THE COMBINED EFFECT OF ROTATION AND MAGNETIC FIELD ON FINITE-AMPLITUDE THERMAL CONVECTION

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Abstract

The combined effect of an imposed rotation and magnetic field on convective transfer in a horizontal Boussinesq layer of fluid heated from below is studied in the mean field approximation. The basic equations are derived by a variational technique and their solutions are then found over a wide range of conditions, in the case of free boundaries, by numerical and analytic techniques, in particular by asymptotic and perturbation methods. The results obtained by the different techniques are shown to be in excellent agreement. As for the linear theory, the calculations predict that the simultaneous presence of a magnetic field and rotation may produce conflicting tendencies.

I. Introduction

The results of an investigation into the combined effect of a magnetic field and rotation on thermal convection have intrinsic interest in many physical situations. The present study, which is primarily concerned with these effects at large values of the Rayleigh number, was undertaken with the field of astrophysics and the solar granulation problem in mind. In the case of the solar convection zone it has been estimated that the Rayleigh number, which is defined in the next section, could have a value as high as 10^{20}. Overall, our principal aim was to investigate the combined influence of rotation and a magnetic field on the total convective heat transport and to establish how the parameters associated with the problem modify this quantity. From the linear theory (Chandrasekhar 1961) it is known that the introduction of a magnetic field alone, or rotation alone, actually inhibits the onset of thermal convection. Corresponding nonlinear studies by Van der Borght *et al.* (1972) and Van der Borght and Murphy (1973) have confirmed that an impressed magnetic field or rotation can significantly modify the total energy flux. However, contrary to expectations the linear theory predicts that when both a magnetic field and rotation act together they may have conflicting tendencies. The results of an investigation into the nonlinear aspects of this problem, involving asymptotic, perturbation, and numerical techniques, are presented in this paper.

The linear problem was first introduced by Chandrasekhar (1961), who established the values of the critical Rayleigh number R_c for the onset of convective instability either as overstability or ordinary cellular convection. Several interesting features of this problem arose; in particular, it was noted that for constant rotation an increasing magnetic field could actually reduce the value of R_c until a minimum was reached but thereafter magnetic inhibition was dominant. Since this situation

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exists for the linear problem it obviously introduces the possibility of an enhanced heat flux in the case of nonlinear convection. Further aspects of the linear problem that are relevant to the present study, together with supporting numerical values, are given at the beginning of the discussion of the numerical results in Section V.

Experimental work undertaken by Nakagawa (1957, 1959) essentially confirms the theoretical estimates for R_c and in addition provides graphic evidence of the predicted change in cell size, at the onset of convective instability, associated with a critical field strength. Unfortunately no experimental results are available giving values of the heat flux when cellular convection has been established in the presence of a magnetic field and rotation, and hence comparisons cannot be made with the theoretical results in this paper.

Besides the Rayleigh number, the two other important dimensionless parameters of the problem are the Chandrasekhar number Q and the Taylor number $\mathscr{T}$, which are determined by the magnetic field strength and the speed of rotation respectively; both of these quantities are defined in the next section. When only a magnetic field is present and in the limit $Q \to \infty$, Chandrasekhar (1961) found $R_c = O(Q)$, while for rotation alone in the limit $\mathscr{T} \to \infty$, he obtained $R_c = O(\mathscr{T}^{2/3})$. The double limit $\mathscr{T} \to \infty$, $Q \to \infty$, with possible applications in astrophysics, has been considered by Eltayeb and Roberts (1970). They established from the linear equations that there are certain value ranges of the ratio $\mathscr{T}/Q$ in which either $\mathscr{T}$ or Q alone essentially determines the value of R_c, and that the values quoted above for R_c are then applicable. Moreover, within the range $O(Q^{3/2}) < \mathscr{T} < O(Q^3)$, where the magnetic field and rotation are equally influential, and in particular when $\mathscr{T} = O(Q^2)$, R_c was found to be $O(\mathscr{T}^{\frac{1}{2}})$. This smaller value of R_c (i.e. rather than $R_c = O(\mathscr{T}^{2/3})$) means that the presence of the magnetic field expedites the onset of cellular convection within these limits. From the present nonlinear solutions we have established that, for $\mathscr{T}$ and Q large with $\mathscr{T} = Q^2$, strong convective motions are maintained and that here the total heat transport attains a maximum value (see Fig. 4 below).

In this study we have taken the axis of rotation and the direction of the gravitational and magnetic fields to be parallel, a situation that would be the exception in astrophysics. However, Eltayeb (1972) has shown for the linear theory that the main conclusions obtained in this case are not affected, apart from numerical factors, by different orientations of the rotation axis and magnetic field direction. In Section II below we consider the equations for convective motions in a horizontal layer of conducting fluid, heated from below, and subject to the simultaneous action of a magnetic field and solid-body rotation about the vertical. The basic system of ordinary differential equations for steady convection are derived by applying the Glansdorff–Prigogine variational method. The Boussinesq approximation is adopted, although for astrophysical applications one should clearly consider the compressible case. However, at this stage the full compressible equations are too difficult to handle numerically and a study of the Boussinesq equations should at least yield an indication of the combined effects of a magnetic field and rotation on the convective processes.

Asymptotic solutions in the cases of large parameter values are found in Sections III and IV, where explicit expressions for the total heat transport are given in terms of the relevant parameters. Details of the numerical approach and the ensuing results are presented in Section V. Since the solutions obtained in the previous sections for parameters appropriate to astrophysical applications do not always reveal the complete interaction between the forces involved, some of the parameter ranges considered in Section V are intended to illustrate the difference between the effects on thermal convection of a magnetic field and rotation acting separately and in combination.

II. Basic Equations

The basic equations of the problem are derived from the Glansdorff–Prigogine variational method (Prigogine and Glansdorff 1964, 1965), which states that the actual flow evolves in such a way as to keep the generalized entropy production a minimum with respect to arbitrary variations from it. A modified form, which is a generalization of the one given by Unno (1969) and takes into account the combined effect of rotation and magnetic field, can be written as

$$\begin{aligned}\delta\psi = \iiint \mathrm{d}x\mathrm{d}y\,\mathrm{d}z \Bigg\{&\frac{\delta p}{\rho}\left(\frac{\partial\rho}{\partial t}+\nabla.(\rho\boldsymbol{u})\right)\\ &+\delta\boldsymbol{u}.\left(\rho\frac{\partial\boldsymbol{u}}{\partial t}+\rho\,\boldsymbol{u}.\nabla\boldsymbol{u}+\nabla p-\rho\boldsymbol{F}+\frac{\mu^*\boldsymbol{H}}{4\pi}\times(\nabla\times\boldsymbol{H})-\mu\{\nabla^2\boldsymbol{u}+\tfrac{1}{3}\nabla(\nabla.\boldsymbol{u})\}\right)\\ &+\frac{\delta T}{T}\left(\rho C_{\mathrm{v}}\frac{\partial T}{\partial t}+\rho C_{\mathrm{v}}\boldsymbol{u}.\nabla T+p\,\nabla.\boldsymbol{u}-K\,\nabla^2 T-\Phi\right)\\ &+\frac{p\,\delta\boldsymbol{H}}{H^2}.\left(\frac{\partial\boldsymbol{H}}{\partial t}+\eta\,\nabla\times(\nabla\times\boldsymbol{H})-\nabla\times(\boldsymbol{u}\times\boldsymbol{H})\right)\Bigg\},\end{aligned} \tag{1}$$

where the body force $\boldsymbol{F}$ is given by

$$\boldsymbol{F} = -\nabla\phi-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})-2(\boldsymbol{\Omega}\times\boldsymbol{u})-\dot{\boldsymbol{\Omega}}\times\boldsymbol{r} \tag{2}$$

and μ, K, C_{v}, η, and μ^* denote respectively the viscosity, conductivity, specific heat at constant volume, resistivity, and permeability of the fluid.

In the following work we make some simplifying assumptions and consider only:

(i) steady solid body rotation about the z axis, i.e.

$$\boldsymbol{\Omega} = (0,0,\Omega_0)\,; \tag{3}$$

(ii) stationary solutions,

$$\frac{\partial\rho}{\partial t} = \frac{\partial T}{\partial t} = 0, \qquad \frac{\partial\boldsymbol{H}}{\partial t} = \frac{\partial\boldsymbol{u}}{\partial t} = \boldsymbol{0}\,; \tag{4}$$

(iii) the incompressible case,

$$\nabla.\boldsymbol{u} = 0, \tag{5}$$

and therefore $\rho = \rho_0 =$ const. everywhere except in the buoyancy term; since we also neglect density variations coupled with the centrifugal acceleration, the following analysis is only valid if the Froude number $F_{\mathrm{R}} = \Omega_0^2 L/g$ is small, where L is some characteristic length scale of the fluid (e.g. the thickness of the convective layer; Greenspan 1968); and

(iv) negligible viscous dissipation, i.e.

$$\Phi = 0. \tag{6}$$

With these assumptions the variational principle can be written as

$$\delta\psi = \iiint \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\left\{\delta\boldsymbol{u}.\left(\rho_0\,\boldsymbol{u}.\nabla\boldsymbol{u}+\nabla p+\rho\,\nabla\phi+\rho_0\{\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})-2(\boldsymbol{\Omega}\times\boldsymbol{u})\}\right.\right.$$

$$\left.+\frac{\mu^*\boldsymbol{H}}{4\pi}\times(\nabla\times\boldsymbol{H})-\mu\nabla^2\boldsymbol{u}\right)$$

$$\left.+\frac{\delta T}{T}\left(\rho C_\mathrm{v}\boldsymbol{u}.\nabla T-K\nabla^2 T\right)+\frac{p\,\delta\boldsymbol{H}}{H^2}.\left(\eta\nabla\times(\nabla\times\boldsymbol{H})-\nabla\times(\boldsymbol{u}\times\boldsymbol{H})\right)\right\}. \tag{7}$$

It is convenient to adopt the following expressions for the velocity $\boldsymbol{u}$ and the strength of the magnetic field $\boldsymbol{H}$ (Chandrasekhar 1961),

$$\boldsymbol{u} = \left(\frac{\mathrm{D}W}{k^2}\frac{\partial f}{\partial x}+\frac{Z}{k^2}\frac{\partial f}{\partial y},\ \frac{\mathrm{D}W}{k^2}\frac{\partial f}{\partial y}-\frac{Z}{k^2}\frac{\partial f}{\partial x},\ Wf\right) \tag{8}$$

and

$$\boldsymbol{H} = \left(\frac{\mathrm{D}h}{k^2}\frac{\partial f}{\partial x}+\frac{X}{k^2}\frac{\partial f}{\partial y},\ \frac{\mathrm{D}h}{k^2}\frac{\partial f}{\partial y}-\frac{X}{k^2}\frac{\partial f}{\partial x},\ H_0+hf\right), \tag{9}$$

where $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}z$. This choice ensures that the continuity equation (5) and the Maxwell equation

$$\nabla.\boldsymbol{H} = 0 \tag{10}$$

are satisfied. In expressions (8) and (9), H_0 represents the z component of the constant impressed magnetic field, W, Z, h, and X are functions of z to be determined, and f satisfies the differential equation

$$\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 = -k^2 f, \tag{11}$$

where k is the horizontal wave number giving the horizontal extent of the convective cells. It should be noted that $\zeta = Zf$ is the z component of the vorticity while $(X/4\pi)f$ is the z component of the current density.

For the temperature we adopt the expression

$$T = T_0 + Ff, \tag{12}$$

where T_0 and F are functions of z to be determined, and for the density

$$\rho = \rho_0 + Pf, \tag{13}$$

where ρ_0 and P are also functions of z.

Let

$$\boldsymbol{M} = \sum_{i=1}^{7} \boldsymbol{M}^i, \tag{14}$$

where

$$\left.\begin{aligned} &\boldsymbol{M}^1 = \rho_0 \boldsymbol{u}.\nabla \boldsymbol{u}, \quad \boldsymbol{M}^2 = \nabla p, \quad \boldsymbol{M}^3 = \rho \nabla \phi, \quad \boldsymbol{M}^4 = \rho_0 \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}), \\ &\boldsymbol{M}^5 = 2\rho_0(\boldsymbol{\Omega} \times \boldsymbol{u}), \quad \boldsymbol{M}^6 = (\mu^*/4\pi)\boldsymbol{H} \times (\nabla \times \boldsymbol{H}), \quad \boldsymbol{M}^7 = -\mu \nabla^2 \boldsymbol{u}, \end{aligned}\right\} \tag{15}$$

and

$$\boldsymbol{N} = \sum_{i=1}^{2} \boldsymbol{N}^i, \tag{16}$$

where

$$\boldsymbol{N}^1 = \eta \nabla \times (\nabla \times \boldsymbol{H}), \qquad \boldsymbol{N}^2 = -\nabla \times (\boldsymbol{u} \times \boldsymbol{H}). \tag{17}$$

On substituting the expressions (8), (9), and (12) for $\boldsymbol{u}$, $\boldsymbol{H}$, and T into the variational principle (7), it is easily seen that the corresponding Euler–Lagrange equations are

$$\langle \rho C_v \boldsymbol{u}.\nabla T - K\nabla^2 T\rangle = 0, \tag{18}$$

$$\langle f(\rho C_v \boldsymbol{u}.\nabla T - K\nabla^2 T)\rangle = 0, \tag{19}$$

$$-k^{-2}\mathrm{D}\left\langle \frac{\partial f}{\partial x} M_x + \frac{\partial f}{\partial y} M_y \right\rangle + \langle f M_z\rangle = 0, \tag{20}$$

$$\left\langle \frac{\partial f}{\partial y} M_x - \frac{\partial f}{\partial x} M_y \right\rangle = 0, \tag{21}$$

$$-k^{-2}\mathrm{D}\left\langle \frac{\partial f}{\partial x} N_x + \frac{\partial f}{\partial y} N_y \right\rangle + \langle f N_z\rangle = 0, \tag{22}$$

$$\left\langle \frac{\partial f}{\partial y} N_x - \frac{\partial f}{\partial x} N_y \right\rangle = 0. \tag{23}$$

Here the angle brackets denote the horizontal averages

$$\langle\,\rangle = A \iint (\,)\, \mathrm{d}x\, \mathrm{d}y, \tag{24}$$

where the constant A is determined by the normalizing condition

$$\langle f^2\rangle = A \iint f^2\, \mathrm{d}x\, \mathrm{d}y = 1. \tag{25}$$

The averages contained in equations (18)–(23) can be determined in a somewhat lengthy but straightforward way. Only the main results will be given here.

Introducing the notation

$$O(M^i) \equiv -\mathrm{D}\left\langle \frac{\partial f}{\partial x} M_x^i + \frac{\partial f}{\partial y} M_y^i \right\rangle + k^2 \langle f M_y^i\rangle, \tag{26}$$

it can be shown that

(i) $$O(M^1) = -\rho_0 C\{W\mathrm{D}(\mathrm{D}^2-k^2)W+2\mathrm{D}W(\mathrm{D}^2-k^2)W+3Z\,\mathrm{D}Z\} \tag{27}$$

where

$$2C = \langle f^3 \rangle\,; \tag{28}$$

which gives $C = 0$ for rolls and square or rectangular planforms of the convection cells and $C = 1/\sqrt{6}$ for hexagonal cells; and also that

(ii) $$O(M^2) = 0\,, \tag{29}$$

(iii) $$O(M^3) = gk^2P\,. \tag{30}$$

Now, for an incompressible fluid the equation of state is

$$\rho = \rho_0 + \alpha\rho_0(T_0 - T)\,, \tag{31}$$

where α is the coefficient of volume expansion. Using equations (12) and (13), we then have

$$Pf = -\alpha\rho_0 Ff\,, \tag{32}$$

and multiplying by f and averaging gives

$$P = -\alpha\rho_0 F\,. \tag{33}$$

Substituting this expression into equation (30) then shows that

$$O(M^3) = -\alpha g\rho_0 k^2 F\,. \tag{34}$$

We further find

(iv) $$O(M^4) = 0\,, \tag{35}$$

(v) $$O(M^5) = -2\rho_0\,\Omega_0\,\mathrm{D}Z\,, \tag{36}$$

(vi) $$O(M^6) = -(\mu^*/4\pi)C\{2\mathrm{D}h(\mathrm{D}^2-k^2)h+h\,\mathrm{D}(\mathrm{D}^2-k^2)h-3X\mathrm{D}X\} \\ -(\mu^*/4\pi)H_0\,\mathrm{D}(\mathrm{D}^2-k^2)h\,, \tag{37}$$

(vii) $$O(M^7) = \mu(\mathrm{D}^2-k^2)^2W\,. \tag{38}$$

In arriving at the above expressions, use has been made of the averages

$$\left\langle\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right\rangle = k^2\,, \tag{39}$$

$$\left\langle f\left\{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right\}\right\rangle = k^2C\,, \tag{40}$$

$$\left\langle\frac{\partial f}{\partial x}\,\frac{\partial f}{\partial y}\,\frac{\partial^2 f}{\partial x\,\partial y}\right\rangle = \tfrac{1}{4}k^4C\,. \tag{41}$$

From equations (27), (29), and (34)–(38) it now follows that the relation (20) can be written as

$$\begin{aligned}\mu(\mathrm{D}^2-k^2)^2 W = {} & +\alpha g \rho_0 k^2 F \\ & +\rho_0 C\{W\mathrm{D}(\mathrm{D}^2-k^2)W+2\mathrm{D}W(\mathrm{D}^2-k^2)W+3Z\,\mathrm{D}Z\}+2\rho_0\,\Omega_0\,\mathrm{D}Z \\ & -(\mu^*/4\pi)C\{2\mathrm{D}h\,(\mathrm{D}^2-k^2)h+h\,\mathrm{D}(\mathrm{D}^2-k^2)h-3X\,\mathrm{D}X\} \\ & -(\mu^*/4\pi)H_0\,\mathrm{D}(\mathrm{D}^2-k^2)h.\end{aligned} \tag{42}$$

If we use the notation

$$Q(\boldsymbol{M}^i) = \left\langle \frac{\partial f}{\partial y}M^i_x - \frac{\partial f}{\partial x}M^i_y \right\rangle, \tag{43}$$

it can be shown that

(i) $$Q(\boldsymbol{M}^1) = C\rho_0(W\mathrm{D}Z - Z\mathrm{D}W); \tag{44}$$

(ii) $$Q(\boldsymbol{M}^j) = 0, \qquad j = 2,3,4; \tag{45}$$

(iii) $$Q(\boldsymbol{M}^5) = -2\rho_0\,\Omega_0\,\mathrm{D}W; \tag{46}$$

(iv) $$Q(\boldsymbol{M}^6) = (\mu^*/4\pi)C(X\mathrm{D}h - h\,\mathrm{D}X) - (\mu^*/4\pi)H_0\,\mathrm{D}X; \tag{47}$$

(v) $$Q(\boldsymbol{M}^7) = -\mu(\mathrm{D}^2-k^2)Z. \tag{48}$$

With the help of these expressions, equation (21) can be written as

$$\begin{aligned}\mu(\mathrm{D}^2-k^2)Z = {} & C\rho_0(W\mathrm{D}Z - Z\mathrm{D}W) - 2\rho_0\,\Omega_0\,\mathrm{D}W \\ & +(\mu^*/4\pi)C(X\mathrm{D}h - h\,\mathrm{D}X) - (\mu^*/4\pi)H_0\,\mathrm{D}X.\end{aligned} \tag{49}$$

Using a similar procedure, the remaining Euler–Lagrange equations (18), (19), (22), and (23) can be shown to reduce respectively to

$$K\mathrm{D}^2 T_0 = \rho_0\,C_{\mathrm{v}}\,\mathrm{D}(FW), \tag{50}$$

$$K(\mathrm{D}^2-k^2)F = \rho_0\,C_{\mathrm{v}}\,W\mathrm{D}T_0 + C\rho_0\,C_{\mathrm{v}}(F\mathrm{D}W + 2W\mathrm{D}F), \tag{51}$$

$$\eta(\mathrm{D}^2-k^2)h - C(W\mathrm{D}h - h\,\mathrm{D}W) + H_0\,\mathrm{D}W = 0, \tag{52}$$

and

$$\eta(\mathrm{D}^2-k^2)X = -C\{2(Z\mathrm{D}h - X\mathrm{D}W) + (h\,\mathrm{D}Z - W\mathrm{D}X)\} - H_0\,\mathrm{D}Z. \tag{53}$$

The dimensionless form of these equations is obtained by making the substitutions

$$\left.\begin{array}{lll} D \to D/d, & k^2 \to a^2/d^2, & W \to (\kappa/d)W, \\ h \to H_0 H, & T_0 \to T_0(\Delta T), & F \to F(\Delta T), \end{array}\right\} \tag{54}$$

where d is the thickness of the layer, $\kappa = K/\rho_0\,C_{\mathrm{v}}$ is the thermal diffusivity, and ΔT

is the temperature drop across the layer. In addition

$$X \to (H_0/d)\mathscr{T}^{\frac{1}{2}} X, \qquad Z \to (\kappa/d^2)\mathscr{T}^{\frac{1}{2}} Z, \tag{55}$$

where the Taylor number $\mathscr{T}$ is defined by

$$\mathscr{T} = 4d^4\Omega_0^2/\nu^2, \tag{56}$$

$\nu = \mu/\rho_0$ being the kinematic viscosity. If we also introduce the Rayleigh number

$$R = g\alpha d^3 \,\Delta T\, \rho_0/\mu\kappa, \tag{57}$$

together with the Chandrasekhar number $Q = \mu^* d^2 H_0^2/4\pi\mu\eta$, the Prandtl number $\sigma = \nu/\kappa$, and the magnetic Prandtl number $\tau = \eta/\kappa$, the fundamental equations of the problem can be readily written as

$$(\mathrm{D}^2-a^2)Z = (C/\sigma)(W\mathrm{D}Z-Z\mathrm{D}W)-\mathrm{D}W+Q\tau C(X\mathrm{D}H-H\mathrm{D}X)-Q\tau\,\mathrm{D}X, \tag{58}$$

$$\begin{aligned}(\mathrm{D}^2-a^2)^2 W = {} & Ra^2F+(C/\sigma)\{W\mathrm{D}(\mathrm{D}^2-a^2)W+2\mathrm{D}W(\mathrm{D}^2-a^2)W+3\mathscr{T}Z\,\mathrm{D}Z\}+\mathscr{T}\mathrm{D}Z\\ & -CQ\tau\{2\mathrm{D}H(\mathrm{D}^2-a^2)H+H\,\mathrm{D}(\mathrm{D}^2-a^2)H-3\mathscr{T}X\mathrm{D}X\}\\ & -Q\tau\,\mathrm{D}(\mathrm{D}^2-a^2)H,\end{aligned} \tag{59}$$

$$\tau(\mathrm{D}^2-a^2)H = C(W\mathrm{D}H-H\mathrm{D}W)-\mathrm{D}W, \tag{60}$$

$$\tau(\mathrm{D}^2-a^2)X = -C\{2(Z\mathrm{D}H-X\mathrm{D}W)+(H\mathrm{D}Z-W\mathrm{D}X)\}-\mathrm{D}Z, \tag{61}$$

$$(\mathrm{D}^2-a^2)F = W\mathrm{D}T_0+C(F\mathrm{D}W+2W\mathrm{D}F), \tag{62}$$

$$\mathrm{D}^2T_0 = \mathrm{D}(FW). \tag{63}$$

In this paper we only consider the problem of thermal convection with square or rectangular planforms ($C = 0$), in which case the above equations assume the considerably simplified forms

$$\mathrm{D}T_0 = FW-N, \tag{64}$$

$$(\mathrm{D}^2-a^2)F = W\mathrm{D}T_0, \tag{65}$$

$$(\mathrm{D}^2-a^2)X = -\mathrm{D}Z, \tag{66}$$

$$(\mathrm{D}^2-a^2)H = -\mathrm{D}W, \tag{67}$$

$$(\mathrm{D}^2-a^2)Z = -\mathrm{D}W-Q\,\mathrm{D}X, \tag{68}$$

$$(\mathrm{D}^2-a^2)^2 W = Ra^2F+\mathscr{T}\,\mathrm{D}Z-Q\,\mathrm{D}(\mathrm{D}^2-a^2)H, \tag{69}$$

where we have now made the substitutions

$$H \to \tau H \qquad \text{and} \qquad X \to \tau X. \tag{70}$$

The constant N in equation (64) is known as the Nusselt number.

The system of equations (64)–(69) has been solved by asymptotic, perturbation, and numerical methods subject to the boundary conditions:

$$W = \mathrm{D}^2 W = F = \mathrm{D}Z = X = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1, \tag{71}$$

$$T_0 = 0 \ \text{at} \ z = 0 \quad \text{and} \quad T_0 = -1 \quad \text{at} \quad z = 1, \tag{72}$$

$$\mathrm{D}H - aH = 0 \ \text{at} \ z = 0 \quad \text{and} \quad \mathrm{D}H + aH = 0 \quad \text{at} \quad z = 1. \tag{73}$$

III. Asymptotic Method

We shall now derive an expression for the Nusselt number N for large values of the Rayleigh number R when the Taylor number $\mathscr{T}$ and the Chandrasekhar number Q are of order unity.

Substitution of equation (67) into (69) yields the expression

$$(\mathrm{D}^2 - a^2)^2 W = Ra^2 F + \mathscr{T}\,\mathrm{D}Z + Q\,\mathrm{D}^2 W. \tag{74}$$

If we now multiply this equation by W and introduce the notation

$$W = (Ra^2 N)^{\frac{1}{2}}\,\psi, \qquad Z = (Ra^2 N)^{\frac{1}{2}}\,\chi, \tag{75}$$

we obtain

$$\psi(\mathrm{D}^2 - a^2)^2 \psi = 1 + \mathscr{T}\psi\,\mathrm{D}\chi + Q\psi\,\mathrm{D}^2\psi. \tag{76}$$

Combining equations (66) and (68) we have

$$(\mathrm{D}^2 - a^2)^2 Z = -\mathrm{D}(\mathrm{D}^2 - a^2) W + Q\,\mathrm{D}^2 Z$$

or, using equations (75),

$$(\mathrm{D}^2 - a^2)^2 \chi = -\mathrm{D}(\mathrm{D}^2 - a^2)\psi + Q\,\mathrm{D}^2\chi. \tag{77}$$

We now expand ψ and χ in the forms

$$\psi = \sum_{k=1,3}^{\infty} A_k \sin k\pi z, \qquad \chi = \sum_{k=1,3}^{\infty} B_k \cos k\pi z. \tag{78}$$

These expansions satisfy the "free" boundary conditions

$$\psi = \mathrm{D}^2\psi = 0, \quad \mathrm{D}\chi = 0 \quad \text{at} \quad z = 0 \text{ and } 1 \tag{79}$$

and are in fact the boundary conditions on W and Z required by (71). Substituting the expansions (78) into equation (77), we obtain

$$\{(k^2\pi^2 + a^2)^2 + Qk^2\pi^2\} B_k = k\pi(k^2\pi^2 + a^2) A_k. \tag{80}$$

Now equation (76) can be written in the form

$$(\mathrm{D}^2 - a^2)^2 \psi - Q\,\mathrm{D}^2\psi = \psi^{-1} + \mathscr{T}\,\mathrm{D}\chi. \tag{81}$$

Retaining only the first two terms in the expansion of ψ, we have

$$\psi^{-1} = \frac{1}{A_1 \sin \pi z}\left(1 - \frac{A_3}{A_1}\frac{\sin 3\pi z}{\sin \pi z}\cdots\right), \tag{82}$$

and substituting this expression together with the expansions (78) into equation (81), multiplying both sides by $2 \sin k\pi z$, and integrating with respect to z from 0 to 1 we find

$$\{(k^2\pi^2+a^2)^2+Qk^2\pi^2\}A_k = 2\int_0^1 \frac{\sin k\pi z}{A_1 \sin \pi z}\left(1 - \frac{A_3(1+2\cos \pi z)}{A_1}\right) dz$$
$$+2\mathscr{T}\int_0^1 (\sin k\pi z)(-\pi B_1 \sin \pi z - 3\pi B_3 \sin 3\pi z)\, dz. \tag{83}$$

Evaluation of the integrals in equation (83) for $k = 1$ and 3, making use of the expression (80) for B_k, leads to two equations in A_1 and A_3 which are found to have the solutions

$$\frac{A_3}{A_1} = \frac{(\pi^2+a^2)^2+Q\pi^2+\pi^2\mathscr{T}(\pi^2+a^2)/\{(\pi^2+a^2)^2+\pi^2 Q\}}{(9\pi^2+a^2)^2+9\pi^2 Q+9\pi^2\mathscr{T}(9\pi^2+a^2)/\{(9\pi^2+a^2)^2+9\pi^2 Q\}} \tag{84}$$

and

$$A_1^2 = \frac{2}{(\pi^2+a^2)^2+Q\pi^2+\pi^2\mathscr{T}(\pi^2+a^2)/\{(\pi^2+a^2)^2+\pi^2 Q\}}$$
$$\times\left(1 - \frac{(\pi^2+a^2)^2+Q\pi^2+\pi^2\mathscr{T}(\pi^2+a^2)/\{(\pi^2+a^2)^2+\pi^2 Q\}}{(9\pi^2+a^2)^2+9\pi^2 Q+9\pi^2\mathscr{T}(9\pi^2+a^2)/\{(9\pi^2+a^2)^2+9\pi^2 Q\}}\right). \tag{85}$$

In the boundary layer we have

$$\psi = k_0 z \tag{86}$$

and it follows from the expansion for ψ in (78) that

$$k_0 = \pi A_1(1+3A_3/A_1). \tag{87}$$

As shown by Howard (1965)

$$N = (2\cdot 124)^{-4/3} k_0^{2/3} (a^2 R)^{1/3}, \tag{88}$$

and making the appropriate substitutions we obtain the following expression for the Nusselt number

$$N = \left(\frac{1}{2\cdot 124}\right)^{4/3}\left(\frac{\pi\sqrt{2}}{[(\pi^2+a^2)^2+Q\pi^2+\pi^2\mathscr{T}(\pi^2+a^2)/\{(\pi^2+a^2)^2+\pi^2 Q\}]^{\frac{1}{2}}}\right)^{2/3}\left(a^2 R\right)^{1/3}$$
$$\times\left(1+\frac{5}{3}\,\frac{(\pi^2+a^2)^2+Q\pi^2+\pi^2\mathscr{T}(\pi^2+a^2)/\{(\pi^2+a^2)^2+\pi^2 Q\}}{(9\pi^2+a^2)^2+9\pi^2 Q+9\pi^2\mathscr{T}(9\pi^2+a^2)/\{(9\pi^2+a^2)^2+9\pi^2 Q\}}\right). \tag{89}$$

A comparison between the values of the Nusselt number obtained by this asymptotic formula and those derived by numerical integration is given in Figures 1 and 3. It can be seen that the agreement at high Rayleigh number is quite good, as long as the Taylor number is not too large.

IV. Perturbation Method

The asymptotic result (89) was derived in the previous section under the assumption that both $\mathscr{T}$ and Q were of order unity and, as can be seen from Figures 1 and 3, it does not yield values of N in good agreement with the numerical results for large values of these parameters. It is therefore important to derive expressions for N as a function of R when either $\mathscr{T}$ or Q is large in order to find solutions in the near-linear case for large Rayleigh numbers.

(a) *R and $\mathscr{T}$ Large, Q of order Unity*

The linear stability theory (Chandrasekhar 1961) predicts that the critical Rayleigh number, under the combined effect of magnetic field and rotation, is given in the case of free boundaries by

$$R = \frac{(n^2\pi^2+a^2)[\{(n^2\pi^2+a^2)^2+Qn^2\pi^2\}^2+\mathscr{T}n^2\pi^2(n^2\pi^2+a^2)]}{a^2\{(n^2\pi^2+a^2)^2+Qn^2\pi^2\}}, \tag{90}$$

the parameter n indicating the order of the mode. For the fundamental mode solution with $\mathscr{T}$ large and $Q = O(1)$, we have

$$R = \frac{\pi^2\mathscr{T}}{a^2\{1+Q\pi^2/(\pi^2+a^2)^2\}}. \tag{91}$$

From the fundamental equations (64)–(69) one can derive the expression

$$(\mathrm{D}^2-a^2)^3W = W^2\{(\mathrm{D}^2-a^2)^2W-\mathscr{T}\,\mathrm{D}Z+Q\,\mathrm{D}(\mathrm{D}^2-a^2)H\}$$
$$-Ra^2NW-\mathscr{T}\,\mathrm{D}^2W-\mathscr{T}Q\,\mathrm{D}^2X+Q\,\mathrm{D}^2(\mathrm{D}^2-a^2)W. \tag{92}$$

If we let

$$R = \lambda\mathscr{T}, \tag{93}$$

with

$$\lambda > \frac{\pi^2}{a^2\{1+Q\pi^2/(\pi^2+a^2)^2\}}, \tag{94}$$

and

$$\gamma = a^2N\lambda, \tag{95}$$

it then follows that for R large and $Q = O(1)$ equation (92) can be written as

$$\mathrm{D}^2W+W^2\,\mathrm{D}Z+\gamma W+Q\,\mathrm{D}^2X = 0. \tag{96}$$

We now introduce the following expansions for the current density X, vorticity Z, and vertical velocity W

$$X = \sum_{k=1,3}^{\infty}\xi_k\sin k\pi z, \quad Z = \sum_{k=1,3}^{\infty}\zeta_k\cos k\pi z, \quad W = \sum_{k=1,3}^{\infty}w_k\sin k\pi z, \tag{97}$$

all of which satisfy the boundary conditions (71). Substitution of these expansions into equations (66) and (68) then leads to the expressions

$$\xi_k = -\frac{k\pi\zeta_k}{k^2\pi^2+a^2} \tag{98}$$

and

$$\zeta_k = \frac{k\pi w_k}{(k^2\pi^2+a^2)\{1+Q(k\pi)^2/(k^2\pi^2+a^2)^2\}}. \tag{99}$$

Introducing the expansions (97) into equation (96), making use of the expressions (98) and (99), and multiplying by $\sin k\pi z$ and integrating from 0 to 1 with respect to z, we finally obtain two equations in w_1 and w_3 by successively taking $k = 1$ and 3. The ratio w_3/w_1 is then given by

$$w_3/w_1 = [3-2HK-\{(2HK-3)^2+4HK\}^{\frac{1}{2}}]/2H, \tag{100}$$

where

$$H = 2+\frac{9(\pi^2+a^2)}{9\pi^2+a^2}\,\frac{1+Q\pi^2/(\pi^2+a^2)^2}{1+9Q\pi^2/(9\pi^2+a^2)^2} \tag{101}$$

and

$$K = \frac{(\gamma-\pi^2)+Q\pi^4/\{(\pi^2+a^2)^2+Q\pi^2\}}{(\gamma-9\pi^2)+81\,Q\pi^4/\{(9\pi^2+a^2)^2+9\pi^2Q\}}, \tag{102}$$

while the value of w_1 becomes

$$w_1^2 = \frac{4(\pi^2+a^2)}{\pi^2}\,\frac{1+Q\pi^2/(\pi^2+a^2)^2}{3-Hw_3/w_1} \times\left((\gamma-\pi)^2+\frac{Q\pi^4}{(\pi^2+a^2)^2\{1+Q\pi^2/(\pi^2+a^2)^2\}}\right). \tag{103}$$

Combination of equations (64) and (65) with the substitution

$$\phi = F/N, \tag{104}$$

gives the equation

$$\mathrm{D}^2\phi = a^2\phi+W^2\phi-W, \tag{105}$$

and using the expansion

$$\phi = \sum_{k=1,3}\phi_k\sin k\pi z \tag{106}$$

it is easily seen that

$$\frac{\phi_3}{\phi_1} = \frac{w_1^2(1-4w_3/w_1)+(w_3/w_1)\{4(\pi^2+a^2)+w_1^2(3-2w_3/w_1)\}}{4(9\pi^2+a^2)+2w_1^2+w_1^2(w_3/w_1)} \tag{107}$$

and

$$\phi_1 = \frac{w_1}{(\pi^2+a^2)+\frac{1}{4}w_1^2\{3-(\phi_3/\phi_1+2w_3/w_1)\}}. \tag{108}$$

The Nusselt number is finally given by

$$N = 1+\tfrac{1}{2}N\phi_1 w_1(1+\phi_3 w_3/\phi_1 w_1). \tag{109}$$

The results obtained by this formula agree very well with those derived by numerical integration for large R and $\mathscr{T}$ (but Q of order unity), as can be seen from Figure 1.

(b) R and Q Large, $\mathscr{T}$ of order Unity

A similar analysis to the above in the case when both R and Q are large but $\mathscr{T}$ is of order unity yields the following expressions for w_3/w_1 and w_1

$$w_3/w_1 = \tfrac{1}{2}\{C_1 \pm (C_1^2+4C_0)^{\frac{1}{2}}\} \tag{110}$$

and

$$w_1^2 = \frac{4\{-\pi^2(\pi^2+a^2)+\gamma\}}{\pi^2(-11\,w_3/w_1+3)}, \tag{111}$$

where

$$C_1 = \frac{3}{11} - \frac{2\{\pi^2(\pi^2+a^2)-\gamma\}}{9\pi^2(9\pi^2+a^2)-\gamma}, \qquad C_0 = \frac{\pi^2(\pi^2+a^2)-\gamma}{11\{(9\pi^2+a^2)9\pi^2-\gamma\}}. \tag{112}$$

The value of N is then given by the transcendental equation (109), keeping in mind that in this case

$$\gamma = a^2NR/Q. \tag{113}$$

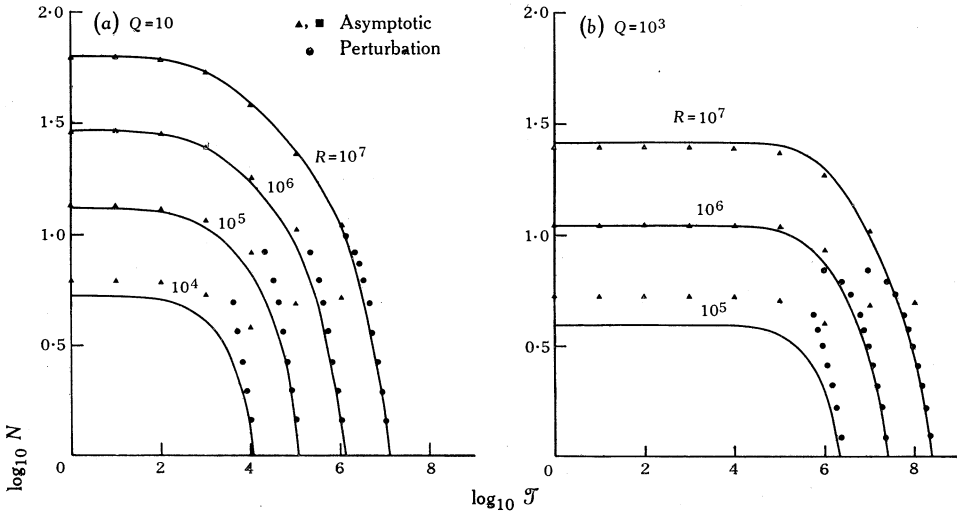
A comparison between the results obtained by these formulae and those predicted by the numerical integrations is given in Figure 3.

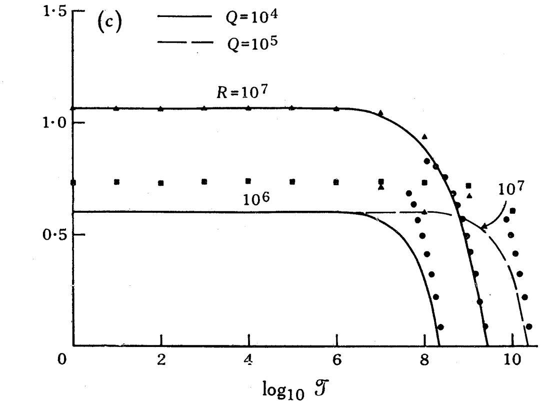
V. Numerical Method and Results

The critical Rayleigh number R_c, as a function of Q and $\mathscr{T}$, and the associated wave number a_c have been calculated by Chandrasekhar (1961) from the linear equations when instability sets in as ordinary cellular convection in the case of free boundaries with $\boldsymbol{g}$, $\boldsymbol{B}$, and $\boldsymbol{\Omega}$ all parallel. For $R \geqslant R_0 \geqslant R_c$, the range of wave numbers that support convection are determined by the condition $\gamma \leqslant 1$, where

$$\gamma = \frac{R_0}{R} = \frac{(\pi^2+a^2)[\{(\pi^2+a^2)^2+\pi^2Q\}^2+\pi^2\mathscr{T}(\pi^2+a^2)]}{a^2\{(\pi^2+a^2)^2+\pi^2Q\}R}. \tag{114}$$

The $R(a)$ curve is of particular interest when both rotation and a magnetic field are present, because in certain cases two minima for R_c result. For example, when $\mathscr{T} = 10^8$ our results show that, for $Q = 10^3$, R has a minimum value of $3\cdot 82\times 10^6$ at $a = 1\cdot 0\pi$ and also $1\cdot 89\times 10^6$ at $a = 8\cdot 8\pi$, while, for the case when $Q = 10^4$, minimum values of R of $5\cdot 74\times 10^5$ and $1\cdot 78\times 10^6$ occur at $a = 1\cdot 2\pi$ and $7\cdot 9\pi$ respectively. In the first example at the onset of cellular convection the cells would be narrow and elongated, but in the second case we would expect large cells corresponding to the lower wave number for R_c. The nonlinear solutions for $N(a)$, which are displayed in Figures 5(a) and 5(b), reflect these features of the linear problem.





Figs. 1(*a*)–1(*c*).—Numerical results (curves) for N as a function of $\mathscr{T}$ when $a = \pi$ for the indicated values of R and increasing Q. The results are compared with asymptotic and perturbation solutions from equations (89) in Section III and (109) in Section IV(*a*) respectively.

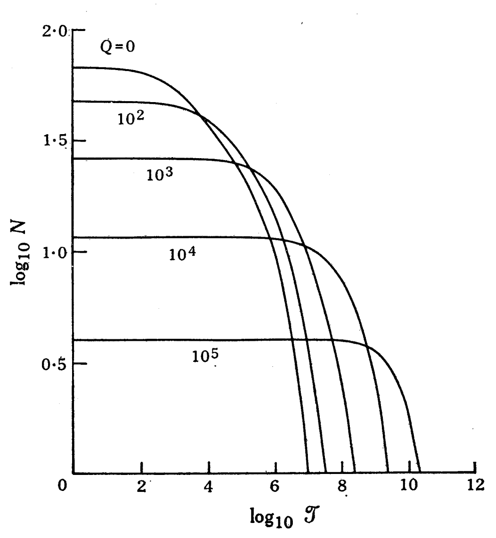
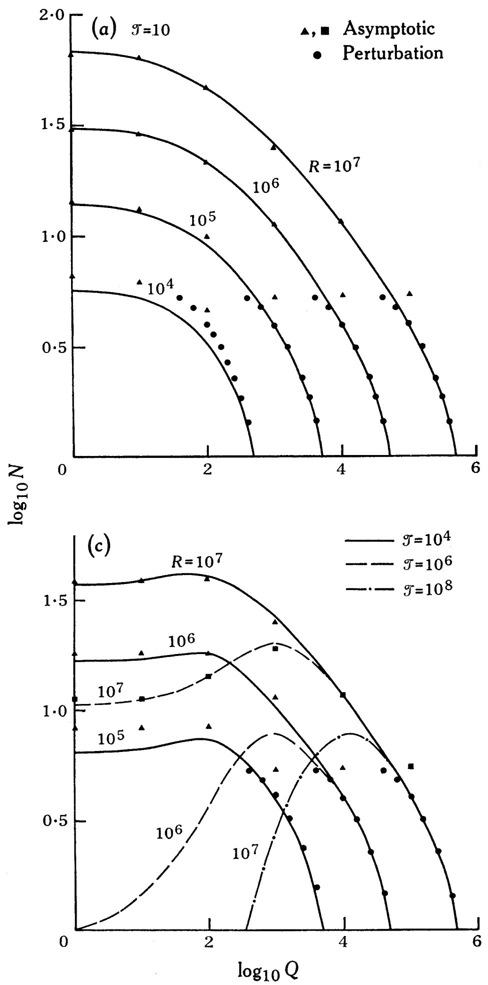


Fig. 2.—Numerical results for N as a function of $\mathscr{T}$ when $a = \pi$ for $R = 10^7$ and the indicated values of Q.



Figs. 3(*a*)–3(*c*).—Numerical results (curves) for N as a function of Q when $a = \pi$ for the indicated values of R and increasing $\mathscr{T}$. The results are compared with asymptotic and perturbation solutions from equations (89) in Section III and (113) in Section IV(*b*) respectively.

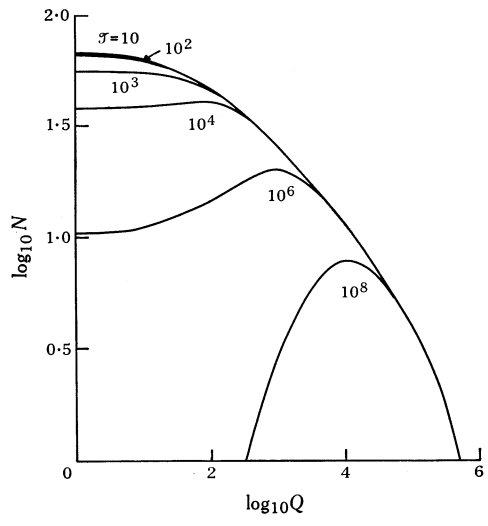
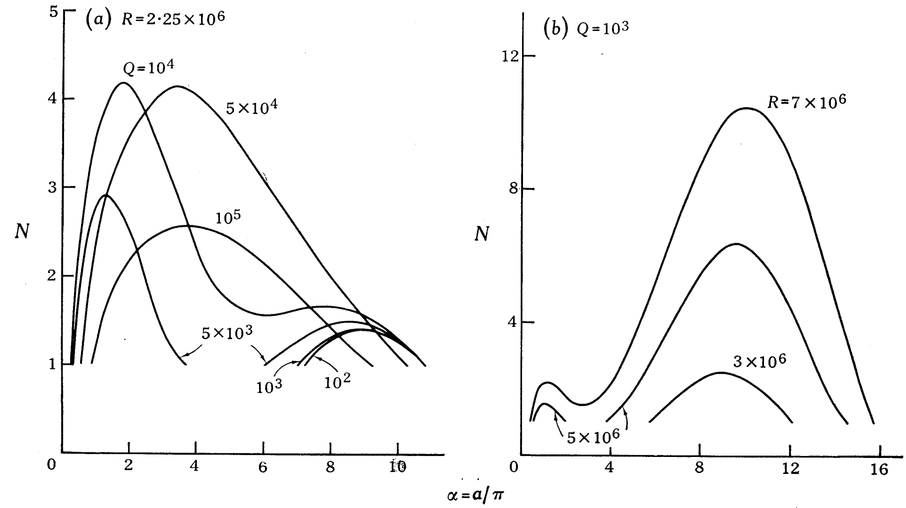


Fig. 4.—Numerical results for N as a function of Q when $a = \pi$ for $R = 10^7$ and the indicated values of $\mathscr{T}$.

We also note from the linear problem that at a particular wave number when $\mathscr{T}$ is large, e.g. when $a = \pi$ and $T = 10^8$, an increase in strength of the applied magnetic field from $Q = 0$ to 10^4 actually decreases the value of R_0 for the onset of thermal instability. However, a further increase in Q results in a larger value of R_0. The corresponding results of the nonlinear calculations given in Figure 4 when $R = 10^7$ confirm that an increase in field strength can facilitate convection.



Figs. 5(a) and 5(b) (*above*).—Dependence of N on $\alpha = a/\pi$ when $\mathscr{T} = 10^8$ for the indicated values of R and Q.

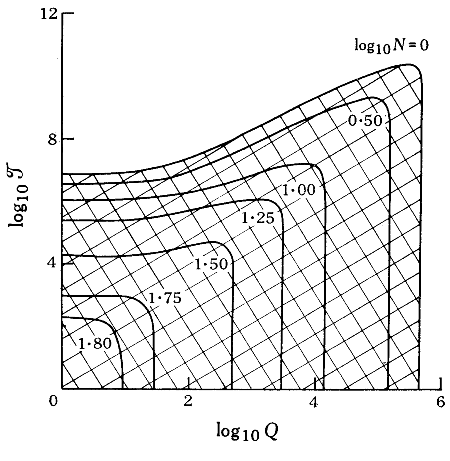


Fig. 6 (*opposite*).—Contour values of N as functions of $\mathscr{T}$ and Q when $R = 10^7$ and $a = \pi$. The cross-hatched region corresponds to convection with values of $N > 1$.

Chandrasekhar (1961) has given a summary of the theoretical and experimental results for R_0 when a rotating layer of mercury heated from below is subjected to an impressed magnetic field. The experimental work undertaken by Nakagawa (1957, 1959) generally confirms the theoretical predictions for the onset of instability and the resulting type, namely overstability or cellular convection. The transition from overstability to cellular convection and the predicted decrease in the Rayleigh number with increasing magnetic field strength were also demonstrated by the experiments. However, it should be noted that the theoretical values used in the above comparisons were based on two free boundaries for the fluid layer. A study of the streak photographs of the convective motions obtained from Nakagawa's experimental investigation also showed that, when Q is increased, a change in cell size accompanies the instability change from overstability to cellular convection. Again this is an interesting confirmation of the theoretical predictions in the case of fluids with low Prandtl numbers.

In the previous sections solutions of the basic equations have been obtained for special ranges in values of the parameters a, R, $\mathscr{T}$, and Q. Numerical solutions of these equations will now be considered, over the parameter ranges that support steady convection, in order to show in more detail the effects of rotation on the total heat transfer across a fluid layer in the presence of a magnetic field. Of particular interest will be the variation of N with respect to a and the value of a for $N_{\max}$, as well as the nature of the solutions for the dependent variables.

Writing equation (64) in the form

$$\mathrm{D}^2 T_0 = \mathrm{D}(FW) \tag{115}$$

and eliminating in turn H and Z from equation (69) using (66), (67), and (68), we find

$$\{(\mathrm{D}^2-a^2)^2 - Q\,\mathrm{D}^2\}^2 W + \mathscr{T}\mathrm{D}^2(\mathrm{D}^2-a^2)W = Ra^2\{(\mathrm{D}^2-a^2)^2 - Q\,\mathrm{D}^2\}F. \tag{116}$$

On introduction of the expansions

$$W(z) = \sum_{n=1}^{M} W_n \sin\{(2n-1)\pi z\}, \tag{117a}$$

$$F(z) = \sum_{n=1}^{M} f_n \sin\{(2n-1)\pi z\}, \tag{117b}$$

$$T_0(z) = \sum_{n=1}^{M} t_n \sin(2n\pi z) - z \tag{117c}$$

into (115), (116), and (65), we then obtain the system of equations:

$$\left\{\left((2n-1)^2+\alpha^2+\frac{Q(2n-1)^2}{\pi^2\{(2n-1)^2+\alpha^2\}}\right)^2+\frac{\mathscr{T}(2n-1)^2}{\pi^4\{(2n-1)^2+\alpha^2\}}\right\}w_n$$

$$= \frac{R_1\alpha^2}{(2n-1)^2+\alpha^2}\left((2n-1)^2+\alpha^2+\frac{Q(2n-1)^2}{\pi^2\{(2n-1)^2+\alpha^2\}}\right)f_n, \tag{118}$$

$$\{(2n-1)^2+\alpha^2\}f_n = w_n - \pi\sum_{p=1}^{M} p t_p\{w_{n+p} + Y(2n-1-2p)\,w_{\frac{1}{2}|2n-1-2p|+\frac{1}{2}}\}, \tag{119}$$

$$nt_n = \tfrac{1}{4}\pi \sum_{p=1}^{M} w_p\{f_{n+p} + Y(2p-1-2n) f_{\frac{1}{2}|2n-2p+1|+\frac{1}{2}}\}, \tag{120}$$

where

$$R_1 = R/\pi^4, \qquad \alpha = a/\pi, \qquad w_n = W_n/\pi^2,$$

and

$$\begin{aligned} Y(n) &= 1 \quad \text{for} \quad n > 0, \\ &= 0 \qquad\qquad n = 0, \\ &= -1 \qquad\qquad n < 0. \end{aligned}$$

The above expansions (117) account for the symmetry of the solutions about $z = \frac{1}{2}$, when $C = 0$, and also for the free boundary conditions which are considered here. The generalized Newton–Raphson method has been used to solve the system of nonlinear equations (118)–(120) for values of the Rayleigh number up to $R = 10^7$ with $M = 90$ to ensure constancy of the Nusselt number.

Now turning to the solution of equation (67) for $H(z)$, with (A) the boundary conditions (73), we find

$$H_A(z) = \tfrac{1}{2}\phi^*(0)\,[\exp\{a(z-1)\} - \exp(-az)] + \phi^*(z), \tag{121}$$

where

$$\phi^*(z) = \sum_{n=1}^{M} \frac{W_n(2n-1)\cos\{(2n-1)\pi z\}}{\pi\{(2n-1)^2+\alpha^2\}}. \tag{122}$$

From the differential equation for the vorticity $Z(z)$, namely

$$\{(D^2-a^2)^2 - Q\,D^2\}Z = -D(D^2-a^2)W, \tag{123}$$

we obtain, in the case of free boundaries,

$$Z(z) = \sum_{n=1}^{M} \frac{\pi(2n-1)\{(2n-1)^2+\alpha^2\}W_n\cos\{(2n-1)\pi z\}}{\pi^2\{(2n-1)^2+\alpha^2\}^2 + Q(2n-1)^2}. \tag{124}$$

In addition, $X(z)$ can be computed from the solution of the equation

$$\{(D^2-a^2)^2 - Q\,D^2\}X = D^2 W \tag{125}$$

subject to the conditions (71), to yield the result

$$X_A(z) = -\sum_{n=1}^{M} \frac{(2n-1)^2 W_n \sin\{(2n-1)\pi z\}}{\pi^2\{(2n-1)^2+\alpha^2\}^2 + Q(2n-1)^2}. \tag{126}$$

An alternative set of boundary conditions (B) applicable to this problem (Chandrasekhar 1961) are

$$H(0) = H(1) = DX(0) = DX(1) = 0.$$

In this case the boundaries are taken to be conducting and the solutions that result for $H(z)$ and $X(z)$ are

$$H_{\mathrm{B}}(z) = \frac{\phi^*(0)}{\exp(a)-\exp(-a)}\Big(\{\exp(-a)+1\}\exp(az)-\{\exp(a)+1\}\exp(-az)\Big)+\phi^*(z), \tag{127}$$

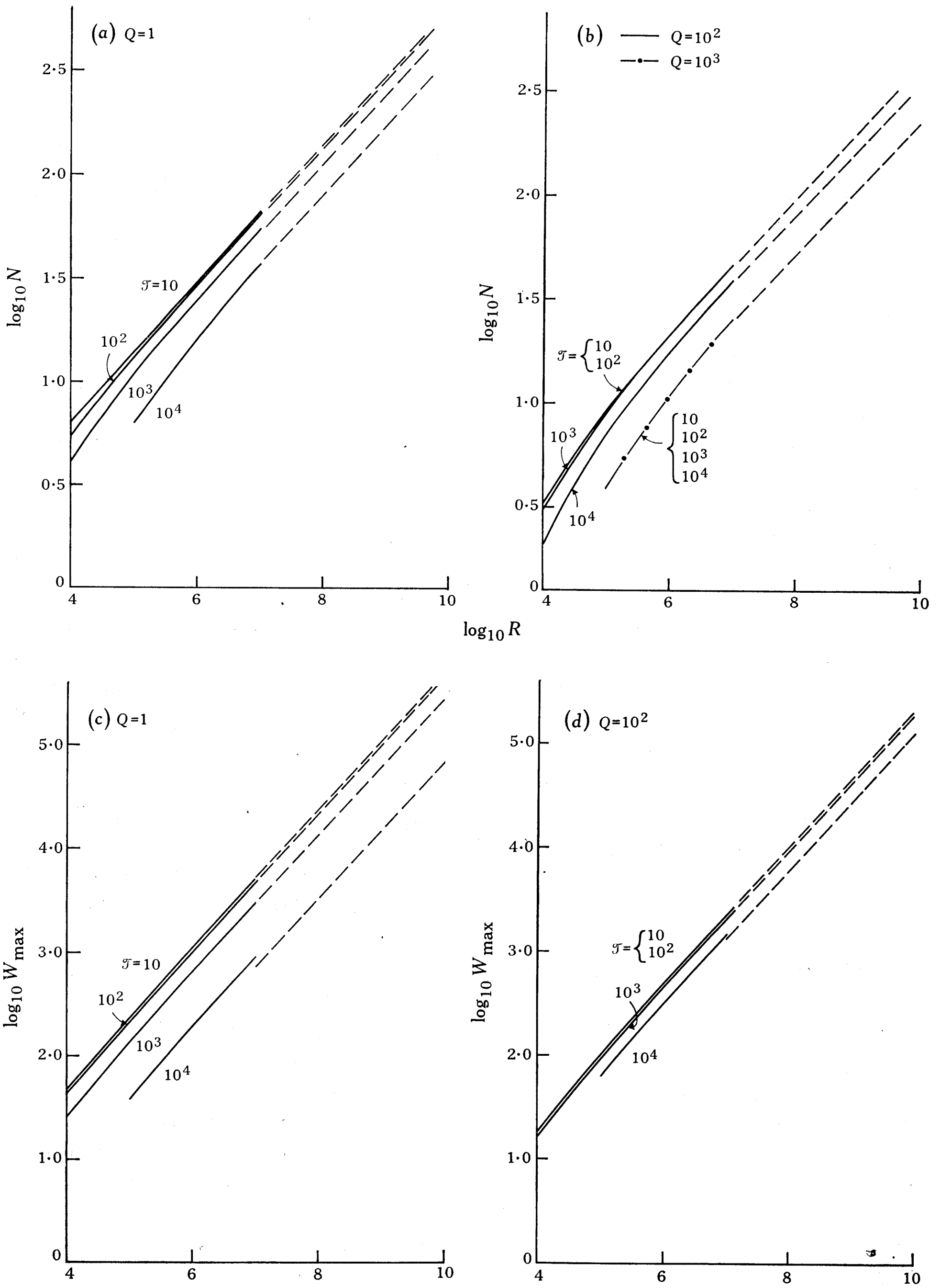
$$X_{\mathrm{B}}(z) = \frac{\psi^*(0)}{a\{\exp(a)-\exp(-a)\}}\Big(\{1+\exp(-a)\}\exp(az)+\{1+\exp(a)\}\exp(-az)\Big)+\psi^*(z) \tag{128}$$

with

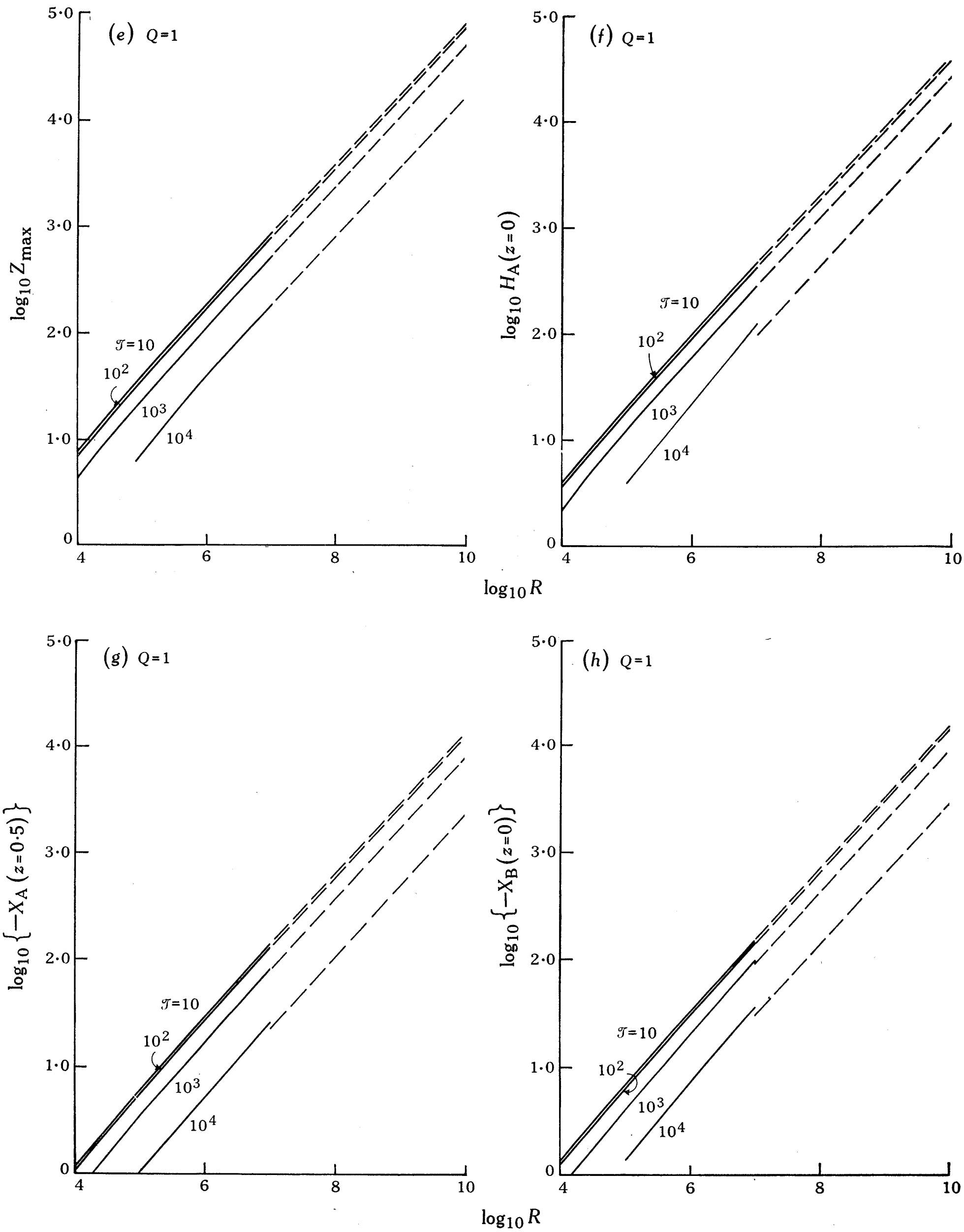
$$\psi^*(z) = -\sum_{n=1}^{M} \frac{(2n-1)^2 W_n \sin\{(2n-1)\pi z\}}{\pi^2\{(2n-1)^2+\alpha^2\}^2+Q(2n-1)^2}. \tag{129}$$

The numerical results for the behaviour of the Nusselt number with respect to the Taylor number are compared in Figure 1(a), for several values of the Rayleigh number when $a = \pi$ and $Q = 10$, with the analytical solutions obtained from the previous sections. It can be seen that agreement is very good at high Rayleigh numbers; in fact at $R = 10^7$ the results from Sections III and IV give a very good representation of N over the whole range of $\mathscr{T}$ for this value of Q. Figures 1(b) and 1(c) illustrate the changes in the $\log N$ versus $\log \mathscr{T}$ dependence for increasing magnetic field strength. An increase in Q clearly reduces the maximum value of N but at the same time the range of $\mathscr{T}$ supporting convection is substantially increased. In Figure 1(b) when $Q = 10^3$ an initial constancy of N with respect to increasing $\mathscr{T}$ can be seen, and this behaviour is extended through to Figure 1(c) where we have the very interesting situation that N is constant for values of the Taylor number from 1 to roughly 10^7 when $R = 10^7$ and $Q = 10^4$. This is a case when a strong magnetic field is present and increasing $\mathscr{T}$ does not affect the convective flux across the layer. Further, these values of $\mathscr{T}$ and Q in the asymptotic formula (89) for N certainly predict this behaviour. Associated with this feature we have the surprising circumstance that the analytic results from Sections III and IV are still in very good agreement with the numerical values. A summary of the numerical results for $R = 10^7$ and $a = \pi$ is given in Figure 2, which shows the inhibiting effect on the heat flux of a magnetic field and rotation acting simultaneously.

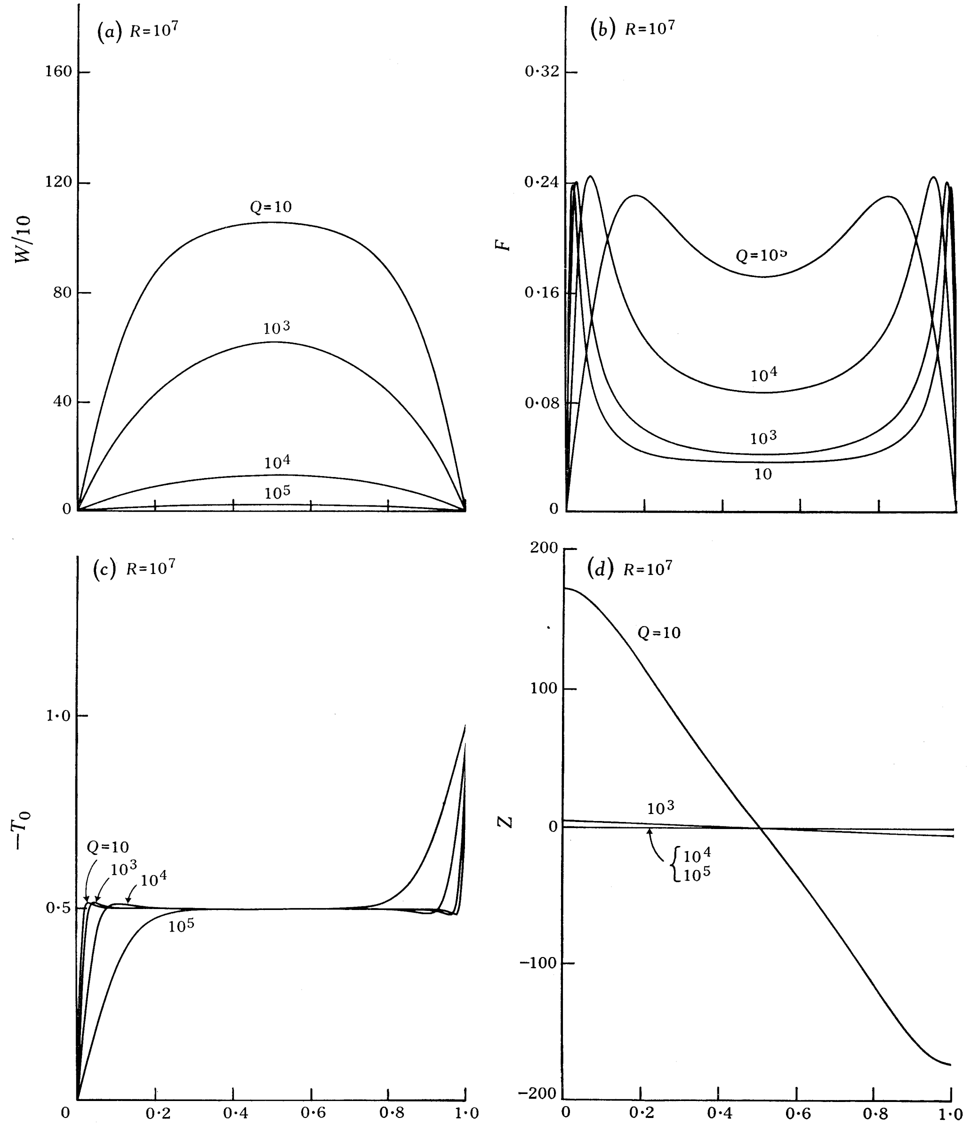
Figures 3(a)–3(c) illustrate the variation of N with respect to the magnetic field strength Q when $a = \pi$ for various values of R and increasing values of $\mathscr{T}$. As before, the numerical results are compared with the analytic values for N obtained from Section III and, in this case, Section IV(b), and good agreement is observed for $R = 10^7$. These results show that increasing the value of $\mathscr{T}$ to 10^3 has little effect on the N–Q dependence. However, for larger $\mathscr{T}$, an increase in Q has the effect of actually enhancing the convective flux across the layer. The maximum values of N in Figure 3(c) occur when $Q/\mathscr{T}^{\frac{1}{2}} \approx 1$, and from the definition of these two quantities this ratio is independent of d and ν. Also of particular interest in Figure 3(c) is that when $R = 10^7$ and $\mathscr{T} = 10^8$ convective heat transport is initially established at $Q = 3{\cdot}5\times 10^2$, a maximum value of N is reached at $Q = 10^4$, and thereafter at large values of Q the high rotation rate ceases to influence the convective processes. Overall this effect is probably best seen in Figure 4, which is a summary of the dependence of N on Q at $R = 10^7$, when all the solutions for different $\mathscr{T}$ have the same value of N for $Q \geqslant 6\times 10^4$.


(a) Q=1
$\mathcal{T}$=10
10^2
10^3
10^4
$\log_{10} N$
(b) Q=10^2 Q=10^3
$\mathcal{T}$= {10, 10^2}
10^3
10^4
{10, 10^2, 10^3, 10^4}
$\log_{10} R$
(c) Q=1
$\mathcal{T}$=10
10^2
10^3
10^4
$\log_{10} W_{max}$
(d) Q=10^2
$\mathcal{T}$= {10, 10^2}
10^3
10^4
$\log_{10} W_{max}$
$\log_{10} R$

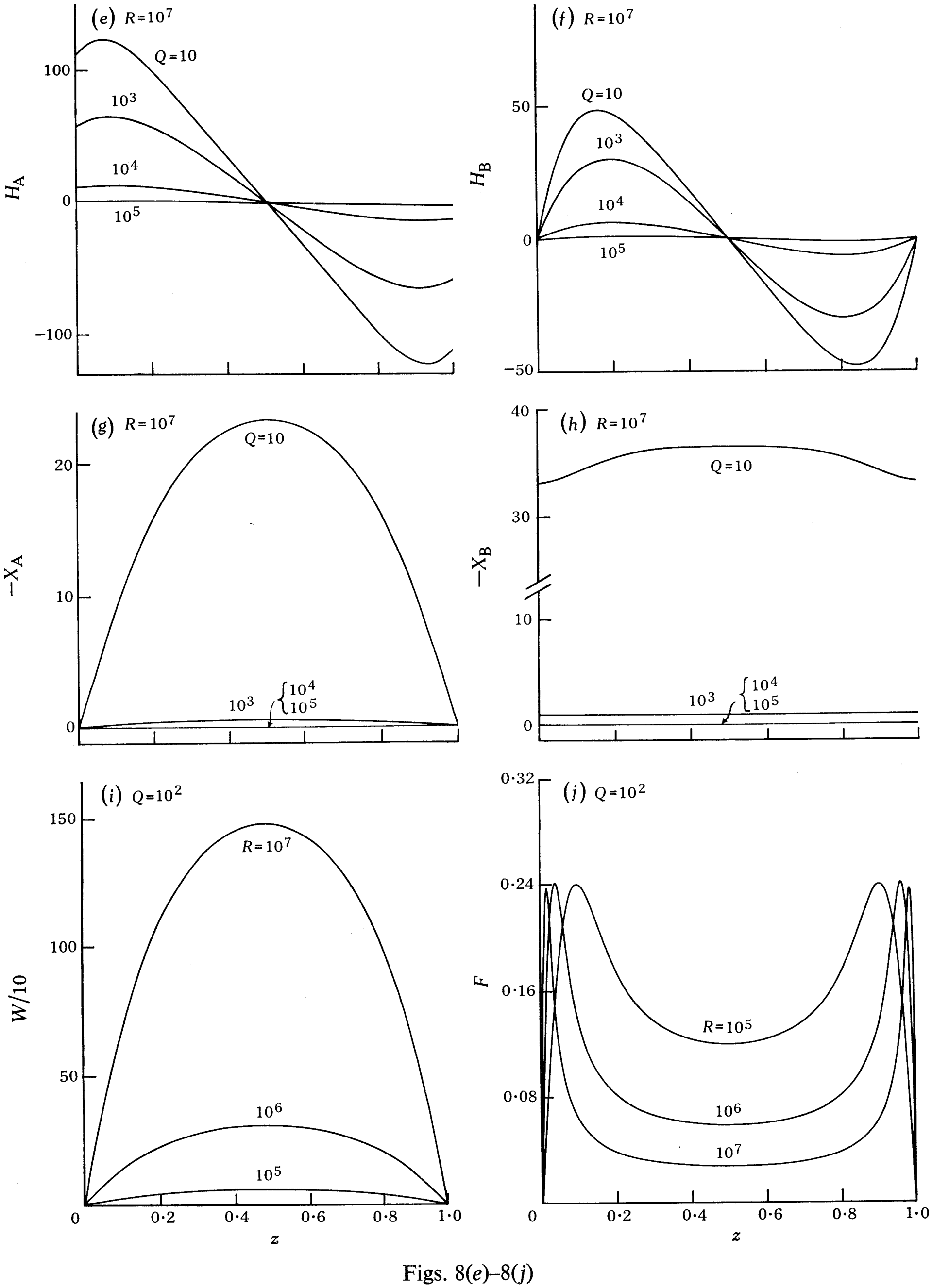
Figs. 7(*a*)–7(*d*)



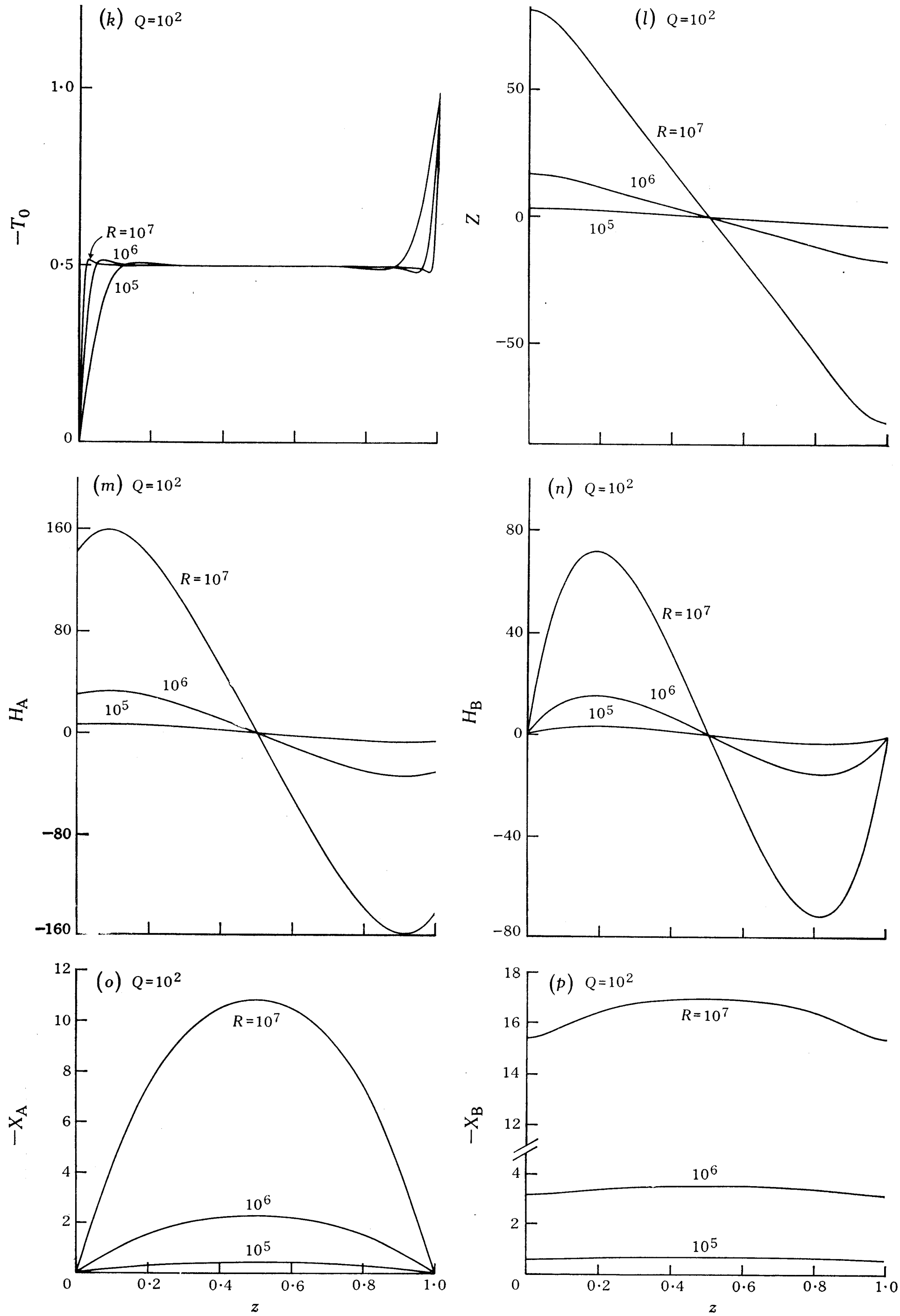
Figs. 7(*a*)–7(*h*).—Numerical results (continuous curves, and dot–dash curve in (*b*)) showing the variation with R of N, W_{max}, Z_{max}, H_A, X_A, and X_B for $a = \pi$ and the indicated values of $\mathscr{T}$ and Q. The dashed extensions to the curves are the asymptotic solutions from Section III.



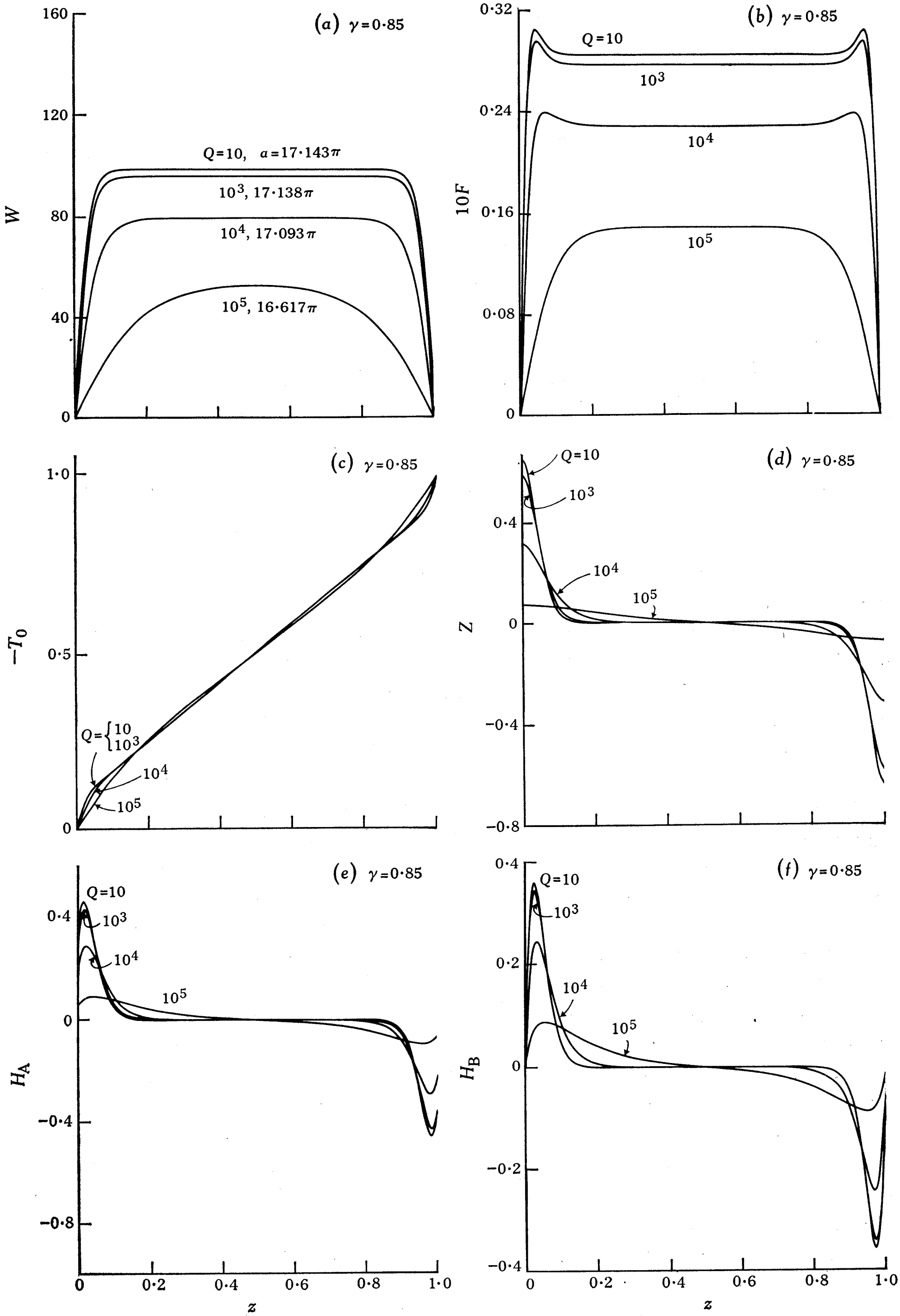
Figs. 8(*a*)–8(*p*).—Numerical results for W, F, T_0, Z, H_A, H_B, X_A, and X_B as functions of z ($0 \leqslant z \leqslant 1$) when $a = \pi$ and $\mathscr{T} = 10^4$ for the indicated values of R and Q.


(e) $R=10^7$
$Q=10$
10^3
10^4
10^5
H_A
100
0
−100
(f) $R=10^7$
$Q=10$
10^3
10^4
10^5
H_B
50
0
−50
(g) $R=10^7$
$Q=10$
10^3
10^4
10^5
$-X_A$
20
10
0
(h) $R=10^7$
$Q=10$
10^3
10^4
10^5
$-X_B$
40
30
10
0
(i) $Q=10^2$
$R=10^7$
10^6
10^5
$W/10$
150
100
50
0
0·2
0·4
0·6
0·8
1·0
z
(j) $Q=10^2$
$R=10^5$
10^6
10^7
F
0·32
0·24
0·16
0·08
0
0·2
0·4
0·6
0·8
1·0
z

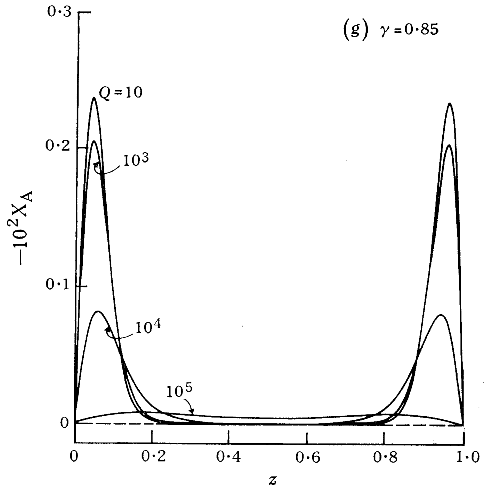
Figs. 8(*e*)–8(*j*)

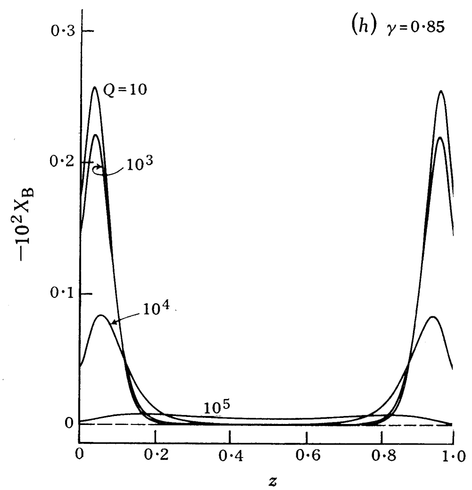

(k) $Q=10^2$
$-T_0$
1·0
0·5
0
$R=10^7$
10^6
10^5
(l) $Q=10^2$
Z
50
0
−50
$R=10^7$
10^6
10^5
(m) $Q=10^2$
H_A
160
80
0
−80
−160
$R=10^7$
10^6
10^5
(n) $Q=10^2$
H_B
80
40
0
−40
−80
$R=10^7$
10^6
10^5
(o) $Q=10^2$
$-X_A$
12
10
8
6
4
2
0
$R=10^7$
10^6
10^5
0·2
0·4
0·6
0·8
1·0
z
(p) $Q=10^2$
$-X_B$
18
16
14
12
4
2
0
$R=10^7$
10^6
10^5
0·2
0·4
0·6
0·8
1·0
z

Figs. 8(*k*)–8(*p*)



Figs. 9(*a*)–9(*f*) (Caption over page.)





Figs. 9(*a*)–9(*h*).—Numerical results for W, F, T_0, Z, H_A, H_B, X_A, and X_B as functions of z ($0 \leqslant z \leqslant 1$) for the indicated values of Q when $\mathscr{T} = 10^4$ and $\gamma = 0{\cdot}85$ with $a \gg \pi$, where γ is defined by equation (114).

The dependence of N on the wave number a for large R, $\mathscr{T}$, and Q is illustrated in Figures 5(*a*) and 5(*b*). Here the computations show that for certain values of the above parameters two distinct ranges of wave numbers support convection and that also there are two values of the horizontal wave number for which the Nusselt number has a maximum. These features are unique to this problem and were not observed in the earlier studies that dealt with the effect on thermal convection of either a magnetic field alone (Van der Borght *et al.* 1972) or rotation alone (Van der Borght and Murphy 1973). Two separate cell sizes are a possibility, although for the values of the parameters given in Figure 5(*b*) it can be expected that thin convective cells would be the most likely to be established.

An alternative way of displaying the dependence of the total heat flux on Q and $\mathscr{T}$ is presented in Figure 6, for $R = 10^7$ and $a = \pi$, where contours of constant N are given as functions of $\mathscr{T}$ and Q. The cross-hatched region corresponds to the pair of $Q, \mathscr{T}$ values that support convection for the given R and a values with $N > 1$.

The variation with respect to R of the Nusselt number N, the maximum value of the vertical velocity $W_{\max}$, the maximum value of the vorticity $Z(z)$, the value of $H_A(z)$ (nonconducting boundaries) at $z = 0$, the value of the current density $X_A(z)$ at $z = \frac{1}{2}$ and $X_B(z)$ (conducting boundaries) at $z = 0$ for $a = \pi$ and the indicated values of $\mathscr{T}$ and Q are given in Figures 7(*a*)–7(*h*). Here the numerical results are indicated by the continuous curves while the dashed extensions correspond to the asymptotic results from Section III. It can be observed that the agreement between all these results is very good at high values of the Rayleigh number. An interesting feature to note is that an increase in Q has a substantially greater effect on $W_{\max}$ than would be expected from the corresponding values of N: on comparing N for $R = 10^7$, $a = \pi$, $\mathscr{T} = 10^4$, and $Q = 10$ or 10^2, we find values of $38{\cdot}57$ and $40{\cdot}28$ respectively while the corresponding values of $W_{\max}$ are $1060{\cdot}6$ and $1478{\cdot}5$.

Figures 8 and 9 show the variation of W, F, T_0, Z, H_A, H_B, X_A, and X_B as functions of z across the fluid layer, $0 \leqslant z \leqslant 1$, and the effects of the basic parameters on these quantities. The solutions possess the same general characteristics as those found in the previous studies (Van der Borght *et al.* 1972; Van der Borght and Murphy 1973). In the present work we now have solutions for the current density $X(z)$ in addition to solutions for both $Z(z)$ and $H(z)$.

One feature of the $\gamma = 0\cdot 85$ solutions displayed in Figure 9 is worth brief comment. For these values of the wave number, corresponding to thin elongated cells, the solutions obtained show that Z, H, and X all have values near zero over the central regions of the fluid layer but strong variations near the upper and lower boundaries. Physically, this means that over the central regions the vorticity does not modify the velocity field, given by equation (8), and that the fluid motion is essentially vertical. Likewise we note from equation (9) that under such circumstances the magnetic field in these regions will correspond to the applied field in strength and direction.

VI. References

CHANDRASEKHAR, S. (1961).—"Hydrodynamic and Hydromagnetic Stability." (Clarendon Press: Oxford.)

ELTAYEB, I. A. (1972).—*Proc. R. Soc.* A **326**, 229.

ELTAYEB, I. A., and ROBERTS, P. H. (1970).—*Astrophys. J.* **162**, 699.

GREENSPAN, H. P. (1968).—"The Theory of Rotating Fluids." (Cambridge Univ. Press.)

HOWARD, L. N. (1965).—Notes from Summer Study Program in Geophysical Fluid Dynamics, Woods Hole Oceanographic Inst., 65–51, Vol. 1, p. 125.

NAKAGAWA, Y. (1957).—*Proc. R. Soc.* A **242**, 81.

NAKAGAWA, Y. (1959).—*Proc. R. Soc.* A **249**, 138.

PRIGOGINE, I., and GLANSDORFF, P. (1964).—*Physica* **30**, 351.

PRIGOGINE, I., and GLANSDORFF, P. (1965).—*Physica* **31**, 1242.

UNNO, W. (1969).—*Publs astr. Soc. Japan* **21**, 240.

VAN DER BORGHT, R., and MURPHY, J. O. (1973).—*Aust. J. Phys.* **26**, 341.

VAN DER BORGHT, R., MURPHY, J. O., and SPIEGEL, E. A. (1972).—*Aust. J. Phys.* **25**, 703.