SHORT COMMUNICATIONS

SAINT–VENANT'S PRINCIPLE IN ANISOTROPIC MEDIA*

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Saint-Venant's principle may be stated (Sternberg 1954): "An equilibrated system of external forces applied to an elastic body, all of the points of application lying within a given sphere, produces deformations of negligible magnitude at distances from the sphere which are sufficiently large compared to its radius."

Although Saint-Venant's principle has been known and used for over 100 years, a general proof and a method of estimating the distance from the disturbance at which deformations become small in specific cases has only recently been given for isotropic media (Boley 1958). Boley showed that Saint-Venant's principle is a consequence of the elliptic character of the governing differential equations, and estimated the "smoothing distance", or distance from the disturbance at which deformations become small, by making use of an upper bound to the Green's function for the problem. In a later paper (Boley 1960), it was shown that in certain cases in which the differential equation of the problem is parabolic Saint-Venant's principle may still apply.

It does not appear to be well known, however, that anisotropy may cause significant changes in the smoothing distance of a disturbance. For example, Lenkhnitskii (1963) remarks in the early pages of his book that he will invoke Saint-Venant's principle in order to simplify the statement of certain problems, but does not comment upon the effect of anisotropy on Saint-Venant's principle.

Because of the high degree of anisotropy of certain structural materials (e.g. fibre-reinforced composites), it appears desirable to investigate the order of magnitude of smoothing distance necessary to the application of Saint-Venant's principle in such media.

We choose in the first instance to examine this problem in a very simple case of steady-state heat transfer. Consider the problem of two-dimensional heat conduction in a semi-infinite bar of width b, thin in the z direction. Let the principal axes of thermal conductivity coincide with the bar axes, and let the sides and edges of the bar be insulated. The differential equation for the problem is (Carslaw and Jaeger 1959, p. 41)

$$K_x \frac{\partial^2 T}{\partial x^2} + K_y \frac{\partial^2 T}{\partial y^2} = 0,$$

and its solution can be shown to be

$$T_{(x,y)} = \sum_{n=0}^{\infty} a_n \exp\left\{-\frac{n\pi y}{b}\left(\frac{K_x}{K_y}\right)^{\frac{1}{2}}\right\} \cos\frac{n\pi x}{b},$$

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where T is temperature, K_x and K_y are the thermal conductivities in the x and y directions respectively, and b is the breadth of the bar.

Let us impose at the end of the bar a disturbance

$$T_{(x,0)} = T_0 \cos\frac{2\pi x}{b},$$

which has null net heat input and, because of its symmetry about the bar centre line, has also null net "moment" of heat input. (This disturbance could be superimposed upon any specified temperature distribution $T^*_{(x,0)}$ at the end of the bar to obtain a new temperature distribution $T^{**}_{(x,0)}$ which would be "statically equivalent" to $T^*_{(x,0)}$.)

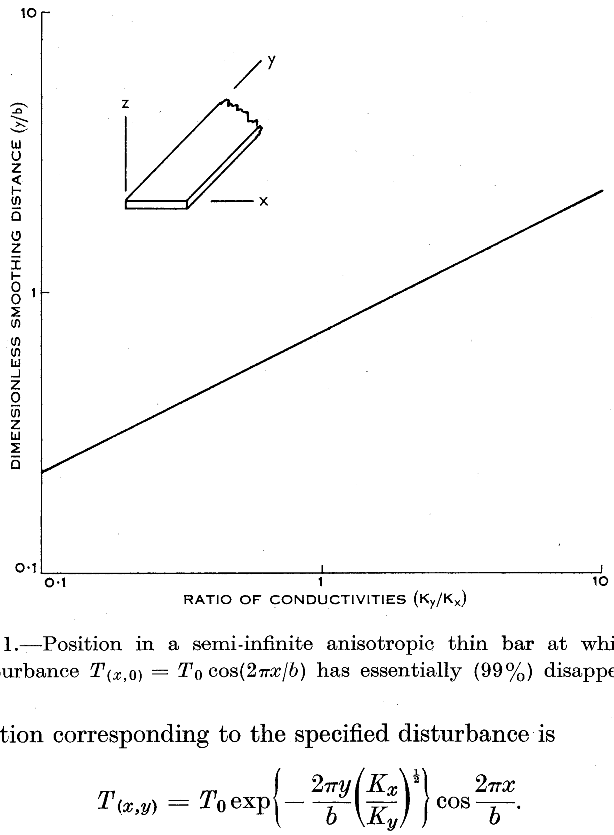

Fig. 1.—Position in a semi-infinite anisotropic thin bar at which the disturbance $T_{(x,0)} = T_0 \cos(2\pi x/b)$ has essentially (99%) disappeared.

The solution corresponding to the specified disturbance is

$$T_{(x,y)} = T_0 \exp\left\{-\frac{2\pi y}{b}\left(\frac{K_x}{K_y}\right)^{\frac{1}{2}}\right\} \cos\frac{2\pi x}{b}.$$

We are particularly interested in the maximum temperature which occurs for any value of x at a specified distance from the end of the bar, and which is given by

$$T_{\max(y)} = T_0 \exp\left\{-\frac{2\pi y}{b}\left(\frac{K_x}{K_y}\right)^{\frac{1}{2}}\right\}.$$

Selecting some particular value for $T_{\max}$ (for example, 1% of T_0), which we will consider to have reduced the disturbance to negligible magnitude, we may calculate the smoothing distance for the problem. These distances are shown in Figure 1.

This simple example is given to illustrate the existence of, and to give some idea of, the magnitude of the effect of anisotropy upon Saint-Venant's principle. In geometrically complicated anisotropic heat conduction problems, where an exact solution is not so easily given, an upper bound to the smoothing distance may be constructed by using the transformation suggested by Carslaw and Jaeger (1959, p. 44) and the method of Boley (1958).

Discussion and Conclusion

Provided that the principal thermal conductivities are non-zero, anisotropy does not change the elliptic character of the equation of steady-state heat conduction, and Saint-Venant's principle applies. It will be noted from Figure 1 that in an isotropic material the disturbance in question has essentially (99%) disappeared in about three-quarters of a bar width.

However, as Figure 1 shows, anisotropy may significantly increase or decrease the smoothing distance, depending upon the orientation of the higher conductivity axis with respect to the bar axis. It may be noted that in the limiting case where K_y becomes zero, Saint-Venant's principle holds with a smoothing distance of zero. If K_x becomes zero, Saint-Venant's principle does not apply.

References

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