solutions of Einstein's equations

$$R_{ab} = 2\Gamma^{c}_{a[c,b]} + 2\Gamma^{d}_{a[c}\Gamma^{c}_{b]d} = 0 \tag{1}$$

(where $a, b, \ldots$ run from 0 to 3) from known vacuum solutions in the following way. Consider a solution to Einstein's vacuum equations that has a non-null Killing vector ξ^a. Without loss of generality ξ^a may be considered timelike, which is equivalent to stating that the solution is independent of time. Define

$$\lambda = \xi^a \xi_a \quad \text{and} \quad \omega_a = \varepsilon_{abcd} \xi^b \xi^{d;c}. \tag{2a, b}$$

When the curl of (2b) is taken, the equation $R_{ab} = 0$ implies that there exists, at least locally, a scalar ω such that $\omega_a = \omega_{,a}$. If

$$h_{ab} = g_{ab} - \lambda^{-1} \xi_a \xi_b \quad \text{and} \quad \tilde{h}_{ab} = \lambda h_{ab}, \tag{3}$$

Einstein's vacuum field equations may be written as

$$\tilde{R}_{ab} = -\tfrac{1}{2}\lambda^{-2}(\lambda_{:a}\lambda_{:b} + \omega_{:a}\omega_{:b}), \tag{4a}$$

$$\lambda\lambda_{:a}{}^{:a} = \lambda_{:a}\lambda^{:a} - \omega_{:a}\omega^{:a}, \tag{4b}$$

$$\lambda\omega_{:a}{}^{:a} = 2\lambda_{:a}\omega^{:a}, \tag{4c}$$

where $\tilde{R}_{ab}$ is the Ricci tensor formed from $\tilde{h}_{ab}$, the colon denotes covariant differentiation with respect to $\tilde{h}_{ab}$ and indices are raised using $\tilde{h}^{ab}$. New solutions of the type $h'_{ab} = g(\lambda, \omega) h_{ab}$, $\lambda' = \lambda'(\lambda, \omega)$ and $\omega' = \omega'(\lambda, \omega)$ can be found, the most general case being

$$\lambda' h'_{ab} = \lambda h_{ab}, \tag{5a}$$

$$\lambda' = \lambda\{(c\omega + d)^2 + c^2\lambda^2\}^{-1}, \tag{5b}$$

$$\omega' = \{(a\omega + b)(c\omega + d) + ac\lambda^2\}\{(c\omega + d)^2 + c^2\lambda^2\}^{-1}, \tag{5c}$$

where a, b, c and d are constant with $ad - bc = 1$. Geroch (1971) prescribes a method for calculating the new metric tensor components g'_{ab} from λ', ω' and h'_{ab}.

If the metric described by g_{ab} is static (non-rotating) such that $\omega = 0$, then equations (5b) and (5c) show that

$$\lambda' = \lambda\{d^2 + c^2\lambda^2\}^{-1}, \qquad \omega' = \{bd + ac\lambda^2\}\{d^2 + c^2\lambda^2\}^{-1}, \tag{6}$$

and the new solution is a stationary (rotating, $\omega' \neq 0$) one. From equations (6) it follows that

$$\lambda'^2 + (\omega' + \omega_0)^2 = A^2 \tag{7}$$

for certain constants ω_0 and A. This is equivalent to equation (11) in a paper by Ehlers (1962) on the generation of stationary metrics from static ones. From equation (7) it follows that only a small class of stationary metrics can be generated from static ones by a transformation of this type. In particular, the metric of a rotating black hole (the Kerr (1963) metric) does not satisfy the property (7) for any of its Killing vectors. The NUT metric (Newman *et al.* 1963), however, does satisfy this property and can be generated from the Schwarzschild metric, as shown by Harrison (1968) and Geroch (1971).

Generation of New Brans–Dicke Solutions from Old

Buchdahl (1972) has described a method of generating static solutions of the vacuum Brans–Dicke equations from static solutions of Einstein's vacuum equations. The vacuum Brans–Dicke equations can be written as

$$R_{ab} = -f\Phi_{,a}\Phi_{,b}, \tag{8}$$

where Φ is a scalar field and f is a coupling constant, equal to $\varpi+\frac{3}{2}$ where ϖ is the coupling constant as it is usually given in the Brans–Dicke theory. The conformal frame is the 'Einstein frame' as discussed by Dicke (1962) and not the 'Brans–Dicke frame' as used by Brans and Dicke (1961). In this frame Buchdahl's (1972) result may be stated as follows. Given a static solution h_{ab}, λ and $\omega = 0$, to Einstein's vacuum equations (Φ = const.), it follows that

$$h'_{ab} = \lambda^{1-k}h_{ab}, \quad \omega' = 0, \qquad \lambda' = \lambda^k, \quad \Phi = n\ln\lambda, \tag{9}$$

where n and k are constants such that

$$k^2+2fn^2 = 1, \tag{10}$$

is a solution to the vacuum Brans–Dicke equations (8). The new solution generated from the Schwarzschild solution in this way is the Brans (1962) class I solution (the Brans–Dicke analogue of the Schwarzschild solution).

The method of generation of new vacuum solutions as given by Geroch (1971) and as outlined above can easily be extended to generate new Brans–Dicke solutions, including the case discussed by Buchdahl (1972). First it will be necessary to write the Brans–Dicke equations in a form analogous to the relations (4). When $R_{ab}\xi^a = 0$ or equivalently $\xi^a\Phi_{,a} = 0$ (e.g. for $\xi^a = \delta^a_0$ and Φ time independent), ω_a as defined by equation (2b) is again curl-free and can be written as $\omega_{,a}$. With λ, h_{ab} and $\tilde{h}_{ab}$ also as defined above, the Brans–Dicke equations can be written

$$\tilde{R}_{ab} = -\tfrac{1}{2}\lambda^{-2}(\lambda_{:a}\lambda_{:b}+\omega_{:a}\omega_{:b})-f\Phi_{:a}\Phi_{:b}, \tag{11a}$$

$$\lambda\lambda_{:a}{}^{:a} = \lambda_{:a}\lambda^{:a}-\omega_{:a}\omega^{:a}, \tag{11b}$$

$$\lambda\omega_{:a}{}^{:a} = 2\lambda_{:a}\omega^{:a}. \tag{11c}$$

It follows immediately that:

Theorem 1. If λ, ω, h_{ab} and Φ are known solutions of equations (11) then λ', ω', h'_{ab} and $\Phi' = \Phi$ are also solutions, where h'_{ab}, λ' and ω' are defined by equations (5).

The question arises as to whether or not stationary vacuum Brans–Dicke solutions can be generated from stationary Einstein vacuum solutions by a method similar to that used by Geroch (1971). Thus a search for solutions of the type

$$h'_{ab} = g(\lambda,\omega)h_{ab}, \qquad \lambda' = \lambda'(\lambda,\omega), \tag{12a, b}$$

$$\omega' = \omega'(\lambda,\omega), \qquad \Phi' = \Phi'(\lambda,\omega) \tag{12c, d}$$

is made. Differential equations for λ' and ω' are obtained with differential expressions as coefficients of $\lambda_{:a}\lambda^{:a}$, $\lambda_{:a}\omega^{:a}$ and $\omega_{:a}\omega^{:a}$. When these coefficients are all zero and the resultant differential equations are solved, h'_{ab}, λ' and ω' are simply the expressions given in equations (5) and Φ' is constant. Since a constant Φ can be ignored as the field equations (8) reduce to Einstein's equations, the following theorem holds.

Theorem 2. No new stationary vacuum Brans–Dicke metric can be generated from a stationary vacuum Einstein metric by a method equivalent to that used by Geroch (1971) in generating Einstein solutions.

If, however, $\omega = 0$ (i.e. a static metric is used as the metric from which a new metric is to be generated), $h'_{ab} = g(\lambda)\,h_{ab}$ and λ' and Φ' are functions of λ only, then the differential equations which result from the method outlined above solve to give

$$\lambda' h'_{ab} = \lambda h_{ab}\,, \qquad \lambda' = c\lambda^k(1+A^2\lambda^{2k})^{-1}\,, \tag{13a, b}$$

$$\omega' = Ac\lambda^{2k}(1+A^2\lambda^{2k})^{-1}\,, \qquad \Phi' = \pm\{\tfrac{1}{2}f^{-1}(1-k^2)\}^{\frac{1}{2}}\ln\lambda\,, \tag{13c, d}$$

where k, c and A are constants. If $A = 0$ the transformations are those given by Buchdahl (1972). The parameter c represents a trivial transformation and can be put equal to unity. The metrics obtained by this method can also be obtained by making a transformation of the type found by Buchdahl followed by a transformation as described in Theorem 1.

The above procedure can be extended to the case when the original metric is a static Brans–Dicke one and the new variables are functions of both λ and Φ. In this case the following theorem holds.

Theorem 3. A vacuum Brans–Dicke metric can be generated from a static vacuum Brans–Dicke metric ($\omega = 0$; $\lambda, \Phi \neq 0$) such that

$$\lambda' h'_{ab} = \lambda h_{ab}\,, \tag{14a}$$

$$\lambda' = \frac{c\exp[\ln\lambda\cos\alpha - (2f)^{\frac{1}{2}}\Phi\sin\alpha]}{1+A^2\exp[2\{\ln\lambda\cos\alpha - (2f)^{\frac{1}{2}}\Phi\sin\alpha\}]}\,, \tag{14b}$$

$$\omega' = \frac{Ac\exp[2\{\ln\lambda\cos\alpha - (2f)^{\frac{1}{2}}\Phi\sin\alpha\}]}{1+A^2\exp[2\{\ln\lambda\cos\alpha - (2f)^{\frac{1}{2}}\Phi\sin\alpha\}]}\,, \tag{14c}$$

$$\Phi' = \pm\{(2f)^{-\frac{1}{2}}\ln\lambda\sin\alpha + \Phi\cos\alpha\}\,, \tag{14d}$$

where A, c and α are constants.

If an attempt is made to find more general solutions than those given in Theorem 3 by writing h'_{ab}, λ', ω' and Φ' as functions of λ, ω and Φ (with $\omega \neq 0$) and then setting the differential coefficients of $\lambda_{:a}\lambda^{:a}$, $\lambda_{:a}\omega^{:a}$, $\lambda_{:a}\Phi^{:a}$ etc. equal to zero, it can be shown that no solutions more general than those given in Theorem 1 result. This is because a lot of the generality has been lost since there are only three independent variables and so not all the six quantities of the type $\lambda_{:a}\lambda^{:a}$ can be independent. It thus could not be expected that for most solutions the coefficients of these terms would all be zero.

It is obvious from the field equations (8) that for every vacuum Brans–Dicke solution with metric tensor defined by g_{ab} and scalar field Φ there is another solution with the same g_{ab} and the same form of Φ but opposite sign. This result is given by Buchdahl (1973) with the difference that he discusses the case in the Brans–Dicke conformal frame and not in the Dicke frame as used here.

An Example: The Brans–Dicke NUT Metric

One example of the application of the above results is the generation of the Brans–Dicke theory metric equivalent of the NUT metric (Newman *et al.* 1963) from the Schwarzschild metric by the use of the results of Theorem 3. Alternatively this equivalent metric may be generated by starting with the Brans class I solution and applying the results of Theorem 1.

The resulting metric is

$$\begin{aligned} \mathrm{d}s^2 &= U'\{\mathrm{d}t+(8kAm/c)\sin^2 \tfrac{1}{2}\theta\,\mathrm{d}\phi\}^2-(U')^{-1}\mathrm{d}r^2 \\ &\quad -(U')^{-1}r^2(1-2m/r)\{\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\}, \end{aligned} \tag{15}$$

where

$$U' = cr^k(r-2m)^k\{r^{2k}+A^2(r-2m)^{2k}\}^{-1} \tag{16}$$

and A, c and k are constants. The scalar field is given by

$$\Phi = \pm\{\tfrac{1}{2}f^{-1}(1-k^2)\}^{\frac{1}{2}}(1-2m/r). \tag{17}$$

If $A = 0$ this is simply the Brans class I solution. If $k = 1$ the transformation

$$l = 2Am/c, \qquad b = 2m-2Al, \qquad r' = r-Al, \qquad c = 1+A^2 \tag{18}$$

puts the metric into the form given by Newman *et al.* (1963).

This procedure cannot be used to generate a Brans–Dicke Kerr metric since, as mentioned above, the Kerr metric cannot be generated from any static metric by the method discussed here.

Case of Stationary Axisymmetric Metric

Any stationary axisymmetric metric can be written in the form

$$\mathrm{d}s^2 = \mathrm{e}^{2\nu}(\mathrm{d}t+\Omega\,\mathrm{d}\phi)^2-\mathrm{e}^{-2\nu}\{\mathrm{e}^{2\mu}\,\mathrm{d}\phi^2+\mathrm{e}^{2\beta}(\mathrm{d}\rho^2+\mathrm{d}z^2)\}, \tag{19}$$

where ν, μ, Ω and β are functions of ρ and z only. When new solutions are generated from (19) by using the procedures discussed above, ν for the given solution is replaced by $\nu+\tilde{\nu}$, where $\tilde{\nu}$ is some new function given by the appropriate transformation, and Ω is replaced by some new function. McIntosh (1974) has discussed a different method for generating new solutions from the metric (19) in which β is replaced by $\beta+\tilde{\beta}$, where $\tilde{\beta}$ is some new function given by the generation procedure, and ν and Ω are left unchanged. Thus in the first method the 3-space hypersurfaces t = const. are left unchanged to within a conformal mapping while in the second method the 2-space with metric $\mathrm{e}^{2\beta}(\mathrm{d}\rho^2+\mathrm{d}z^2)$ is left unchanged to within a conformal mapping and the 2-surfaces ρ = const. and z = const. are left unchanged. The two methods are thus completely different. The second method can obviously be extended to some families of metrics other than (19).

Conclusions

An extension of Geroch's (1971) method of generating new vacuum Einstein metrics has been developed to generate, in the most general case, stationary vacuum Brans–Dicke metrics. The theory can easily be extended to generate new metrics from known metrics when both have a spacelike rather than a timelike Killing vector. Unfortunately this method cannot be used to generate a rotating Brans–Dicke metric from the Kerr metric, as indicated by Theorem 2. On the other hand, it can be used to generate a Brans–Dicke NUT metric from the Schwarzschild metric.

Many other new solutions of the Brans–Dicke field equations may be generated by the method developed here. Such solutions will help in the study of the Brans–Dicke gravitational theory and the relationship between this theory and that of Einstein.

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