

ENERGY AND DISPERSION IN ELECTROMAGNETIC SYSTEMS

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Summary

The total average electromagnetic energy of a loss-free system is expressed in terms of the frequency dependence of the fields on an arbitrary mathematical surface enclosing the system. By introducing scattering, admittance, and impedance matrices the surface integral is reduced and the energy written as quadratic forms involving the frequency derivatives of these matrices. The arguments are of sufficient generality to apply in the presence of inhomogeneous, dispersive, and anisotropic or gyrotropic media. The requirement that the total energy be positive leads to important restrictions on the frequency dependence of the parameter matrices. The application to extensions of Foster's reactance theorem is discussed.

I. INTRODUCTION

Energy storage and dispersion in electromagnetic systems are closely related, high energy storage being associated with high dispersion. A detailed examination of the total average electromagnetic energy of loss-free systems shows it to be expressible completely in terms of the frequency dependence of field quantities on an enclosing mathematical surface. The surface integral is reducible to forms involving the frequency dependence of the impedance, admittance, or scattering matrices defined with reference to a set of basis functions on the surface. The result for the simplest system consisting of a single reactance is well known in a.c. circuit theory, namely

$$W = \frac{1}{2} I^* I dX/d\omega, \quad (1)$$

where W is the total average energy when the reactance X is excited by a complex r.m.s. current I of angular frequency ω . The results to be obtained are generalizations of equation (1) for arbitrary linear loss-free electromagnetic systems.

The systems to be considered are general except for the requirement that the media are to have negligible loss. The actual media present may be both anisotropic and gyrotropic (non-reciprocal), and intrinsically dispersive. The Krönig-Kramers dispersion relations imply that dissipation at *some* frequencies is to be associated with dispersion. It will be assumed that the losses at all frequencies of interest are negligible, even though the dispersion is still significant. The limiting case occurs when the entire loss associated with the dispersion is concentrated at a set of resonant frequencies. The inclusion of the possibility of the presence of gyrotropic material leads to very little additional complication to that for anisotropic materials, and the effects of possible conduction currents are described by an effective permittivity tensor. The form of the expression for the energy density which accounts for these conduction currents and for any dispersion present in the basic permittivity and

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permeability tensors is of special importance. The discussion of Landau and Lifshitz (1960, Sec. 61) is made the basis for the evaluation of the total average energy.

Montgomery, Dicke, and Purcell (1948, Sec. 5.21) have given expressions in the form of the final results but under very restricted conditions of excitation and without regard for the implications of the presence of dispersive or other more general media. The results are obtained by considering the frequency dependence of the complex Poynting vector integrated over an enclosing mathematical surface. The surface integral expression obtained for the total energy is of the same form as that obtained in a discussion of average electromagnetic forces (Smith 1961) and may be reduced to impedance, admittance, and scattering matrix (Smith 1964a) forms in the same way.

Equation (1) is used in the proof of Foster's reactance theorem (Foster 1924; Montgomery, Dicke, and Purcell 1948, Sec. 5.24) on the form of the analytical dependence of a single reactance X on ω . The relevance of the general results to extensions of Foster's theorem is noted.

II. FUNDAMENTAL EQUATIONS

The system considered is assumed to be excited by exterior sources of angular frequency ω . All field vectors are complex corresponding to the assumed time dependence $\exp(i\omega t)$. It will be convenient for the amplitudes to be taken as root-mean-square values. The Maxwell equations become (in MKSA rationalized units)

$$\nabla \times \mathbf{E} = -i\omega \mathbf{B}, \quad (2)$$

$$\nabla \times \mathbf{H} = i\omega \mathbf{D} + \mathbf{J}, \quad (3)$$

$$\nabla \cdot \mathbf{D} = \rho, \quad (4)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (5)$$

where $\mathbf{E}$, $\mathbf{H}$, $\mathbf{B}$, and $\mathbf{D}$ are the complex r.m.s. vectors of the electric field, the magnetic intensity, the magnetic induction, and the electric displacement respectively. To preserve generality a current density $\mathbf{J}$ and charge density ρ are included, although for loss-free systems there are severe restrictions on the form $\mathbf{J}$ may take.

The usual linear constitutive equations relating $\mathbf{D}$ and $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{H}$ are assumed for the media present, together with Ohm's law for the current density, namely

$$\mathbf{D} = [\kappa] \cdot \mathbf{E}, \quad (6)$$

$$\mathbf{B} = [\mu] \cdot \mathbf{H}, \quad (7)$$

$$\mathbf{J} = [\sigma] \cdot \mathbf{E}, \quad (8)$$

where $[\kappa]$, $[\mu]$, and $[\sigma]$ are the second-rank tensors of permittivity, permeability, and conductivity. These tensors may all be dependent on both position and frequency of excitation. For reciprocal media each of these tensors is symmetric but for general gyrotropic media any of them may be asymmetric.

By introducing an effective permittivity tensor $[\epsilon]$ defined by

$$[\epsilon] = [\kappa] + [\sigma]/i\omega, \quad (9)$$

the second Maxwell equation (3) may be written

$$\nabla \times \mathbf{H} = i\omega \mathbf{D}', \quad (10)$$

where

$$\mathbf{D}' = [\epsilon] \cdot \mathbf{E} \quad (11)$$

is the effective displacement vector. This device collects together in $i\omega \mathbf{D}'$ all current densities apart from the magnetization current density which is already included in $\mathbf{H}$.

Now consider the surface integral of the complex Poynting vector $\mathbf{E} \times \mathbf{H}^*$ over an arbitrary closed surface S enclosing a volume V of medium, the *inward* normal being regarded as positive:

$$\begin{aligned} \int_S \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{A} &= - \int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}^*) d\tau \\ &= \int_V (\mathbf{E} \cdot \nabla \times \mathbf{H}^* - \mathbf{H}^* \cdot \nabla \times \mathbf{E}) d\tau \\ &= i\omega \int_V (\mathbf{H}^* \cdot \mathbf{B} - \mathbf{E} \cdot \mathbf{D}'^*) d\tau, \end{aligned} \quad (12)$$

(using the Maxwell equations (2) and (10)),

$$= i\omega \int_V (\mathbf{H}^* \cdot [\mu] \cdot \mathbf{H} - \mathbf{E} \cdot [\epsilon]^* \cdot \mathbf{E}^*) d\tau, \quad (13)$$

(using equations (7) and (11)).

For a system including only loss-free media the surface integral of the complex Poynting vector must always be purely imaginary. Equation (13) then shows that the quadratic forms $\mathbf{H}^* \cdot [\mu] \cdot \mathbf{H}$ and $\mathbf{E} \cdot [\epsilon]^* \cdot \mathbf{E}^*$ are always real. This implies that $[\epsilon]$ and $[\mu]$ are Hermitian. In component form,

$$\mu_{kl} = \mu_{lk}^*, \quad \epsilon_{kl} = \epsilon_{lk}^*.$$

For reciprocal systems in which $[\epsilon]$ and $[\mu]$ are also symmetric this corresponds to real $[\epsilon]$ and $[\mu]$. From equation (9) the Hermitian $[\epsilon]$ corresponds to an anti-hermitian $[\sigma]$. That is,

$$\sigma_{kl} = -\sigma_{lk}^*. \quad (14)$$

A simple illustrative case of equation (14) occurs when conduction is by charge carriers whose only restraint to motion is from inertia or some conservative field of force. The presence of a static magnetic field gives rise to Hall currents and the full generality of equation (14) is then required.

It is clear that, if only because of the inclusion of the effects of conduction currents in the tensor $[\epsilon]$, the field energy density may not be taken simply as

$$U = \frac{1}{2}(\mathbf{E}^* \cdot \mathbf{D}' + \mathbf{H}^* \cdot \mathbf{B}). \quad (15)$$

In general there will also be dispersion from both $[\kappa]$ and $[\mu]$. When dispersion

occurs equation (15) does not give the field energy density. However, Landau and Lifshitz (1960, Sec. 61) have derived an expression for the total average energy density when conduction and dispersion are present. Although their treatment refers to isotropic materials, it is readily adapted to general anisotropic materials giving (Landau and Lifshitz 1960, Sec. 76)

$$U = \frac{1}{2}[\mathbf{H}^* \cdot \{d(\omega[\boldsymbol{\mu}])/d\omega\} \cdot \mathbf{H} + \mathbf{E}^* \cdot \{d(\omega[\boldsymbol{\epsilon}])/d\omega\} \cdot \mathbf{E}], \quad (16)$$

which corresponds to (15) in the absence of dispersion. Equation (16) automatically accounts for all energy associated with the application of the electromagnetic field including any mechanical energy associated with the conduction or polarization processes.

III. BEHAVIOUR OF A SYSTEM WITH VARIATIONS OF FREQUENCY

Suppose the closed mathematical surface S encloses the whole system of interest but not the sources. Consider an integration of the complex Poynting vector $\mathbf{E} \times \mathbf{H}^*$ over S reckoning the *inward* normal positive, together with integrations over the surfaces $S_1, S_2, \dots$ of any perfect conductors included inside S regarding the *outward* normals as positive, then

$$\int_{S, S_1, S_2, \dots} \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{A} = - \int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}^*) d\tau,$$

where V is the active volume of the system. However, $\mathbf{E}$ is normal to the surfaces $S_1, S_2, \dots$, therefore

$$\begin{aligned} \int_S \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{A} &= - \int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}^*) d\tau \\ &= i\omega \int_V (\mathbf{H}^* \cdot \mathbf{B} - \mathbf{E} \cdot \mathbf{D}^*) d\tau, \end{aligned} \quad (17)$$

as in equation (12).

A variation of equation (17) corresponding to a small change $\delta\omega$ of the excitation frequency gives

$$\delta \int_S \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{A} = i\omega \int_V \{\delta(\mathbf{H}^* \cdot \mathbf{B}) - \delta(\mathbf{E} \cdot \mathbf{D}^*)\} d\tau + i\delta\omega \int_V (\mathbf{H}^* \cdot \mathbf{B} - \mathbf{E} \cdot \mathbf{D}^*) d\tau. \quad (18)$$

Some of the details of the following argument are similar to those used previously in deriving an expression for radiation-pressure forces (Smith 1961), but account must now be taken of the lesser symmetry of $[\boldsymbol{\mu}]$ and $[\boldsymbol{\epsilon}]$.

From equation (11),

$$\begin{aligned} \delta(\mathbf{E} \cdot \mathbf{D}^*) &= \mathbf{E} \cdot \delta\mathbf{D}^* + \delta\mathbf{E} \cdot [\boldsymbol{\epsilon}]^* \cdot \mathbf{E}^* \\ &= \mathbf{E} \cdot \delta\mathbf{D}^* + \mathbf{E}^* \cdot [\boldsymbol{\epsilon}] \cdot \delta\mathbf{E}, \end{aligned}$$

since $[\boldsymbol{\epsilon}]$ is Hermitian,

$$= \mathbf{E} \cdot \delta\mathbf{D}^* + \mathbf{E}^* \cdot \delta\mathbf{D}' - \mathbf{E}^* \cdot [\delta\boldsymbol{\epsilon}] \cdot \mathbf{E}. \quad (19)$$

Also, from (7),

$$\begin{aligned}\delta(\mathbf{H}^* \cdot \mathbf{B}) &= \delta \mathbf{H}^* \cdot \mathbf{B} + \mathbf{H}^* \cdot [\delta \boldsymbol{\mu}] \cdot \mathbf{H} + \mathbf{H}^* \cdot [\boldsymbol{\mu}] \cdot \delta \mathbf{H} \\ &= \delta \mathbf{H}^* \cdot \mathbf{B} + \mathbf{H}^* \cdot [\delta \boldsymbol{\mu}] \cdot \mathbf{H} + \delta \mathbf{H} \cdot [\boldsymbol{\mu}]^* \cdot \mathbf{H}^*, \\ &\quad \text{since } [\boldsymbol{\mu}] \text{ is Hermitian,} \\ &= \delta \mathbf{H}^* \cdot \mathbf{B} + \delta \mathbf{H} \cdot \mathbf{B}^* + \mathbf{H}^* \cdot [\delta \boldsymbol{\mu}] \cdot \mathbf{H}.\end{aligned}\quad (20)$$

The terms in $[\delta \boldsymbol{\mu}]$ and $[\delta \boldsymbol{\varepsilon}]$ are direct effects of dispersion in the media.

Substitution of equations (19) and (20) in equation (18) gives

$$\begin{aligned}\delta \int_S \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{A} &= i \delta \omega \int_V (\mathbf{H}^* \cdot \mathbf{B} - \mathbf{E} \cdot \mathbf{D}'^*) d\tau + i \omega \int_V (\mathbf{H}^* \cdot [\delta \boldsymbol{\mu}] \cdot \mathbf{H} + \mathbf{E}^* \cdot [\delta \boldsymbol{\varepsilon}] \cdot \mathbf{E}) d\tau \\ &\quad + i \omega \int_V \{(\delta \mathbf{H}^* \cdot \mathbf{B} - \mathbf{E} \cdot \delta \mathbf{D}'^*) + (\mathbf{B}^* \cdot \delta \mathbf{H} - \mathbf{E}^* \cdot \delta \mathbf{D}')\} d\tau.\end{aligned}\quad (21)$$

The Maxwell equations may now be used to express the last term in equation (21) in a more useful form. The variation of equation (10) gives

$$i \omega \delta \mathbf{D}' = \nabla \times \delta \mathbf{H} - i \delta \omega \mathbf{D}',$$

which together with equation (2) allows the term to be written

$$\begin{aligned}&\int_V \{(\mathbf{E} \cdot \nabla \times \delta \mathbf{H}^* - \delta \mathbf{H}^* \cdot \nabla \times \mathbf{E}) + (\delta \mathbf{H} \cdot \nabla \times \mathbf{E}^* - \mathbf{E}^* \cdot \nabla \times \delta \mathbf{H})\} d\tau \\ &\quad + i \delta \omega \int_V (\mathbf{E} \cdot \mathbf{D}'^* + \mathbf{E}^* \cdot \mathbf{D}') d\tau \\ &= - \int_V \nabla \cdot (\mathbf{E} \times \delta \mathbf{H}^* - \mathbf{E}^* \times \delta \mathbf{H}) d\tau + i \delta \omega \int_V (\mathbf{E} \cdot \mathbf{D}'^* + \mathbf{E}^* \cdot \mathbf{D}') d\tau \\ &= \int_{S_1, S_2, S_3, \dots} (\mathbf{E} \times \delta \mathbf{H}^* - \mathbf{E}^* \times \delta \mathbf{H}) \cdot d\mathbf{A} + i \delta \omega \int_V (\mathbf{E} \cdot \mathbf{D}'^* + \mathbf{E}^* \cdot \mathbf{D}') d\tau \\ &= \int_S (\mathbf{E} \times \delta \mathbf{H}^* - \mathbf{E}^* \times \delta \mathbf{H}) \cdot d\mathbf{A} + i \delta \omega \int_V (\mathbf{E} \cdot \mathbf{D}'^* + \mathbf{E}^* \cdot \mathbf{D}') d\tau,\end{aligned}\quad (22)$$

since $\mathbf{E}$ is normal to the conducting surfaces $S_1, S_2, \dots$.

Equations (21) and (22) when combined give

$$\begin{aligned}\int_S (\delta \mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \delta \mathbf{H}) \cdot d\mathbf{A} &= i \delta \omega \int_V (\mathbf{H}^* \cdot \mathbf{B} + \mathbf{E}^* \cdot \mathbf{D}') d\tau \\ &\quad + i \omega \int_V (\mathbf{H}^* \cdot [\delta \boldsymbol{\mu}] \cdot \mathbf{H} + \mathbf{E}^* \cdot [\delta \boldsymbol{\varepsilon}] \cdot \mathbf{E}) d\tau \\ &= i \delta \omega \int_V [\mathbf{H}^* \cdot \{d(\omega[\boldsymbol{\mu}])/d\omega\} \cdot \mathbf{H} + \mathbf{E}^* \cdot \{d(\omega[\boldsymbol{\varepsilon}])/d\omega\} \cdot \mathbf{E}] d\tau.\end{aligned}\quad (23)$$

The volume integral on the right-hand side of equation (23) is recognized as being twice the total average energy W derived from the energy density of equation (16). Therefore,

$$2i \delta \omega W = \int_S (\delta \mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \delta \mathbf{H}) \cdot d\mathbf{A},\quad (24)$$

which expresses the total average energy W in terms of the frequency dependence

(dispersion) of the fields on an arbitrary enclosing surface S . The variations $\delta\mathbf{E}$ and $\delta\mathbf{H}$ on S , of course, are not independent and in the following section reduced forms of the basic equation (24) are obtained.

IV. REDUCTION OF EQUATION (24) TO PARAMETER FORMS

The surface integral in equation (24) is of the same form as that relating to average electromagnetic forces (Smith 1961) and may be reduced in the same ways (Smith 1961, 1964a). In the present paper no restriction to reciprocal media has been applied so that care is required in writing the reduced forms, as the scattering, impedance, and admittance matrices $\mathbf{S}$, $\mathbf{Z}$, and $\mathbf{Y}$ are not necessarily symmetric.

For loss-free systems the requirement that the complex power to the system be purely imaginary leads to the usual unitarity of $\mathbf{S}$, that is

$$\mathbf{S}^{-1} = \mathbf{S}^\dagger, \quad (\dagger \text{ denotes Hermitian conjugate}), \quad (25)$$

and the corresponding antihermitian nature of $\mathbf{Z}$ and $\mathbf{Y}$,

$$\mathbf{Z}^\dagger = -\mathbf{Z}, \quad \mathbf{Y}^\dagger = -\mathbf{Y},$$

which for reciprocal systems implies that $\mathbf{Z}$ and $\mathbf{Y}$ are purely imaginary.

Equation (24) reduces to the following parameter matrix forms:

(i) *Scattering Matrix Form (Smith 1964a)*

$$iW = -\mathbf{b}^\dagger d\mathbf{S}/d\omega \mathbf{a} \quad (\mathbf{b} = \mathbf{S}\mathbf{a}), \quad (26)$$

$\mathbf{a}$ and $\mathbf{b}$ being column vectors of the incident and scattered partial wave r.m.s. amplitudes. Alternatively equation (26) may be written

$$\begin{aligned} iW &= -\mathbf{a}^\dagger \mathbf{S}^\dagger d\mathbf{S}/d\omega \mathbf{a} \\ &= -\mathbf{a}^\dagger \mathbf{S}^{-1} d\mathbf{S}/d\omega \mathbf{a} \quad (\text{from (25)}). \end{aligned} \quad (27)$$

(ii) *Impedance and Admittance Matrix Forms (Smith 1961)*

$$2iW = \mathbf{I}^\dagger (d\mathbf{Z}/d\omega) \mathbf{I}, \quad (28)$$

$$2iW = \mathbf{V}^\dagger (d\mathbf{Y}/d\omega) \mathbf{V}, \quad (29)$$

where

$$\mathbf{V} = \mathbf{Z}\mathbf{I}, \quad (30)$$

$$\mathbf{I} = \mathbf{Y}\mathbf{V}, \quad (31)$$

and $\mathbf{V}$ and $\mathbf{I}$ are the "voltage" and "current" column vectors.

The equivalence of equations (26), (28), and (29) is a result of the relationships between admittance or impedance matrices and the scattering matrix (Montgomery, Dicke, and Purcell 1948; Smith 1964b). In terms of the reactance matrix $\mathbf{X}$, $i\mathbf{X}$ being defined generally as the loss-free (antihermitian) part of $\mathbf{Z}$, we have

$$2W = \mathbf{I}^\dagger (d\mathbf{X}/d\omega) \mathbf{I} \quad (32)$$

or

$$2W = \mathbf{V}^\dagger (d\hat{\mathbf{B}}/d\omega) \mathbf{V}, \quad (33)$$

where $\hat{\mathbf{B}}$ is the susceptance matrix related similarly to $\mathbf{Y}$. Clearly $\mathbf{X}$ and $\hat{\mathbf{B}}$ are Hermitian in general, and real for reciprocal systems. For the loss-free systems under consideration,

$$\mathbf{X} = -\hat{\mathbf{B}}^{-1}.$$

V. DISCUSSION AND APPLICATIONS

The total average energy for sinusoidal excitation of a general loss-free system has been expressed as a surface integral of field quantities, and alternatively in algebraic forms involving the frequency dependence of admittance, impedance, reactance, susceptance, or scattering matrices. There has been no restriction on the frequency of excitation, neither has there been any restriction to reciprocal or non-dispersive media. The parameter forms of Section IV illustrate the close connection between energy storage and dispersion, high dispersion implying high energy storage for similar conditions of excitation.

Montgomery, Dicke, and Purcell (1948, Sec. 5.21) have also obtained these parameter forms but for isotropic, non-dispersive media with special excitation such that the fields everywhere are of constant phase. In low-frequency electric circuit theory, results corresponding to equations (32) and (33) for a two-terminal network (one-port), where there is just a single component to each $\mathbf{V}$ and $\mathbf{I}$, are well known (Guillemin 1957).

Since the average energy density U is always positive for non-vanishing fields (Landau and Lifshitz 1960, Sec. 61), the total average energy W is necessarily positive. This leads to important conditions for the parameter matrices. Thus from equations (32) and (33) the matrices $d\mathbf{X}/d\omega$ and $d\hat{\mathbf{B}}/d\omega$ are positive definite. For a two-terminal network these conditions on $d\mathbf{X}/d\omega$ and $d\hat{\mathbf{B}}/d\omega$ together with general considerations of the analytic behaviour and symmetry of reactance functions at complex frequencies (Cauer 1958, Chs. 4 and 5; Landau and Lifshitz 1960, Secs. 58 and 62) lead to Foster's reactance theorem (Bode 1945, Sec. 9.4; Montgomery, Dicke, and Purcell 1948, Sec. 5.24) which expresses $X(\omega)$, $\hat{B}(\omega)$ in the form

$$X(\omega) = - \sum_n r_n \left(\frac{1}{\omega + \omega_n} + \frac{1}{\omega - \omega_n} \right), \quad (34)$$

where r_n and ω_n^2 are real and positive. (If $X(\omega)$ has a pole at infinity a further term $a\omega$, with $a > 0$, may be required in equation (34).)

The results of the present paper immediately extend Foster's theorem to systems containing dispersive or gyrotropic materials. More generally, it may be reasoned similarly that the eigenvalues of $\mathbf{X}$ and $\hat{\mathbf{B}}$ are all of the form (34). Since $\mathbf{X}$ and $\hat{\mathbf{B}} = -\mathbf{X}^{-1}$ are Hermitian, they may be brought simultaneously to diagonal form by a unitary transformation of the terminal voltages $\mathbf{V}$ and currents $\mathbf{I}$ without disturbing power or energy relationships such as equations (32) and (33). The resulting diagonal elements (eigenvalues) of $\mathbf{X}$ and $\hat{\mathbf{B}}$ are real. With this choice of $\mathbf{V}$ and $\mathbf{I}$, $d\mathbf{X}/d\omega$ and $d\hat{\mathbf{B}}/d\omega$ need not be diagonal, but by a familiar result from

matrix theory (Landau and Lifshitz 1959, Sec. 72) their diagonal values are the derivatives of the eigenvalues of $\mathbf{X}$ and $\hat{\mathbf{B}}$. These values must be positive for the quadratic forms of equation (32) and (33) to be positive definite. Foster's theorem may then be extended to apply to each eigenvalue of $\mathbf{X}$ and $\hat{\mathbf{B}}$. For reciprocal systems ($\mathbf{X}$, $\hat{\mathbf{B}}$ real and symmetric) the required diagonalizing transformation is real and orthogonal rather than unitary. Cauer (1931, 1958) and Carlin (1955) have discussed extensions of Foster's theorem for n -terminal pair lumped-element systems.

Zeros of the eigenvalues of $\mathbf{X}$ correspond to poles of the eigenvalues of $\hat{\mathbf{B}}$, and vice versa. The critical frequencies ω_n of equation (34) are resonant frequencies corresponding to zeros of the eigenvalues of $\hat{\mathbf{B}}$, i.e. resonant frequencies with respect to the surface S . It has been assumed in the reductions of Section IV that critical resonant or anti-resonant (zeros of the eigenvalues of $\mathbf{X}$) frequencies are to be avoided in writing equations (30) and (31). The form given in equation (34) may also be obtained by expansion of a general field in terms of the eigenfunction solutions of the system with specified boundary conditions on S . The ω_n^2 are the corresponding eigenvalues of ω^2 . This technique is employed for deriving the Wigner-Eisenbud many-level formula for the R matrix in formal nuclear scattering theory (Blatt and Weisskopf 1952; Lane and Thomas 1958). The Wigner-Eisenbud formula of quantum theory giving the energy dependence of the parameter matrices corresponds to Foster's reactance theorem in electromagnetism, although there are important differences of detail.

The result from the scattering matrix formulation (equation (27)) corresponding to the positive reactance or susceptance derivatives is best expressed in terms of a phase shift matrix $\boldsymbol{\eta}$. Since $\mathbf{S}$ is unitary it may be diagonalized by a unitary transformation of the partial wave amplitudes, and each of the eigenvalues is of unit modulus. Thus $\mathbf{S}$ may be expressed as

$$\mathbf{S} = \exp 2i\boldsymbol{\eta},$$

where $\boldsymbol{\eta}$ is Hermitian with real eigenvalues η_k . Since the total average energy is positive, equation (27) shows that the matrix $i\mathbf{S}^{-1}d\mathbf{S}/d\omega$ is positive definite, so that its diagonal elements are always positive. In particular, the diagonal elements in a representation in which $\mathbf{S}$ is diagonal, are positive, that is

$$(i\mathbf{S}^{-1}d\mathbf{S}/d\omega)_{kk} > 0.$$

But

$$\begin{aligned} (i\mathbf{S}^{-1}d\mathbf{S}/d\omega)_{kk} &= i \exp(-2i\eta_k) \{d(\exp 2i\boldsymbol{\eta})/d\omega\}_{kk} \\ &= i \exp(-2i\eta_k) \cdot d(\exp 2i\eta_k)/d\omega \\ &= -2 d\eta_k/d\omega, \end{aligned}$$

therefore,

$$d\eta_k/d\omega < 0. \quad (35)$$

Equation (35) corresponds to the positive derivatives of the eigenvalues of the

reactance or susceptance matrices. In fact the interrelation of $\mathbf{X}$ and $\hat{\mathbf{B}}$ with $\mathbf{S}$ gives

$$\mathbf{X} = \cot \boldsymbol{\eta}, \quad \hat{\mathbf{B}} = -\tan \boldsymbol{\eta},$$

from which the equivalence follows by differentiation in a diagonal representation.

The time dependence has been taken as harmonic in all of the considerations. This is not an important restriction since the time-averaged energy may be computed by summing over all Fourier components present, it being assumed that the system is without loss at all of these frequencies. Under periodic excitation the average energy is simply the energy averaged over a single period.

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