SPACE CHARGE EFFECTS IN THE TOWNSEND–HUXLEY SWARM TECHNIQUE

By B. S. Liley*

[*Manuscript received May* 22, 1967]

Summary

An analytical treatment of space charge effects in the Townsend–Huxley swarm technique is given. The general case of a mixed swarm of equally charged ions and electrons drifting through a gas in which attachment is possible is considered. The case of a pure electron or pure ion swarm is treated as a particular case of the general one. The main result is an expression for $R(r,z)$, this being the ratio of the current collected by a disk of radius r, in the plane $z = z$, to the total swarm current. This ratio is determined as an explicit function of D/μ, the total current, the electron (or ion) current, and the attachment coefficient. It is shown that space charge effects can be of importance for large currents, low E/p, and low gas temperatures. Moreover, in practice such effects are only likely to be of interest in thermal swarms.

I. Introduction

Under certain conditions, space charge effects may be of major importance in the Townsend–Huxley swarm technique. At first sight this is a surprising statement, since in this technique the swarm currents are only of the order of 10^{-13}–10^{-12} A. However, the fact that such effects can be significant is readily understood from the following argument. It is to be expected (confirmed in Section IV(*c*)) that the ratio $\Delta\phi/V$ is a measure of the magnitude of space charge effects in the Townsend–Huxley technique. $\Delta\phi$ is the potential, due to space charge, between the axis and some characteristic radius of the swarm and V is the mean thermal energy of a particle in the swarm. For large values of this ratio the lateral spreading of the beam is determined by space charge effects, while for small values diffusion processes dominate. Apart from a geometrical factor, $\Delta\phi$ is proportional to N, where N is the number of particles per unit length. Again, the swarm current I is of order New, w being the drift speed and e the particle charge, and it follows that $\Delta\phi/V$ is proportional to I/wV. Obviously, for a given gas temperature and current this ratio may be made as large as desired by making w, that is, E/p, sufficiently small. Expressed in physical terms, the lower the value of w the larger the value of N and thus of $\Delta\phi$.

In general, the ratio I/wV defines those conditions for which space charge effects may be important. They are large currents, low E/p, and low gas temperatures. Again, due to the inverse dependence of this ratio on w, it is clear that for a given E/p such effects will be much greater in an ion swarm than in an electron swarm.

* Department of Engineering Physics, Australian National University, Canberra.

In a general mixture of equally charged electrons and ions $N = N_i + N_e$, where N_i and N_e are the ion and electron line densities respectively. That is, $\Delta\phi$ is proportional to $(I_i/w_i + I_e/w_e)$, where I_e and I_i are the electron and ion currents, and w_e and w_i the respective drift speeds. Since w_i/w_e is of the order of 10^{-2}, it follows that the ion current need only be a small fraction of the total current for the space charge field to be almost entirely determined by the ions; that is, in an electron swarm "contaminated" by ions marked space charge effects can occur, the fields associated with the relatively slow moving ions being sufficient to perturb the faster moving electrons. Such contamination occurs in an electron swarm moving through a gas in which attachment phenomena are possible.

In practice, space charge phenomena need not invalidate experimental accuracy. The presence of such effects is easily recognized, various parameters such as D/μ exhibiting an apparent current dependence, and limiting values as the current tends to zero may be readily determined. Even so, a detailed and accurate analysis of the phenomena is warranted. There are two main reasons for this. The first is to guarantee that any experimentally observed current dependence can be adequately explained in terms of space charge effects. The second is that the results of such a treatment can be used to determine drift speeds and attachment coefficients at low values of E/p and at low temperatures.

In the subsequent analysis, the general case of a swarm of equally charged electrons and ions drifting through a gas in which attachment is possible is treated, the case of a pure electron or pure ion swarm being simply a particular example of the general one. In the following section, the basic equations are discussed. These are complex and nonlinear, and certain approximations are inevitable. These approximations are introduced in Section III and a solution to the problem is obtained. Since the usefulness of the results is obviously dependent on their accuracy, the limitations imposed by the approximations are discussed at some length in Section IV. Also in that section, the real physical conditions for which space charge effects are likely to be of importance are considered. Finally, in Section V, the main results and conclusions of the analysis are briefly summarized.

Wherever applicable, an MKS system of units is used. Again, certain references are made to Watson's (1944) text on Bessel functions. Such references are denoted by WBF followed by the appropriate page number.

Applications of the results of this analysis are to be given elsewhere (Crompton, to be published).

II. General

(a) *Statement of the Problem*

The essential elements of the experimental arrangement are shown in Figure 1, details being given by Crompton and Jory (1962). The diffusion chamber is bounded by two plane electrodes separated by a distance h. The anode is circumferentially split at a radius b. The axial electric field E_a is maintained at a uniform value at a radius c by means of suitably placed additional electrodes. For theoretical purposes the radius c is assumed to be infinite.

Electrons are generated by a heated filament and enter the diffusion chamber through a small hole in the cathode. In the absence of space charge effects and attachment processes these electrons drift towards the anode, diffusing radially in transit. The experimental results are given in terms of R, the ratio of the current received by the central anode disk to the total anode current, as a function of E/p. From this ratio the electron diffusion coefficient, and hence the electron temperature, may be deduced.

For large electron currents or very low E/p, space charge effects become important, radial spreading of the swarm being appreciably greater than that due to diffusion alone. In the presence of negative ions this effect is enhanced, the relatively slow moving ions leading to a degree of charge "accumulation".

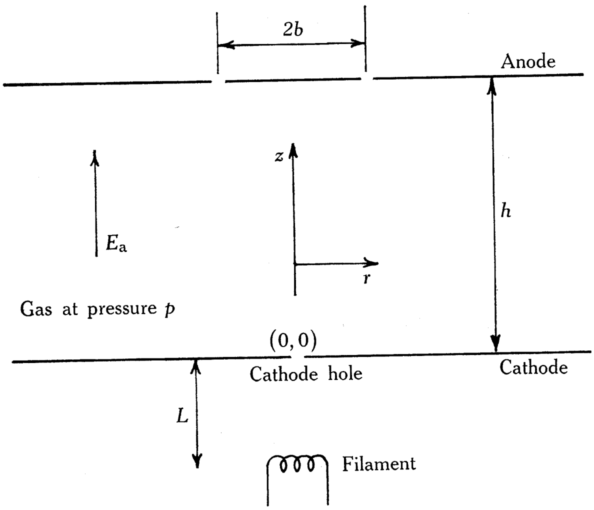

Fig. 1.—Experimental arrangement and geometry.

Therefore, for electrons drifting through a gas in which attachment is possible, the ratio R besides being a function of E/p is also a function of the attachment coefficient α and the total current I. The prime object of the subsequent theory is to determine this function.

(b) *Basic Equations*

The basic equations governing the drift and diffusion of the electrons are the continuity and flux equations

$$\operatorname{div} \mathbf{j}_{\mathrm{e}} = -\alpha' n_{\mathrm{e}} \tag{1}$$

and

$$\mathbf{j}_{\mathrm{e}} = n_{\mathrm{e}} \mu_{\mathrm{e}} \mathbf{E} - D_{\mathrm{e}} \operatorname{grad} n_{\mathrm{e}}, \tag{2}$$

where $\mathbf{j}_{\mathrm{e}}$ is the electron flux density, n_{e} the electron number density, α' the attachment coefficient, μ_{e} the electron mobility coefficient, D_{e} the electron diffusion coefficient, and $\mathbf{E}$ the electric field vector. It is assumed that the electron temperature (and hence α', μ_{e}, and D_{e}) is uniform throughout the diffusion chamber.

The electric field vector may be written as the sum of two components

$$\mathbf{E} = \mathbf{E}_{\mathrm{a}}+\mathbf{E}_{\mathrm{s}}\,,$$

where $\mathbf{E}_{\mathrm{a}}$ is the applied uniform axial electric field, and $\mathbf{E}_{\mathrm{s}}$ is the field due to space charge. For the cylindrically symmetric problem considered here,

$$\mathbf{E}_{\mathrm{a}} \equiv (0,0,E_{\mathrm{a}}); \qquad \mathbf{E}_{\mathrm{s}} \equiv (E_{\mathrm{s}r},0,E_{\mathrm{s}z})\,.$$

On substituting equation (2) in (1), a diffusion equation for the electrons may be obtained. This is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n_{\mathrm{e}}}{\partial r}\right)+\frac{\partial^2 n_{\mathrm{e}}}{\partial z^2}-2\lambda_{\mathrm{e}}\frac{\partial n_{\mathrm{e}}}{\partial z}-2\alpha\,\lambda_{\mathrm{e}}\,n_{\mathrm{e}} = \frac{2\lambda_{\mathrm{e}}}{E_{\mathrm{a}}}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r(n_{\mathrm{e}}\,E_{\mathrm{s}r})\right)+\frac{\partial}{\partial z}\left(n_{\mathrm{e}}\,E_{\mathrm{s}z}\right)\right\}, \tag{3}$$

where

$$\lambda_{\mathrm{e}} = w_{\mathrm{e}}/2D_{\mathrm{e}}\,, \qquad w_{\mathrm{e}} = \mu_{\mathrm{e}}\,E_{\mathrm{a}}\,, \qquad \alpha = \alpha'/w_{\mathrm{e}}.$$

A similar equation may be obtained for the negative ions, this being

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n_{\mathrm{i}}}{\partial r}\right)+\frac{\partial^2 n_{\mathrm{i}}}{\partial z^2}-2\lambda_{\mathrm{i}}\frac{\partial n_{\mathrm{i}}}{\partial z}+2\alpha\,\lambda_{\mathrm{i}}\frac{w_{\mathrm{e}}}{w_{\mathrm{i}}}n_{\mathrm{e}} = \frac{2\lambda_{\mathrm{i}}}{E_{\mathrm{a}}}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r(n_{\mathrm{i}}\,E_{\mathrm{s}r})\right)+\frac{\partial}{\partial z}\left(n_{\mathrm{i}}\,E_{\mathrm{s}z}\right)\right\}, \tag{4}$$

where the definitions of λ_{i}, n_{i}, and w_{i} correspond to those given for the electrons.

The term $2\alpha\,\lambda_{\mathrm{e}}\,n_{\mathrm{e}}$ in equation (3) accounts for the loss of electrons due to attachment processes, while the corresponding term in equation (4) accounts for the gain in ions due to the same processes.

To obtain a closed set, it is necessary to include with equations (3) and (4) the Poisson equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rE_{\mathrm{s}r}\right)+\frac{\partial}{\partial z}\left(E_{\mathrm{s}z}\right) = \frac{(n_{\mathrm{i}}+n_{\mathrm{e}})e_{\mathrm{e}}}{\epsilon}, \tag{5}$$

where ϵ is the permittivity of the medium, and e_{e} is the electronic charge.

(c) *Boundary Conditions*

Equations (3), (4), and (5) have to be solved subject to certain boundary conditions. Provided thermal velocities are appreciably greater than drift velocities, an appropriate set of boundary conditions for the electrons is (Hurst and Liley 1965, Section III(c))

$$n_{\mathrm{e}}\ \text{finite}, \qquad j_{\mathrm{e}z}(r,0) = \tfrac{1}{2}n_{\mathrm{e}}(r,0)w_{\mathrm{e}}+P_{\mathrm{e}}(r,0), \qquad n_{\mathrm{e}}(r,h) = 0.$$

$P_{\mathrm{e}}(r,0)$ is the axial source term. For the problem in hand this is a point source at the origin $(0,0)$ and

$$P_{\mathrm{e}}(r,0) = p'(r,0)\,\delta(r)\,,$$

where

$$2\pi\int_0^\infty p'(r,0)\,\delta(r)\,r\,\mathrm{d}r = P\,.$$

P is the total axial flux at the source hole.

Similar boundary conditions apply to the ions.

With respect to equation (5), the relevant boundary conditions are not discussed since they are not needed.

(d) *The Ratio* $R(r,z)$

It is convenient to define a general ratio $R(r,z)$ by the relation

$$R(r,z) = R_{\mathrm{e}}(r,z)+R_{\mathrm{i}}(r,z)\,, \tag{6}$$

where

$$R_{\mathrm{e}}(r,z) = (2\pi/I)\int_0^r e_{\mathrm{e}}\, j_{\mathrm{e}z}(r,z)\, r\,\mathrm{d}r \tag{7}$$

and

$$R_{\mathrm{i}}(r,z) = (2\pi/I)\int_0^r e_{\mathrm{e}}\, j_{\mathrm{i}z}(r,z)\, r\,\mathrm{d}r\,, \tag{8}$$

I being the total swarm current.

III. The Ratio $R(r,z)$

(a) *An Equation for* $R(r,z)$

In order to simplify the mathematics certain approximations will be made. In particular it will be assumed that both diffusion and space charge effects in the axial direction may be neglected. The limitations imposed by these approximations on the validity of the final results are discussed in Section IV. Subject to the approximations, equations (3), (4), and (5) become

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n_{\mathrm{e}}}{\partial r}\right)-2\lambda_{\mathrm{e}}\frac{\partial n_{\mathrm{e}}}{\partial z}-2\alpha\,\lambda_{\mathrm{e}}\,n_{\mathrm{e}} = \frac{2\lambda_{\mathrm{e}}}{E_{\mathrm{a}}}\frac{1}{r}\frac{\partial}{\partial r}\Big(r(n_{\mathrm{e}}\,E_{\mathrm{s}r})\Big)\,, \tag{9}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n_{\mathrm{i}}}{\partial r}\right)-2\lambda_{\mathrm{i}}\frac{\partial n_{\mathrm{i}}}{\partial z}+2\alpha\,\lambda_{\mathrm{i}}\frac{w_{\mathrm{e}}}{w_{\mathrm{i}}}n_{\mathrm{e}} = \frac{2\lambda_{\mathrm{i}}}{E_{\mathrm{a}}}\frac{1}{r}\frac{\partial}{\partial r}\Big(r(n_{\mathrm{i}}\,E_{\mathrm{s}r})\Big)\,, \tag{10}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(rE_{\mathrm{s}r}\Big) = (n_{\mathrm{i}}+n_{\mathrm{e}})e_{\mathrm{e}}/\epsilon\,, \tag{11}$$

while

$$j_{\mathrm{e}z} = n_{\mathrm{e}}\,w_{\mathrm{e}}\,; \qquad j_{\mathrm{i}z} = n_{\mathrm{i}}\,w_{\mathrm{i}}\,. \tag{12}$$

These equations can be rewritten in terms of R and R_{e}.

Integration of equation (11), after some rearrangement in which equations (12) are used in (6) and (7), gives

$$E_{\mathrm{s}r} = \frac{I}{2\pi\epsilon w_{\mathrm{i}}}\frac{1}{r}\left\{R-\left(1-\frac{w_{\mathrm{i}}}{w_{\mathrm{e}}}\right)R_{\mathrm{e}}\right\}. \tag{13}$$

Again, noting that n_{e} can now be expressed in the form

$$n_{\mathrm{e}} = \frac{I}{2\pi e_{\mathrm{e}}\,w_{\mathrm{e}}}\frac{1}{r}\frac{\partial R_{\mathrm{e}}}{\partial r}\,,$$

multiplication of equation (9) by r and integration over the range 0 to r gives

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial R_{\mathrm{e}}}{\partial r}\right)-2\lambda_{\mathrm{e}}\frac{\partial R_{\mathrm{e}}}{\partial z}-2\lambda_{\mathrm{e}}\,\alpha R_{\mathrm{e}} = 2\lambda_{\mathrm{e}}\frac{E_{\mathrm{s}r}}{E_{\mathrm{a}}}\frac{\partial R_{\mathrm{e}}}{\partial r}\,. \tag{14}$$

Similarly, on multiplying equation (10) by rw_{i} and integrating from 0 to r an equation for R_{i} may be obtained. Adding this equation to (14), which is then used to eliminate all terms in α, an equation for the total ratio R may also be obtained. This is

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial R}{\partial r}\right)-2\lambda_{\mathrm{i}}\frac{\partial R}{\partial z} = 2\lambda_{\mathrm{i}}\frac{E_{\mathrm{s}r}}{E_{\mathrm{a}}}\frac{\partial R}{\partial r}-\left(\frac{\lambda_{\mathrm{i}}}{\lambda_{\mathrm{e}}}-1\right)r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial R_{\mathrm{e}}}{\partial r}\right). \tag{15}$$

Using equation (13) to eliminate $E_{\mathrm{s}r}$ from (14) and (15), the problem is reduced to one of finding a solution of two simultaneous second-order partial differential equations. In general, further simplification is impossible. If, however,

$$\lambda_{\mathrm{i}} = \lambda_{\mathrm{e}} \equiv \lambda, \tag{16}$$

reduction to a single equation readily follows. Since, as implied in the Introduction and further discussed in Section IV, imposition of this condition is unlikely to limit the practical usefulness of the results, the subsequent analysis is restricted to this one particular case. Subject to (16), equation (15) becomes

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial R}{\partial r}\right)-2\lambda\frac{\partial R}{\partial z} = 2\lambda\frac{E_{\mathrm{s}r}}{E_{\mathrm{a}}}\frac{\partial R}{\partial r}. \tag{17}$$

Referring now to equation (14), if the substitution

$$R_{\mathrm{e}} = \exp(-\alpha z)\,R_{\mathrm{e}0}$$

is made, then it is found that $R_{\mathrm{e}0}$ satisfies exactly the same equation as R, namely, equation (17). Therefore, since R and $R_{\mathrm{e}0}$ will satisfy identical boundary conditions, it follows that

$$R_{\mathrm{e}} = f_{\mathrm{e}0}\exp(-\alpha z)\,R, \tag{18}$$

where $f_{\mathrm{e}0}$ (a constant) is that fraction of the total current carried by the electrons at $z = 0$ ($r = 0$).

Using this result in equation (13), and eliminating $E_{\mathrm{s}r}$ from (17), a single equation determining R is obtained. This is

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial R}{\partial r}\right)-2\lambda\frac{\partial R}{\partial z} = 2\gamma\left\{1-\beta\exp(-\alpha z)\right\}\frac{R}{r}\frac{\partial R}{\partial r}, \tag{19}$$

where

$$\beta = f_{\mathrm{e}0}(1-w_{\mathrm{i}}/w_{\mathrm{e}}) \tag{20}$$

and

$$\gamma = \lambda I/2\pi\epsilon w_{\mathrm{i}}E_{\mathrm{a}}. \tag{21}$$

Finally, certain particular cases of this equation should be noted. For $\alpha = 0$, equation (19) also applies to a general swarm of equally charged ions and electrons, with the limiting cases of a pure electron swarm for $f_{\mathrm{e}0} = 1$ and a pure ion swarm for $f_{\mathrm{e}0} = 0$.

(b) *Transformation to Integral Form*

The imposition of boundary conditions on the solution of equation (19) is most easily performed by transforming this equation to integral form. On introducing a flux density J, defined by

$$R = 2\pi\int_0^r Jr\,\mathrm{d}r, \tag{22}$$

differentiation of (19) with respect to r leads to the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial J}{\partial r}\right)-2\lambda\frac{\partial J}{\partial z} = -4\pi\rho(r,z)\,, \tag{23}$$

where

$$\rho(r,z) \equiv -\frac{\gamma}{4\pi^2}\left\{1-\beta\exp(-\alpha z)\right\}\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{R}{r}\frac{\partial R}{\partial r}\right). \tag{24}$$

Equation (23) is a standard type of diffusion equation with a continuously distributed source term.

If instead of being given by (24), $\rho(r,z)$ is given by

$$\rho(r,z) = \delta(r)\,\delta(z)\,,$$

where

$$2\pi\int_0^\infty \delta(r)\,r\,\mathrm{d}r = 1\,,$$

then a solution of (23) is

$$J_1(r,z) = z^{-1}\exp(-\lambda r^2/2z)\,; \qquad z \geqslant 0\,.$$

Similarly, for a point source of unit strength at the point (r',θ',z') a solution is

$$g(r,\theta,z;r',\theta',z') = Z^{-1}\exp(-\lambda R^2/2Z)\,, \tag{25}$$

where

$$R^2 = r^2+r'^2-2rr'\cos(\theta-\theta')$$

and

$$Z = z-z' \quad (\geqslant 0)\,.$$

Therefore, for a source of strength $\rho(r',z')\,r'\,\mathrm{d}\theta'\,\mathrm{d}r'\,\mathrm{d}z'$ the contribution to J at the point (r,θ,z) is

$$\delta J_\mathrm{p} = \rho(r',z')\,gr'\,\mathrm{d}\theta'\,\mathrm{d}r'\,\mathrm{d}z'$$

and the total flux density due to all such sources is

$$J_\mathrm{p}(r,z) = \int_0^z\int_0^\infty\int_0^{2\pi}\rho(r',z')\,gr'\,\mathrm{d}\theta'\,\mathrm{d}r'\,\mathrm{d}z'\,. \tag{26}$$

Since ρ is independent of θ, integration over θ' is straightforward. Noting that (WBF, p. 79)

$$I_0(x) = \pi^{-1}\int_0^\pi \exp(x\cos\theta)\,\mathrm{d}\theta\,,$$

on substituting (25) for g in the integral (26)

$$J_\mathrm{p}(r,z) = 2\pi\int_0^z\int_0^\infty \rho(r',z')\,Z^{-1}\exp\left(-\lambda\frac{r^2+r'^2}{2Z}\right)I_0\left(\frac{rr'}{Z}\right)r'\,\mathrm{d}r'\,\mathrm{d}z'\,. \tag{27}$$

With ρ given by (24) this is a "solution" of equation (23). However, to this solution a further term, corresponding to the real point source at the origin ($r' = 0$, $z' = 0$) must be added. Apart from a constant this term is given by (25) and

$$J(r,z) = (A/z)\exp(-\lambda r^2/2z)+J_\mathrm{p}(r,z)\,. \tag{28}$$

Inserting this expression for J in (22) leads to an integral equation for R. Furthermore, since $R(\infty, 0) = 1$ while $J_p(r, 0) = 0$, the constant A may be determined from this equation and

$$R = \{1-\exp(-\nu)\}+2\pi\int_0^r J_p(r,z)\,r\,\mathrm{d}r, \tag{29}$$

where

$$\nu = \lambda r^2/2z \tag{30}$$

and ρ is given by (24).

(c) *Solution for Small* γ

In most cases of practical interest γ is a small quantity. Therefore, in seeking a solution of (29) we expand R as a power series in γ, that is, we put

$$R = \sum_{n=0}^{\infty} \gamma^n R_n. \tag{31}$$

On substituting this expression for R in equation (29) and equating powers of γ,

$$R_0 = \{1-\exp(-\nu)\}, \tag{32}$$

$$R_n = -A_n(r,z) \qquad n \geqslant 1, \tag{33}$$

where from equations (29), (27), and (24)

$$A_n(r,z) = \int_0^r\int_0^z Z^{-1}\{1-\beta\exp(-\alpha z')\}\exp(-\lambda r^2/2Z)\,F_n(r,Z)\,\mathrm{d}z'\,r\,\mathrm{d}r, \tag{34}$$

$$F_n(r,Z) = \tfrac{1}{2}\int_0^\infty \exp\left(-\frac{\lambda r'^2}{2Z}\right)\left[\frac{\partial}{\partial r'}\left\{\frac{1}{r'}\frac{\partial}{\partial r'}\left(\sum_s R_s R_{n-1-s}\right)\right\}\right] I_0\left(\frac{\lambda r r'}{Z}\right)\mathrm{d}r', \tag{35}$$

it being understood that for $k \neq 0$, $R_{-k} = 0$. Therefore, as inspection confirms, R_n may be determined in terms of $R_{n-1}, R_{n-2}, \ldots R_0$ and hence a complete solution may be found. However, as pointed out in Section IV, a solution to first order in γ is adequate and to this approximation

$$R = \{1-\exp(-\nu)\}-\gamma A_1, \tag{36}$$

the relevant F_1 integral being

$$F_1 = \tfrac{1}{2}\int_0^\infty \exp\left(-\frac{\lambda r'^2}{2Z}\right)\left[\frac{\partial}{\partial r'}\left\{\frac{1}{r'}\frac{\partial}{\partial r'}\left(1-\exp(-\nu')\right)^2\right\}\right] I_0\left(\frac{\lambda r r'}{Z}\right)\mathrm{d}r'. \tag{37}$$

Using the integral (WBF, p. 393),

$$\int_0^\infty I_0(at)\exp(-p^2t^2)\,t\,\mathrm{d}t = (1/2p^2)\exp(a^2/4p^2),$$

integration over r' is straightforward and F_1 is readily determined. Substituting the result in (34) and reversing the order of integration, i.e. integrating over r first,

$$A_1(r,z) = \exp(-\nu)\int_0^z \frac{1-\beta\exp(-\alpha z')}{z'}\left\{1-\exp\left(-\nu\frac{z'}{2z-z'}\right)\right\}\mathrm{d}z'. \tag{38}$$

Making the substitution

$$z' = 2zy/(\nu+y),$$

$A_1(r,z)$ may be rewritten as

$$A_1(r,z) = \nu\exp(-\nu)\int_0^\nu \left\{1-\beta\exp\left(\frac{-2azy}{\nu+y}\right)\right\}\frac{1-\exp(-y)}{y(y+\nu)}\,\mathrm{d}y. \tag{39}$$

To evaluate this integral it is necessary to expand the integrand in powers of $2az$. On doing this it is found that

$$A_1(r,z) = (1-\beta)a_{10}(\nu) - \beta\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}a_{1n}(\nu)\,(2az)^n, \tag{40}$$

where the coefficients $a_{1n}(\nu)$ are given by

$$a_{1n}(\nu) = \nu\exp(-\nu)\int_0^\nu y^{n-1}\{1-\exp(-y)\}/(y+\nu)^{n+1}\,\mathrm{d}y. \tag{41}$$

As shown in the Appendix, these integrals may be reduced to tabulated form, the first four being

$$a_{10} = \{1-\exp(-\nu)\}\ln 2 + E(0,\nu)\{1+\exp(-\nu)\} - E(0,2\nu), \tag{42}$$

$$a_{11} = \nu\{\ln 2 + E(0,\nu) - E(0,2\nu)\} - \tfrac{1}{2}\exp(-\nu)\{1-\exp(-\nu)\}, \tag{43}$$

$$a_{12} = a_{11}(1+\tfrac{1}{2}\nu) + \tfrac{1}{8}\exp(-\nu)\{1-\exp(-\nu)\} - \tfrac{1}{4}\exp(-\nu)\,\nu, \tag{44}$$

$$a_{13} = a_{11}(1+\nu+\tfrac{1}{6}\nu^2) + \tfrac{1}{24}(5+\nu)\exp(-\nu)\{1-\exp(-\nu)\} - \tfrac{1}{24}(9+2\nu)\exp(-\nu)\,\nu, \tag{45}$$

with

$$E(0,x) = \int_0^x \{1-\exp(-u)\}/u\,\mathrm{d}u \tag{46}$$

being a tabulated function (Harvard Computation Laboratory 1949).

At first sight these coefficients and the corresponding series appear disappointingly complicated. However, due to the alternating character of the series, for $2az$ of the order of unity or less convergence is rapid and only the first few terms need be retained. In fact, as will be shown, the series may be approximated by a simple exponential function.

From the integral expressions (41) for the coefficients, comparison of the integrands over the range of integration shows that

$$a_{1n+1} < \tfrac{1}{2}a_{1n}.$$

Therefore, for $2az$ of the order of unity or less, neglect of all terms in the series higher than the nth involves an error no greater than

$$\mathscr{E}_n = \frac{a_{1n+1}}{a_{11}}\frac{(2az)^n}{(n+1)!} < \frac{1}{2^n(n+1)!}(2az)^n. \tag{47}$$

In particular for $n = 3$, $2az \leqslant 1$, the error is already less than 1%. In fact, as will be seen, the real error is even smaller than this. Noting that for small x

$$E(0,x) \sim x, \tag{48}$$

it can be shown from the expressions given for a_{11}, a_{12}, and a_{13} that as $\nu \to 0$

$$a_{12}/a_{11} \to 0\cdot 35(\ldots); \qquad a_{13}/a_{12} \to 0\cdot 39(\ldots). \tag{49}$$

Furthermore, on calculating these ratios for other values of ν it is found that they are surprisingly constant over a wide range, at least over the range $\nu = 0$ to $\nu = 2$, differing by only a few per cent from these limiting values. Therefore, in particular, from (47)

$$\mathscr{E}_3 = \frac{a_{14}}{a_{11}}\frac{1}{4!}(2az)^3 = \frac{a_{14}a_{13}a_{12}}{a_{13}a_{12}a_{11}}\frac{1}{4!}(2az)^3 < 0\cdot02(az)^3,$$

and even for $az = 1$ the error involved in neglecting all terms higher than the third

TABLE 1

PARTIAL TABULATION OF COEFFICIENTS a_{10} AND a_{11}

ν	$a_{10}(\nu)$	$a_{11}(\nu)$	ν	$a_{10}(\nu)$	$a_{11}(\nu)$
0·00	0	0	1·00	0·2085	0·0542
0·05	0·0326	0·0091	1·20	0·1977	0·0507
0·10	0·0614	0·0170	1·40	0·1824	0·0462
0·20	0·1087	0·0298	1·60	0·1651	0·0413
0·40	0·1707	0·0462	1·80	0·1472	0·0364
0·60	0·2012	0·0538	2·00	0·1299	0·0317
0·80	0·2115	0·0557			

is only of the order of 2%. Because of this $A_1(r,z)$ may be reduced to a relatively simple expression. The series term may be rearranged to read

$$a_{11}\frac{a_{11}}{a_{12}}\left\{\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n!}\frac{a_{12}}{a_{11}}\frac{a_{1n}}{a_{11}}(2az)^n\right)\right\}. \tag{50}$$

However,

$$\frac{a_{12}}{a_{11}}\frac{a_{12}}{a_{11}} = (0\cdot35)^2,$$

$$\frac{a_{12}}{a_{11}}\frac{a_{13}}{a_{11}} = (0\cdot35)^2(0\cdot39) \simeq (0\cdot35)^3,$$

and, to the degree of approximation required, it may be assumed in general that

$$\frac{a_{1n}}{a_{11}}\frac{a_{12}}{a_{11}} = (0\cdot35)^n.$$

This means that the series (50) may be approximated by

$$(a_{11}/a_{12})a_{11}\{\exp(-0\cdot7\,az)-1\} = \tfrac{20}{7}a_{11}\{\exp(-0\cdot7\,az)-1\}$$

and that

$$A_1(r,z) = (1-\beta)a_{10}+\tfrac{20}{7}\beta a_{11}\{1-\exp(-0\cdot7\,az)\}.$$

Therefore for $az \leqslant 1$, $\nu \leqslant 2$, the final solution for R is

$$R = \{1-\exp(-\nu)\}-\gamma(1-\beta)a_{10}-\tfrac{20}{7}\gamma\beta a_{11}\{1-\exp(-0\cdot7\,az)\}, \tag{51}$$

the accuracy being of the order of 1%. The coefficients a_{10} and a_{11} are given exactly by (42) and (43), and have been partially tabulated in Table 1. For most practical cases this solution should be adequate. However, if greater accuracy or a wider range of ν or az is required then the series term in $A_1(r,z)$ must be calculated exactly.

(d) *Solution for Large* γ

For very large γ radial diffusion is negligible, the lateral spreading of the swarm being governed entirely by space charge effects. For this particular case equation (19) becomes

$$\frac{\partial R}{\partial z} = -\frac{\gamma}{\lambda}\left\{1-\beta\exp(-az)\right\}\frac{R}{r}\frac{\partial R}{\partial r}. \tag{52}$$

On seeking a solution of the form

$$R = U(z)\,r^2, \qquad r \leqslant b;$$
$$R = 1, \qquad r \geqslant b;$$

it is found that

$$(1/U^2)\mathrm{d}U/\mathrm{d}z = -(2\gamma/\lambda)\{1-\beta\exp(-az)\}.$$

For a point source at the origin this equation is to be solved subject to

$$1/U(0) = 0.$$

The relevant solution is

$$U(z) = \left\{\frac{2\gamma}{\lambda}z\left(1-\frac{\beta}{az}\{1-\exp(-az)\}\right)\right\}^{-1},$$

giving

$$\left.\begin{aligned} R(r,z) &= r^2\left\{\frac{2\gamma}{\lambda}z\left(1-\frac{\beta}{az}\{1-\exp(-az)\}\right)\right\}^{-1}, && r \leqslant b,\\ &= 1, && r \geqslant b. \end{aligned}\right\} \tag{53}$$

The two solutions for $r = b$ define b.

(e) *Special Cases*

There are several cases of special interest, namely,

(i) $a = 0$, with $f_{e0} = 0$ or $f_{e0} = 1$;

(ii) $\nu \ll 1$, $az \ll 1$;

(iii) $az \gg 1$.

(i) *Case* $a = 0$

For small γ the solution for $a = 0$ is

$$R = \{1-\exp(-\nu)\}-\gamma(1-\beta)a_{10}, \tag{54}$$

where, as defined by equations (20) and (21),

$$\beta = f_{e0}(1-w_i/w_e)$$

and

$$\gamma = \lambda I/2\pi\epsilon w_i E_a \qquad (\lambda \equiv \lambda_i = \lambda_e).$$

This result applies to a mixture of equally charged ions and electrons in the absence of attachment processes. Furthermore, this result emphasizes an important point. Since $(1-f_{e0})I$ is the current due to the ions in the swarm and w_i/w_e is only of the order of 10^{-2}, the ionic current need only be a small fraction of the total for the ions to completely determine the space charge field.

If $f_{e0} = 0$, the swarm consists entirely of ions and

$$R = \{1-\exp(-\nu)\}-\gamma_i a_{10}, \tag{55}$$

with

$$\gamma_i = \lambda_i I/2\pi\epsilon w_i E_a. \tag{56}$$

γ_i is always a positive quantity and no matter what the ionic charge this solution is applicable.

If $f_{e0} = 1$, the swarm consists entirely of electrons and

$$R = \{1-\exp(-\nu)\}-\gamma_e a_{10}, \tag{57}$$

with

$$\gamma_e = \lambda_e I/2\pi\epsilon w_e E_a. \tag{58}$$

It is to be noted that the solutions (55) and (57) are valid no matter what the values of λ_i and λ_e.

Corresponding results for large γ may be obtained from (53).

(ii) *Case* $\nu \ll 1$, $az \ll 1$

Using (48), for small ν and az, equation (51) becomes

$$R \simeq \Big(1-\gamma[\{(1-\beta)+2az\beta\}\ln 2 -az\beta]\Big)\nu. \tag{59}$$

Noting that to the same approximation azI is the ionic current, this result once again emphasizes the dominance of the space charge fields associated with the ions. Even for $f_{e0} = 1$, $2az$ need only be of the order of w_i/w_e ($\sim 10^{-2}$) for these fields to dominate those of the electrons.

In general of course, f_{e0} is not equal to unity. In drifting between the filament and the cathode electrons are lost and ions are produced by attachment processes. If conditions in this region are similar to those in the diffusion chamber proper, then

$$f_{e0} = \exp(-aL) \simeq 1-aL, \tag{60}$$

where L is the axial distance between the filament and the cathode.

(iii) *Case* $az \gg 1$

For $az \gg 1$ the solution (51) is inadequate. However, on physical grounds alone it is to be expected that the result will be the same as that given in (55). For sufficiently large az, this may be confirmed by inspection of (39).

IV. Discussion

In this section the implications and limitations of the various approximations made in the preceding analysis are discussed. In particular, the conditions for which the axial diffusion and the axial component of the space charge field may be neglected are established. Again, the physical significance of γ and the expansion of R in terms of this parameter are discussed, and finally, for the general case, the physical limitations imposed on the results by the restriction $\lambda_i = \lambda_e$ are considered.

(a) Axial Diffusion

The axial component of the flux density equation is

$$j_z = n\mu E_z - D\,\partial n/\partial z\,, \tag{61}$$

subscripts being ignored since this equation applies to both electrons and ions.

The condition

$$n\mu E_z \gg \partial n/\partial z \tag{62}$$

states that the motion of the swarm in the axial direction is essentially independent of diffusion. It is not, however, a criterion for neglecting axial diffusion. Since axial and radial diffusion are coupled by the continuity equation, to ignore axial diffusion on the grounds that (62) is satisfied could lead to a serious miscalculation of the degree of diffusion in the radial direction. Referring to equations (3) and (4), the true criterion is

$$\left|\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n}{\partial r}\right)\right| \ll \left|\frac{\partial^2 n}{\partial z^2}\right|.$$

This condition is, however, too stringent since it cannot be satisfied for all r and z. Therefore, in order to establish a suitable criterion appropriate to the previous analysis a more exact approach is adopted. In the absence of space charge effects, the zero-order solution with axial diffusion included is compared directly with the case when such is neglected.

Defining

$$J = J_{\rm i}+J_{\rm e}\,, \qquad J_{\rm e} = n_{\rm e}w_{\rm e}\,, \qquad J_{\rm i} = n_{\rm i}w_{\rm i}\,, \tag{63}$$

equations (3) and (4) may be expressed in terms of $J_{\rm e}$ and $J_{\rm i}$. On putting $\lambda_{\rm i} = \lambda_{\rm e}$ and ignoring the space charge terms, the addition of these two equations gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial J}{\partial r}\right)+\frac{\partial^2 J}{\partial z^2}-2\lambda\frac{\partial J}{\partial z} = 0\,. \tag{64}$$

Again, from (61) it may be shown that

$$j_z = j_{{\rm e}z}+j_{{\rm i}z} = J-(1/2\lambda)\partial J/\partial z\,. \tag{65}$$

Equations (64) and (65) are also obviously true in the limiting cases of $J = J_{\rm e}$, $J = J_{\rm i}$. Furthermore, from Section II(*c*) it may be shown that J, $J_{\rm e}$, and $J_{\rm i}$ all satisfy the same types of boundary conditions. Therefore the solutions for J and the limiting cases $J = J_{\rm i}$, $J = J_{\rm e}$ are analytically the same and, in particular, the ratio R_0 for all three cases is given by the Huxley–Crompton relationship (Hurst and Liley 1965)

$$R_0 = 1-z(z^2+r^2)^{-\frac{1}{2}}\exp[-\lambda\{(z^2+r^2)^{\frac{1}{2}}-z\}]\,. \tag{66}$$

On the other hand, the zero-order solution for the case in which axial diffusion is ignored is

$$R_0 = 1-\exp(-\lambda r^2/2z)\,. \tag{67}$$

For $z^2/r^2 \gg 1$ the difference between these two solutions is negligible. However, in using (67) to calculate higher order approximations in the presence of space charge, the condition $z^2/r^2 \gg 1$ must be satisfied for relatively large r and not just for those values near the axis. Therefore, in order to satisfy this requirement, a criterion for ignoring axial diffusion must be of the form

$$z^2/r^2 \gg 1, \qquad \lambda r^2/2z \sim 1.$$

These two conditions may be combined to give

$$2/\lambda z \ll 1. \tag{68}$$

The results derived in Section III are subject to this criterion.

(*b*) *Axial Component of Space Charge Field*

The discussion in this subsection is similar to that given in the preceding one. The condition

$$E_{\mathrm{a}} \gg E_{\mathrm{s}z},$$

is a statement that motion in the axial direction is essentially unaffected by space charge fields. On the other hand, since $E_{\mathrm{s}z}$ and $E_{\mathrm{s}r}$ are coupled by Poisson's equation, the criterion for neglecting $E_{\mathrm{s}z}$ is

$$\left|\frac{\partial E_{\mathrm{s}z}}{\partial z}\right| \ll \left|\frac{1}{r}\frac{\partial}{\partial r}\left(r E_{\mathrm{s}r}\right)\right|.$$

Unfortunately, unlike the diffusion case, an exact solution is not possible. Therefore, this criterion must be adopted as such. It may, however, be rewritten in the alternative form

$$\left|\frac{\partial E_{\mathrm{s}z}}{\partial z}\right| \ll \left|\frac{(n_{\mathrm{i}}+n_{\mathrm{e}})e_{\mathrm{e}}}{\epsilon}\right|. \tag{69}$$

In Section II(*a*) it was stated that the electric field is maintained at a uniform value $E_z(c,z) = E_{\mathrm{a}}$ at the radius c. If $\phi(r,z)$ is the potential at the point (r,z), it follows that

$$\phi(r,z) = \phi(c,0) - E_{\mathrm{a}}\, z - \int_c^r E_{\mathrm{s}r}(r,z)\,\mathrm{d}r, \tag{70}$$

whence

$$\partial E_{\mathrm{s}z}/\partial z = -\partial^2\phi/\partial z^2 = \int_c^r \partial^2 E_{\mathrm{s}r}/\partial z^2\,\mathrm{d}r. \tag{71}$$

Restricting the subsequent discussion to an ion swarm, on neglecting $E_{\mathrm{s}z}$, $E_{\mathrm{s}r}$ is given by (equation (13))

$$E_{\mathrm{s}r} = \frac{I}{2\pi\epsilon w_{\mathrm{i}}}\frac{1}{r}R, \tag{72}$$

where, to the same degree of approximation,

$$R = 1 - \exp(-\lambda r^2/2z). \tag{73}$$

On substituting (72) in (71) it is found that

$$\left|\frac{\partial E_{\mathrm{s}z}}{\partial z}\right| \simeq \frac{I}{2\pi\epsilon w_{\mathrm{i}}}\frac{1}{2z^2}\left\{\left(1-\frac{\lambda r^2}{2z}\right)\exp\left(-\frac{\lambda r^2}{2z}\right) - \left(1-\frac{\lambda c^2}{2z}\right)\exp\left(-\frac{\lambda c^2}{2z}\right)\right\}. \tag{74}$$

On the other hand, since

$$n_{\mathrm{i}} = \frac{I}{2\pi w_{\mathrm{i}}\, e_{\mathrm{i}}}\frac{1}{r}\frac{\partial R}{\partial r},$$

$$\frac{n_{\mathrm{i}}\, e_{\mathrm{i}}}{\epsilon} = \frac{I}{2\pi\epsilon w_{\mathrm{i}}}\frac{\lambda}{z}\exp\left(-\frac{\lambda r^2}{2z}\right). \tag{75}$$

Therefore, for $\lambda c^2/2z$ appreciably greater than unity, (69) becomes

$$|(1/2\lambda z)(1-\lambda r^2/2z)| \ll 1 .$$

For $\lambda r^2/2z$ very large this cannot be satisfied. However, since r cannot be greater than c, while in actual fact $|\partial E_z/\partial z| = 0$ for $r = c$ (see equation (74)), an adequate criterion is

$$|(1/2\lambda z)(1-c^2/2z)| \ll 1 ,$$

or taking $\lambda c^2/2z = 5$, to give a condition identical to that obtained in the preceding subsection,

$$2/\lambda z \ll 1 . \tag{76}$$

This criterion has been derived only for an ion swarm. It may be confirmed that it is also adequate for the more general case.

(c) *The Parameter* γ

The radial component of the flux density equation is

$$j_r = n\mu E_r - D\,\partial n/\partial r ,$$

or, putting $E_r = -\partial\phi/\partial r$,

$$j_r = -n\mu\frac{\partial}{\partial r}\left\{\frac{D}{\mu}\ln\left(\frac{n}{n_0}\right) + \Delta\phi\right\},$$

where n_0 is the number density at $r = 0$ and

$$\Delta\phi = \phi(r,z) - \phi(0,z) .$$

Therefore, taking $n = n_0\exp(-1)$, a relative measure of the importance of space charge effects is the ratio

$$\frac{\Delta\phi}{D/\mu} \sim \frac{e\,\Delta\phi}{kT} \equiv \frac{\Delta\phi}{V}, \tag{77}$$

where

$$\Delta\phi = -\int_0^{r_0} E_{\mathrm{s}r}\,\mathrm{d}r .$$

r_0 is that value of r for which $n = n_0\exp(-1)$. From (13), (18), and (20)

$$E_{\mathrm{s}r} = \frac{I}{2\pi\epsilon w_{\mathrm{i}}}\frac{R}{r}\{1-\beta\exp(-\alpha z)\},$$

and on using equations (73) and (75) it may be shown that

$$\Delta\phi = \frac{I}{4\pi w_{\mathrm{i}}\,\epsilon}\{1-\beta\exp(-\alpha z)\}\,E(0,1) . \tag{78}$$

It follows, from (77) and (21) that

$$\Delta\phi/V \simeq \gamma\{1-\beta\exp(-az)\}. \tag{79}$$

Comparison of this result with equation (51) confirms the assertions made in the Introduction that $\Delta\phi/V$ is a good measure of the relative importance of space charge phenomena. In fact, as a little thought soon shows, the expansion of R, although formally expanded in powers of γ, is really in terms of a parameter of order $\Delta\phi/V$.

As given by (78), the dominance of the ions in determining the space charge field is not immediately apparent. That this is the case, however, may be readily confirmed. Let I_e and I_i be the electron and ion currents respectively, with initial values I_{e0} and I_{i0}. Then

$$I = I_e+I_i = I_{e0}+I_{i0}. \tag{80}$$

However,

$$I_e = I_{e0}\exp(-az) \equiv f_{e0}I\exp(-az) \tag{81}$$

and it follows from the conservation equation (80) that

$$I_i = I_{i0}+I_{e0}\{1-\exp(-az)\} \equiv (1-f_{e0})I+f_{e0}I\{1-\exp(-az)\}. \tag{82}$$

Therefore, in particular,

$$I_i+(w_i/w_e)I_e = I\{1-\beta\exp(-az)\}.$$

Since w_i/w_e is a small quantity, this result immediately emphasizes the dominant role of the ions.

In Section III(c) the solution for R was found only to first order in γ. In general, such a solution is adequate, since it should be always possible to determine R and $\partial R/\partial I$ in the limit as I tends to zero. Nevertheless, it is of interest to note the orders involved. For $I = 10^{-12}$ A, γ is of the order of $10^{-2}/w_i V$ (with w_i in metres per second and V in volts). For ions in hydrogen at room temperature, an E/p of $0\cdot1\ \mathrm{V\,cm^{-1}\,torr^{-1}}$ gives a w_i of order 10 and $V \sim 1/40$. For these values γ is $0\cdot04$. Therefore, for an ion swarm $\Delta\phi/V$ is approximately equal to $0\cdot04$, and small but significant effects should be observable. However, if the swarm consisted entirely of electrons, $\Delta\phi/V$ is roughly 100 times smaller and space charge effects would be negligible. On the other hand, if the temperature were lowered to $V = 1/160$ (liquid nitrogen temperatures), γ would approximately equal $0\cdot16$ and large effects should be observable in an ion swarm. Furthermore, E/p would only have to be decreased by one order and the current increased to 5×10^{-12} A for very significant effects to occur in an electron swarm. In fact, even at room temperatures, for these latter values of E/p $(0\cdot01)$ and I (5×10^{-12}) an electron swarm need only be contaminated by ions to the extent of 2% (that is, $I_i \sim 10^{-13}$ A) for $\Delta\phi/V$ to be of the order of 5%.

These estimates indicate the importance of space charge effects in the Townsend–Huxley swarm technique. They also imply, however, that for most cases of practical interest $\Delta\phi/V$ is a small quantity and that a solution to first order in γ is not only adequate but also quite accurate.

(d) *The Restriction* $\lambda_i = \lambda_e$

This restriction is by no means as severe as it appears at first sight. As already emphasized, the results in the case of a swarm consisting of a single particle type are valid for all values of λ. Again, as implied in the preceding subsection, space charge effects are only likely to be of importance for extremely low values of E/p. For these values the temperature of the particles in the swarm is the same as that of the gas, and under such conditions λ_i is in fact equal to λ_e.

There is, however, one important point to note. For small γ, the associated space charge term in equation (15) is also small. Therefore, it may appear that even minor departures from the condition $\lambda_i = \lambda_e$, leading to retention of the term

$$\left(\frac{\lambda_i}{\lambda_e}-1\right) r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial R_e}{\partial r}\right),$$

could drastically change the nature of the results. Fortunately, this is not the case. On retaining this term and ignoring space charge effects it may be shown that the solution for R, although dependent on α, is independent of I. Therefore, small departures from the restriction $\lambda_i = \lambda_e$ will not drastically alter the space charge terms in the result, since such terms are primarily current dependent.

V. Conclusions

In the Townsend–Huxley swarm technique space charge effects may be of importance for large currents, low gas temperatures, and low E/p. In most practical cases, however, such effects are only of interest in thermal swarms. The experimental technique consists of determining the ratio $R(r,z)$ as a function of various controllable parameters, where $R(r,z)$ is the ratio of the current passing through a disk of radius r, in the plane $z = z$, to the total swarm current I.

If

$$2/\lambda z \ll 1, \qquad \lambda \equiv \lambda_i = \lambda_e,$$

then for a swarm consisting of equally charged electrons and ions, the ions possibly being produced by attachment processes, $R(r,z)$ satisfies the differential equation

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial R}{\partial r}\right)-2\lambda\frac{\partial R}{\partial z} = 2\gamma\left\{1-\beta\exp(-\alpha z)\right\}\frac{R}{r}\frac{\partial R}{\partial r},$$

where

$$\lambda_e = w_e/2D_e, \qquad \lambda_i = w_i/2D_i,$$

$$\gamma = \lambda I/2\pi\epsilon w_i E_a, \qquad \beta = f_{e0}(1-w_i/w_e).$$

w_e and w_i are the drift speeds of the electrons and ions respectively, with D_e and D_i the associated diffusion coefficients, ϵ is the permittivity (MKS units) of the medium concerned, and α is the attachment coefficient. $f_{e0}I$ is the initial value of the electron current at the point source $r = 0$, $z = 0$. If L is the distance between the filament, producing the electrons, and the cathode (see Fig. 1), $f_{e0} \simeq \exp(-\alpha L)$.

For small γ, a solution for $R(r,z)$ to first order in γ is

$$R = \{1-\exp(-\nu)\}-\gamma[(1-\beta)a_{10}+\tfrac{20}{7}\beta a_{11}\{1-\exp(-0\cdot 7\,\alpha z)\}],$$

where

$$\nu \equiv \lambda r^2/2z.$$

a_{10} and a_{11} are functions of ν, being given explicitly in Section III(c) and partially tabulated in Table 1. The solution for R is inexact in the term $\exp(-0\cdot 7\,\alpha z)$. The inaccuracy involved, however, is only of the order of 1% provided $\alpha z \leqslant 1$, $\nu \leqslant 2$. For other values of αz and ν or for a greater degree of accuracy the exact solution given in Section III(c) must be used.

Limiting cases for $f_{e0} = 1$, $\alpha = 0$, corresponding to a swarm of electrons, and $f_{e0} = 0$, corresponding to a swarm of ions, may be found from this solution. The solution in the case of the ions is valid no matter what the ionic charge.

In the limit $I \rightarrow 0$, these results may be used to determine λ_i and λ_e, and, depending on the experimental accuracy, w_e, w_i, and α.

For very large γ, space charge effects completely dominate and these solutions are inadequate. The appropriate solution in this case is given in Section III(d).

VI. Acknowledgment

The author would like to thank Dr. R. W. Crompton for many helpful discussions and for suggesting this problem.

VII. References

Crompton, R. W., and Jory, R. L. (1962).—*Aust. J. Phys.* **15**, 451.

Harvard Computation Laboratory (1949).—"Tables of the Generalised Exponential-integral Functions." (Harvard Univ. Press.)

Hurst, C. A., and Liley, B. S. (1965).—*Aust. J. Phys.* **18**, 521.

Watson, G. N. (1944).—"A Treatise on the Theory of Bessel Functions." 2nd Ed. (Cambridge Univ. Press.)

Appendix

In evaluating the integrals

$$a_{1n} = \nu \mathrm{e}^{-\nu} \int_0^{\nu} \frac{y^{n-1}(1-\mathrm{e}^{-y})}{(y+\nu)^{n+1}}\,\mathrm{d}y,$$

the generalized exponential integral

$$E(0,x) = \int_0^x (1-\mathrm{e}^{-u})/u\,\mathrm{d}u \tag{A1}$$

is basic. In particular, it is to be noted that

$$\begin{aligned}\int_{\nu}^{2\nu} \mathrm{e}^{-u}/u\,\mathrm{d}u &= \int_{\nu}^{2\nu} u^{-1}\,\mathrm{d}u - \int_{\nu}^{2\nu} (1-\mathrm{e}^{-u})/u\,\mathrm{d}u \\ &= \ln 2 + E(0,\nu) - E(0,2\nu)\,. \end{aligned} \tag{A2}$$

For $n = 0$

$$\begin{aligned} a_{10} &= \nu\,\mathrm{e}^{-\nu} \int_0^{\nu} (1-\mathrm{e}^{-y})/y(y+\nu)\,\mathrm{d}y \\ &= \mathrm{e}^{-\nu} E(0,\nu) - \mathrm{e}^{-\nu} \int_0^{\nu} (1-\mathrm{e}^{-y})/(y+\nu)\,\mathrm{d}y \\ &= \mathrm{e}^{-\nu} E(0,\nu) - \mathrm{e}^{-\nu} \ln 2 + \mathrm{e}^{-\nu} \int_0^{\nu} \mathrm{e}^{-y}/(y+\nu)\,\mathrm{d}y \\ &= \mathrm{e}^{-\nu} E(0,\nu) - \mathrm{e}^{-\nu} \ln 2 + \int_{\nu}^{2\nu} \mathrm{e}^{-s}/s\,\mathrm{d}s. \end{aligned}$$

The final form given in Section III(c) follows from (A2).

For $n \geqslant 1$ put $y = s-\nu$, then

$$a_{1n} = \nu\,\mathrm{e}^{-\nu}\int_{\nu}^{2\nu}\{(s-\nu)^{n-1}/s^{n+1}\}(1-\mathrm{e}^{-(s-\nu)})\,\mathrm{d}s$$

$$= \nu\,\mathrm{e}^{-\nu}\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\nu^k I_k(\nu)\,,$$

where

$$I_k(\nu) \equiv \int_{\nu}^{2\nu}(1-\mathrm{e}^{-(s-\nu)})/s^{k+2}\,\mathrm{d}s$$

$$= \int_{\nu}^{2\nu}1/s^{k+2}\,\mathrm{d}s-\mathrm{e}^{\nu}\int_{\nu}^{2\nu}\mathrm{e}^{-s}/s^{k+2}\,\mathrm{d}s.$$

Repeated integration by parts reduces these integrals to

$$I_k(\nu) = \frac{1}{k+1}\left(\frac{1}{\nu^{k+1}}-\frac{1}{(2\nu)^{k+1}}\right)+\mathrm{e}^{\nu}\frac{(-1)^{k+2}}{(k+1)!}\Big(\ln 2+E(0,\nu)-E(0,2\nu)\Big)$$

$$-\frac{1}{\nu^{k+1}}\sum_{m=0}^{k}\frac{(-1)^m\nu^m(1-2^{m-k-1}\mathrm{e}^{-\nu})}{(k+1)k\ldots(k+1-m)}.$$

In particular,

$$a_{11} = \nu\,\mathrm{e}^{-\nu}I_0,$$

$$a_{12} = \nu\,\mathrm{e}^{-\nu}[I_0-\nu I_1],$$

$$a_{13} = \nu\,\mathrm{e}^{-\nu}[I_0-2\nu I_1+\nu^2 I_2],$$

the final forms being given in Section III(c).