APPLICATION OF REACTION MATRIX THEORY TO MESON PROCESSES

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Summary

The possible application of reaction matrix theory to pion–nucleon processes is discussed. It is found that suitable boundary conditions can remove the effects of the nucleon–pion cloud, leading to the conventional R-matrix formalism. A simple dynamical channel-coupling scheme is proposed which permits full analysis of experiments.

I. Introduction

The reaction matrix theory of nuclear interaction processes has proved highly satisfactory in explaining many features of nucleon–nuclear collisions, despite the great complexity of the intermediate compound nucleus states. Even though the Wigner and Eisenbud (1947) theory does not, for example, predict the energy eigenvalues at which resonances occur, it does display specifically how the cross sections should be parameterized, and lends physical meaning to the parameters.

A similar if not wholly analogous situation exists today with meson–nucleon interactions. There is no many-body theory of the complex forces involved and, owing to the large magnitude of the pion–nucleon coupling constant (Bethe and de Hoffman 1955), the perturbation theory developed by Feynman (1949) failed to give useful answers. On the other hand, dispersion theory (recently reviewed by Hamilton 1967) has made great progress towards an understanding of the intermediate processes and the relationship between two-body channels (Mandelstam 1958), but seems to lack one essential feature: it does not predict exactly how the contributions from various states should be parameterized. Donnachie and Hamilton (1964) came closest to a derivation from first principles with variational calculations, achieving considerable success using Layson's (1961) form for the P33 phase shift, unitarity, and dispersion relations. They obtained mutually consistent sets of phase shifts.

Let us consider how reaction matrix theory might be made compatible with current theories of high energy processes. Presumably, if the perturbation series with all conceivable intermediate states could be summed, one would then have a type of wave equation, such as the Bethe and Salpeter (1951) equation, which has appropriate interaction terms derived from field theory and whose solutions possess eigenvalues in energy corresponding to resonant states. Such a task has not been carried out fully, owing to the prohibitive number of intermediate processes that require consideration and an uncertainty about the eventual convergence of the perturbation series.

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The only successful field theoretical model to date is the static theory of Chew and Low (1956), which fits the P33 low energy behaviour quite adequately but fails to predict most of the other phase shifts correctly. Jackson's (1958) presentation of this theory exhibits the close connection with conventional quantum mechanics, and we shall assume that a meaningful wave equation exists for the problem in its entirety, although Jackson assumes the pion to be a conventional quantum mechanical particle and treats the nucleon as a quantized field.

It is shown in the present paper that, under very general assumptions, the source terms in such an equation still provide a simple form for the reaction matrix, although mixtures of states can contribute to a given resonance level, and the energies corresponding to the poles in the R-matrix do not necessarily correspond to eigenvalues of the wave equation. The effect of variations in the radial channel parameter is investigated.

II. Green's Theorem and Distribution of Eigenvalues

We assume that the wave function of the compound pion–nucleon system satisfies an equation of the Bethe–Salpeter type, and that the basic amplitude for the two-body state has the character of a spinor with the following properties.

(i) In the region of interaction the amplitude satisfies the two equations

$$(\boldsymbol{\gamma}.\mathbf{P}+M)\Psi = \Lambda_1 \tag{1a}$$

and

$$(\Box^2-\mu^2)\Psi = \Lambda_2\,, \tag{1b}$$

where Λ_1 and Λ_2 are source terms describing the quantized field effects, the $\boldsymbol{\gamma}$'s are conventional Dirac matrices, $\mathbf{P}$ is the four-momentum of the nucleon, M is the mass of the nucleon, μ is the mass of the meson, and each equation becomes the free-particle equation under the conditions

$$\Lambda_1 \to 0\,, \quad \Lambda_2 \to 0 \qquad \text{as} \qquad |\mathbf{r}_1-\mathbf{r}_2| \to \infty\,,$$

where $\mathbf{r}_1 = (\boldsymbol{r}_1, t_1)$ are the nucleon four-vector coordinates and $\mathbf{r}_2 = (\boldsymbol{r}_2, t_2)$ the pion four-vector coordinates, t_2, t_1 being the time coordinates of the pion and nucleon respectively.

(ii) It is assumed that the spinor $\Psi(\mathbf{r}_1, \mathbf{r}_2)$ is factorizable in the above asymptotic region and in the interaction region as

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \phi\big((M\mathbf{r}_1+\mu\mathbf{r}_2)/(M+\mu)\big)\,\Psi_R(\mathbf{r}_1-\mathbf{r}_2)\,, \tag{2}$$

a property found by Cutkosky (1954) and Wick (1954).

(iii) In the centre-of-mass system, the time derivatives in the wave equation of the two-body system can be separated in such a way that the spinor Ψ_R of equation (2) satisfies

$$(\nabla_R^2+\boldsymbol{k}^2)\,\Psi_R(\mathbf{R}, \mathbf{k}) = \rho(\mathbf{R}, \mathbf{k})\,, \tag{3}$$

where $\mathbf{R} = \mathbf{r}_1-\mathbf{r}_2$, $\mathbf{k} = (\boldsymbol{k}, ik_0)$, ρ is the pion field source, and $\eta = M\mu/(M+\mu)$.

(iv) Should many multiparticle channels be involved, we assume that the amplitude Ψ still obeys an equation with the structure

$$\left(\sum_i \frac{\mathbf{P}_i^2}{2M_i} - \tfrac{1}{2}\sum_i M_i\right)\Psi = \Lambda,$$

where $\mathbf{P}_i$ is the particle four-momentum and M_i is the particle mass, which is such that a separation of the relative time derivatives yields an equation

$$\sum_c \left(\frac{\nabla_{Rc}^2}{2\eta_c} + \frac{\boldsymbol{k}_c^2}{2\eta_c}\right)\Psi_R = \sum_c \frac{\rho_c}{\eta_c} = \frac{\rho}{\eta}, \tag{4}$$

η_c being the reduced mass in channel c.

(v) The wave functions X_λ representing the eigenfunctions in the region of interaction obey

$$\left(\sum_c \frac{\nabla_{Rc}^2}{2\eta_c} + \frac{\boldsymbol{k}_\lambda^2}{2\eta}\right)X_\lambda = \frac{\rho_\lambda}{\eta}, \qquad \frac{1}{\eta} = \sum_i \frac{1}{M_i}, \tag{5}$$

where

$$\sum_c (\boldsymbol{k}_{\lambda c}^2/\eta_c) = \boldsymbol{k}_\lambda^2/\eta \qquad \text{and} \qquad \sum_c (\boldsymbol{k}_c^2/\eta_c) = \boldsymbol{k}^2/\eta,$$

for each state X_λ for which there is an eigenvalue λ.

The Ψ_R and X_λ are chosen with the relative time coordinate put equal to zero to define a series of completely space-like surfaces in the centre-of-mass system. One expects from this non-invariant separation that the source terms ρ_λ will involve recoil corrections.

(vi) If the superposition principle is to hold everywhere we must have

$$\Psi_R(\boldsymbol{r}, \boldsymbol{k}) = \sum_\lambda A_\lambda(\boldsymbol{k})\, X_\lambda(\boldsymbol{r}, \boldsymbol{k}) \tag{6a}$$

and

$$\rho(\boldsymbol{r}, \boldsymbol{k}) = \sum_\lambda A_\lambda(\boldsymbol{k})\, \rho_\lambda(\boldsymbol{r}, \boldsymbol{k}). \tag{6b}$$

Now the appropriate boundary conditions involve the value of the logarithmic derivatives of the radial wave functions in each channel c (Preston 1962) and partition α, which is defined at each channel radius a_c as

$$\psi_{\lambda c}(a_c) = \int \Phi_c^* X_\lambda \, \mathrm{d}s, \tag{7}$$

where

$$\Phi_c = r_\alpha^{-1} \mathrm{i}^l \, \mathrm{Y}_l^m(\boldsymbol{\omega})\, \psi_{\alpha s \mu},$$

$\mathrm{Y}_l^m(\boldsymbol{\omega})$ is the conventional spherical harmonic, the integral is taken over the surface defined by a sphere of radius a_c, and $\psi_{\alpha s \mu}$ is a normalization constant.

Proceeding in the standard way from (5), it follows that

$$\int \mathrm{d}\tau \left(\sum_c (X_\lambda^* \nabla_{Rc}^2 \Psi_R - \Psi_R \nabla_{Rc}^2 X_\lambda^*)\eta/\eta_c\right) + (k^2 - k_\lambda^2) \int \mathrm{d}\tau\, X_\lambda^* \Psi_R$$

$$= \int \mathrm{d}\tau\, (X_\lambda^* \rho - \Psi_R \rho_\lambda^*)$$

$$= \sum_\mu A_\mu \int \mathrm{d}\tau\, (X_\lambda^* \rho_\mu - X_\mu \rho_\lambda^*)$$

$$= \sum_\mu G_{\lambda\mu} A_\mu, \tag{8}$$

the integral being taken over the volume of the region of interaction of the finite range forces. Usually X_λ and ρ_λ are real functions and so $G_{\lambda\mu}$ is an antisymmetric real matrix. More generally,

$$G_{\lambda\mu} = -G_{\mu\lambda}^*. \tag{9}$$

The components of **G** are related to transition matrix elements for processes involving field sources. Notice that for purely lamellar interactions of the form

$$\rho_\lambda = V(R) X_\lambda, \tag{10}$$

where V is a scalar potential function, equations (8) lead to a null matrix **G**, showing the absence of the field source type of transition.

It is shown in Section VI that, if the X_λ are normalized in such a way that

$$\int \mathrm{d}\tau\, X_\lambda^* X_\mu = \delta_{\lambda\mu} \qquad \text{and} \qquad A_\lambda = \int \mathrm{d}\tau\, X_\lambda^* \Psi_R, \tag{11}$$

then $G_{\lambda\mu}$ also becomes a null matrix if the derivatives of the internal radial wave functions vanish at the channel radii.

Using Green's theorem in equations (8), we get

$$\sum_c \left(\psi_{\lambda c}^* \Psi'_{Rc} - \Psi_{Rc} \psi_{\lambda c}'^*\right)_{a_c} \frac{\eta}{\eta_c} + (k^2 - k_\lambda^2) A_\lambda = \sum_\mu G_{\lambda\mu} A_\mu, \tag{12}$$

where the $\psi_{\lambda c}$ are the radial wave functions in channel c with eigenvalues k_λ and the primes denote differentiation with respect to r_c, Ψ_{Rc} being the appropriate channel wave function.

By using boundary condition matrices B_c (Preston 1962) one can solve (12) for A_λ to obtain

$$A_\mu = \sum_\lambda (C_{\lambda\mu})^{-1} \sum_c \left(\psi_{\lambda c}^* \Psi'_{Rc} - \frac{B_c}{a_c} \psi_{\lambda c}^* \Psi_{Rc}\right)_{a_c} \frac{\eta}{\eta_c}, \tag{13}$$

where

$$a_c \psi'_{\lambda c} = B_c \psi_{\lambda c}, \qquad \text{and} \qquad C_{\lambda\mu} = \delta_{\lambda\mu}(k_\lambda^2 - k^2) + G_{\lambda\mu}. \tag{14}$$

The inverse of $C_{\lambda\mu}$ becomes diagonal in the case of lamellar forces.

In terms of equations (6) and (13) the wave function in channel c becomes

$$\Psi_{Rc}(r_c) = \sum_{c'} \left(\Psi'_{Rc'}(a_{c'}) - \frac{B_{c'}}{a_{c'}} \Psi_{Rc'}(a_{c'}) \right) \sum_{\lambda} \sum_{\mu} \overset{*}{\psi}_{\lambda c'}(a_{c'}) \psi_{\mu c}(r_c) \frac{\eta}{\eta_c} (C_{\lambda\mu})^{-1} . \tag{15}$$

III. Reaction Matrix

The reaction matrix is defined as the relationship between the wave functions and their derivatives at the channel radii a_c, according to

$$(\eta_c a_c)^{-\frac{1}{2}} \Psi_c(a_c) = \sum_{c'} R_{cc'} (\eta_{c'} a_{c'})^{-\frac{1}{2}} (a_{c'} \Psi'_{c'} - B_{c'} \Psi_{c'})_{a_{c'}} . \tag{16}$$

Here η_c is the reduced mass in channel c and the suffix R has been omitted for convenience. Evaluating $\Psi_c(r_c)$ at a_c in (15), we find by comparison with (16) that

$$R_{cc'} = \sum_{\lambda} \sum_{\mu} \overset{*}{\gamma}_{\lambda c'} \gamma_{\mu c} (C_{\lambda\mu})^{-1} , \tag{17}$$

where

$$\gamma_{\lambda c} = \hbar (\eta / \eta_c a_c)^{\frac{1}{2}} \psi_{\lambda c}(a_c) .$$

Equation (17) is the expression for the reaction matrix in the presence of source terms. To illustrate how it differs from the conventional form, let $\mathbf{C}$ be a 2×2 matrix denoting the existence of two eigenvalues to the field equation (5):

$$\mathbf{C} = \begin{pmatrix} E_1 - E & G \\ -\overset{*}{G} & E_2 - E \end{pmatrix}, \qquad \begin{aligned} E_\lambda &= k_\lambda^2 , \\ G &= G_{12} = -\overset{*}{G}_{21} . \end{aligned}$$

We shall assume that there is no diagonal component coming from $\mathbf{G}$. Hence

$$\mathbf{C}^{-1} = \frac{1}{(E_1 - E)(E_2 - E) + |G|^2} \begin{pmatrix} E_2 - E & -G \\ \overset{*}{G} & E_1 - E \end{pmatrix} . \tag{18}$$

Let $C' = C = 1$ denote the elastic scattering channel, giving, for real G,

$$R_{11} = \frac{\overset{2}{\gamma}_{11}}{E_1 - E + |G|^2/(E_1 - E)} + \frac{\overset{2}{\gamma}_{21}}{E_2 - E + |G|^2/(E_2 - E)} . \tag{19}$$

The conventional expression for R_{11} is obtained by putting $|G|^2$ equal to zero.

IV. Way of Reducing to Conventional Form

It is probable that the off-diagonal elements of $\mathbf{C}$ may be impossible to isolate from the experimental data. This is because they occur as part of a scalar product in $R_{cc'}$ and not as separate terms. For a large number of eigenvalues, we can write

$$R_{cc'} = (1/D(E))(\gamma_{1c} \gamma_{1c'} N_{11} + \gamma_{1c} \gamma_{2c'} N_{12} + \ldots) ,$$

where the N_{ij} are the appropriate cofactors of each element of $\mathbf{C}$ and

$$D(E) = \det \mathbf{C} . \tag{20}$$

The resonances occur when $D(E)$ vanishes, so let E'_λ be the q roots of $D(E) = 0$. By the well-known theorem of partial fractions (Gradshteyn and Ryzhik 1965) we can write

$$(C_{ij})^{-1} = \frac{N_{ij}}{D(E)} = \frac{A_{ij}}{E'_1 - E} + \frac{B_{ij}}{E'_2 - E} + \ldots,$$

where

$$A_{ij} = N_{ij}(E'_1)/D'(E'_1), \quad B_{ij} = N_{ij}(E'_2)/D'(E'_2), \quad \ldots$$

and

$$D'(E) = \mathrm{d}D(E)/\mathrm{d}E.$$

Thus

$$R_{cc'} = \sum_{ij} \gamma_{ic}\gamma_{jc'} \frac{N_{ij}(E'_\lambda)}{D'(E'_\lambda)} \frac{1}{E'_\lambda - E} \tag{21}$$

is a form exhibiting the poles and residues of $R_{cc'}$. It only remains to redefine the partial widths such that

$$\gamma'_{\lambda c}\gamma'_{\lambda c'} = \sum_{ij} \gamma_{ic}\gamma_{jc'} N_{ij}(E'_\lambda)/D'(E'_\lambda), \tag{22}$$

to yield the conventional form for $R_{cc'}$ of

$$R_{cc'} = \sum_{\lambda} \gamma'_{\lambda c}\gamma'_{\lambda c'}/(E'_\lambda - E) \tag{23}$$

given first by Wigner and Eisenbud (1947).

This result shows that we can admit a wider class of wave equations than the Schrödinger equation with lamellar potentials without changing the structure of reaction matrix theory. However, the poles of the R-matrix no longer correspond to the eigenvalues of the wave equations in the interior region, even after taking account of the usual level shifts.

V. Boundary Condition Matrix

Consider Green's theorem for two of the eigenfunctions X_λ, and an interaction of the type in equation (4),

$$\int \mathrm{d}s \left(X^*_\lambda \frac{\partial X_\mu}{\partial r} - X_\mu \frac{\partial X^*_\lambda}{\partial r} \right)_{r=a} = G_{\lambda\mu} + (k^2_\mu - k^2_\lambda) \int X^*_\lambda X_\mu \, \mathrm{d}\tau$$

$$= \sum_c \left(\psi^*_{\lambda c} \frac{\partial \psi_{\mu c}}{\partial r} - \psi_{\mu c} \frac{\partial \psi^*_{\lambda c}}{\partial r} \right)_{r=a}. \tag{24}$$

When the X_λ are orthogonal, equation (24) yields a relation connecting $G_{\lambda\mu}$ to the derivatives at the channel radii. By adopting the boundary condition (13) the matrix elements can be made to vanish, and this leads to the condition from (8) that

$$\int X^*_\lambda \rho_\mu \, \mathrm{d}\tau = \int X_\mu \rho^*_\lambda \, \mathrm{d}\tau. \tag{25}$$

An inspection of Jackson's P-wave equation in the static model reveals that condition (25) is indeed satisfied by the source terms. As well as taking account of such effects as the diffusiveness of the nucleon surface, the boundary condition matrices can diagonalize $C_{\lambda\mu}$, showing that the Wigner–Eisenbud reaction matrix takes into account meson field sources when applied to nuclear problems.

As an alternative, if such general boundary conditions cannot be found, one may work with the derivatives themselves as parameters. In general, one will have

$$A_\mu = \sum_\lambda \sum_c (C_{\lambda\mu})^{-1}(\psi^*_{\lambda c}\psi'_c - \psi_c \psi'^*_{\lambda c})\eta/\eta_c ,$$

whence

$$\psi_c = \sum_{c'} \sum_\lambda \sum_\mu (C_{\lambda\mu})^{-1}(\psi^*_{\lambda c'}\psi'_{c'} - \psi_{c'}\psi'^*_{\lambda c'})\psi_{\mu c'}\,\eta/\eta_c ,$$

which yields

$$\begin{aligned}\psi_c \hbar\left(\frac{\eta}{\eta_c a_c}\right)^{\frac{1}{2}} &= \sum_{c''} \psi'_{c''}\left\{\left(\sum_\lambda \sum_\mu (C_{\lambda\mu})^{-1}\frac{\hbar^2\eta}{(\eta_c \eta_{c'} a_c a_{c'})^{\frac{1}{2}}}\psi^*_{\lambda c}\psi_{\mu c'}\right)\left(\delta_{cc'}+F_{cc'}\right)^{-1}\right\} \\ &= \sum_{c''} R_{cc''}\hbar(\eta/\eta_{c''}a_{c''})^{\frac{1}{2}}\psi'_{c''} ,\end{aligned} \tag{26}$$

where $\delta_{cc'}$ is the Kronecker delta function. Hence

$$R_{cc''} = \sum_\lambda \sum_\mu \sum_{c'} (C_{\lambda\mu})^{-1}\gamma_{\lambda c''}\gamma_{\mu c'}(\delta_{cc'}+F_{cc'})^{-1} ,$$

where

$$F_{cc'} = \hbar^2 \sum_\lambda \sum_\mu (C_{\lambda\mu})^{-1}\psi'^*_{\lambda c'}\psi_{\mu c} . \tag{27}$$

The derivatives appear as parameters in $F_{cc'}$, and this is the most general form for $R_{cc'}$, without the greatly simplifying assumption of the boundary conditions (13). In applications to experiment we have used this assumption with satisfactory results.

VI. Coupling Assumptions

To examine the question of the effect of unitarity upon phase shifts for multi-particle processes, Cook (1967) assumed that the partial wave transition amplitudes in all channels had the same phase. This assumption is a far too restrictive dynamical condition to be of much practical use, but can be broadened a little by observing that, in low energy neutron nuclear collisions, the inverse R-matrix always has very much larger elements than the R-matrix itself and that, where single-level fits are adequate, the inverse R-matrix can be considered to be singular. That is

$$\det(R_{cc'}) \simeq 0 . \tag{28}$$

In fact, every 2×2 minor of **R** can be considered to be zero. This is essentially a dynamical constraint connecting the asymptotic behaviour of the various channel wave functions and implies that the reduced widths for each channel are in approximately constant ratio, or that one particular term in $R_{cc'}$ is dominant. It is a most

useful assumption in multichannel analysis, as it allows us to connect all inelastic channels to the real and imaginary parts of the scattering phase shift. If we write

$$\text{(i)}\ \mathbf{Q} = (1/\mathrm{i})(\mathbf{1}+\mathbf{S}_1)^{-1}(\mathbf{S}_1-\mathbf{1}), \qquad \mathbf{S}_1 = \boldsymbol{\Omega}^{-1}\mathbf{P}^{-\frac{1}{2}}\mathbf{S}\mathbf{P}^{\frac{1}{2}}\boldsymbol{\Omega}^{-1},$$

and

$$\text{(ii)}\ \mathbf{R} = \mathbf{Q}(\mathbf{s}\mathbf{Q}+\mathbf{P})^{-1}, \tag{29}$$

where $\boldsymbol{\Omega}$ is the hard sphere phase matrix, $\mathbf{P}$ the penetration factor matrix, and $\mathbf{s}$ the level shift matrix, these quantities being diagonal matrices (as defined by Preston 1962), we get

$$\det \mathbf{Q} = \det(\mathbf{S}_1-\mathbf{1}) \simeq 0. \tag{30}$$

From this condition we obtain

$$\mathbf{S}_1-\mathbf{1} \simeq (2\mathrm{i}/\Delta)\mathbf{Q} = 2\mathrm{i}\mathbf{T}_1,$$

where $\mathbf{T}_1$ is the reduced transition matrix and

$$\det \mathbf{S} = \Delta^*/\Delta, \qquad \Delta = \det(\mathbf{1}-\mathrm{i}\mathbf{Q}).$$

For example, in the two-channel case we can parameterize $\mathbf{S}_1$ as

$$\mathbf{S}_1 = \begin{pmatrix} \eta e^{2i\delta} & (1-\eta^2)^{\frac{1}{2}}e^{2i\phi} \\ (1-\eta^2)^{\frac{1}{2}}e^{2i\phi} & -\eta e^{2i(2\phi-\delta)} \end{pmatrix}, \tag{31}$$

where η is the absorption coefficient, δ the scattering phase shift, and ϕ the production phase shift. Hence

$$-\det \mathbf{S}_1 \simeq e^{4i\phi} \simeq (1-\eta e^{2i\delta})/(1-\eta e^{-2i\delta}), \tag{32}$$

from which the phase shift for production can be determined. This assumption was tested and found to give no contradiction with experiment.

VII. Conclusions

Some complications that might arise when applying R-matrix theory to pion field interactions have been investigated. It has been found that Lorentz invariance imposes minor modifications of underlying concepts, and that boundary condition matrices can remove off-diagonal elements of the R-matrix in the space of eigenfunctions. Finally, a simple dynamical coupling assumption has been proposed which simplifies analysis of multichannel experiments.

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