INELASTIC COLLISIONS FOR SMALL MOMENTUM TRANSFERS*

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*Choice of Variables*

Let us denote by $p_i$, $i = 1, 2, \ldots, 5$, the four-momenta of the *internal* lines of the five-point single-loop graph ($p_i^2 = \mu_i^2$). Each vertex may then be labelled by two suffices and we shall denote the four-momenta of the *external* particles (measured formally as incoming) by $p_{12} = P_1, p_{23} = P_2, \ldots, p_{51} = P_5$ ($P_i^2 = M_i^2$). Scalar product variables may be constructed as follows

$$s_{ij} = (P_i+P_j)^2, \tag{1.1}$$

where we must select out of 10 possible invariants an independent set of 5 variables. The linear dependences are

$$\sum_{\substack{j=1\\ j\neq i, i<j}}^{5} s_{ij} = M_i^2 + \sum_{j=1}^{5} M_i^2, \qquad i = 1, 2, \ldots 5. \tag{1.2}$$

An independent set of variables is

$$s_{12},\ s_{13},\ s_{24},\ s_{25},\ s_{34},$$

and we shall examine the analytic properties of the amplitude as a function of the energy variables $s_{12}$ and $s_{34}$ with the momentum transfer variables $s_{13}$, $s_{24}$, and $s_{25}$ fixed at the physical values $c_{13}$, $c_{24}$, and $c_{25}$. Let us write

$$\begin{array}{llll} s_{12} = v, & s_{13} = c_{13}, & s_{14} = k_{14}-u-v, & s_{15} = k_{15}+u, \\ & s_{23} = k_{23}-v, & s_{24} = c_{24}, & s_{25} = c_{25}, \\ & & s_{34} = u+v, & s_{35} = k_{35}-u, \\ & & & s_{45} = k_{45}, \end{array}$$

where, in the pion–nucleon case ($M_1 = M_3 = M_{\mathrm{n}}$, $M_2 = M_4 = M_5 = M_\pi$), we have

$$\begin{aligned} k_{14} &= M_\pi^2+2M_{\mathrm{n}}^2+c_{25}-c_{13}, \\ k_{15} &= 2M_\pi^2+M_{\mathrm{n}}^2-c_{25}, \\ k_{23} &= 4M_\pi^2+2M_{\mathrm{n}}^2-c_{24}-c_{25}, \\ k_{35} &= M_{\mathrm{n}}^2-M_\pi^2+c_{24}+c_{25}-c_{13}, \\ k_{24} &= 3M_\pi^2+c_{13}-c_{24}-c_{25}. \end{aligned}$$

Normally one wishes to fix $s_{34}$ at a physical value while one discusses analyticity in $v$ but we shall leave ourselves free to fix *any* linear combination of $u$ and $v$.

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*Normal Thresholds*

It has been shown (Landshoff and Trieman 1961) that, if $s_{34} = u+v$ is fixed at a physical value, normal threshold cuts in the variable $s_{12} = v$ overlap and the scattering amplitude *cannot* be the boundary value of an analytic function in $v$. If, however, we plot the normal thresholds as lines in the $(u, v)$ plane it is clear that, for some fixed linear combinations of $u$ and $v$, the cuts need not overlap. A necessary (though insufficient) condition for non-overlapping cuts is, in the pion-nucleon case,

$$(\mu_1+\mu_3)^2+(\mu_2+\mu_4)^2 \geqslant 4M_\pi^2+2M_{\mathrm{n}}^2-(c_{24}+c_{25}). \tag{2.1}$$

If we insist that nucleon number is conserved through the diagram and set the other internal masses equal to the pion mass, condition (2.1) becomes

$$c_{24}+c_{25} \geqslant 2M_\pi(M_\pi-2M_{\mathrm{n}}), \tag{2.2}$$

which can be realized for small physical momentum transfers.

*Complex Singularities*

Some contracted diagrams do not depend on the variable $u$ and these lead to complex singularities previously discovered by various authors (Kim 1961; Landshoff and Trieman 1961).

Two vertex diagrams which arise as contractions of the five-point single-loop graph correspond, in the pion–nucleon case, to singularities located at

$$\lambda_{24}v = c_{12}\beta-\gamma\pm\mathrm{i}\{(\beta^2-1)(1-\alpha_{12}^2)\}^{\frac{1}{2}}, \tag{3.1}$$

and

$$\lambda_{13}v = \mu_{13}-\alpha_{34}\beta\pm\mathrm{i}\{(\beta^2-1)(1-\alpha_{12}^2)\}^{\frac{1}{2}}, \tag{3.2}$$

where

$$\begin{aligned}
\lambda_{ij} &= (2\mu_i\mu_j)^{-1},\\
\mu_{ij} &= (\mu_i^2+\mu_j^2)\lambda_{ij},\\
\alpha_{12} &= \mu_{12}-\lambda_{12}M_{\mathrm{n}}^2,\\
\alpha_{34} &= \mu_{34}-\lambda_{34}M_{\mathrm{n}}^2,\\
\beta &= \mu_{14}-\lambda_{14}c_{25},\\
\gamma &= \mu_{25}-\lambda_{25}(2M_\pi^2+M_{\mathrm{n}}^2)+\lambda_{25}c_{25}.
\end{aligned}$$

Since physical values of $c_{25}$ are negative, $\beta^2 \geqslant 1$, and $\alpha_{12}^2 \leqslant 1$ is a mass stability condition for the five-point diagram.

The author has shown elsewhere (Cunningham 1964) that, regardless of whether or not $R$-products are used in the formulation of perturbation theory, complex singularities of this type always appear in the physical sheet.

The same conclusion can be reached concerning complex singularities of fourth order which are roots of the quartic equation

$$\begin{vmatrix} 1 & \alpha_{12} & \mu_{13}-\lambda_{13}v & \beta \\ \alpha_{12} & 1 & \alpha_{23} & \gamma+\lambda_{24}v \\ \mu_{13}-\lambda_{13}v & \alpha_{23} & 1 & \alpha_{34} \\ \beta & \gamma+\lambda_{24}v & \alpha_{34} & 1 \end{vmatrix} = 0, \tag{3.3}$$

where

$$\beta_{23} = \mu_{23} - \lambda_{23} M_{\pi}^2,$$
$$\alpha_{34} = \mu_{34} - \lambda_{34} M_{\mathrm{n}}^2.$$

*Conclusions*

For suitable small physical momentum transfers the amplitude $A(s_{12}(v),\ s_{34}(v))$ is the boundary value of a function analytic in the complex $v$-plane with suitable cuts. Whether or not a given physical value of $v$ is associated with a physical energy $s_{34}$ depends on the particular linear combination of $u$ and $v$ which we have chosen to fix; for each such choice, pairs $(s_{12},\ s_{34})$ exist with two, one, and no physical members.

From the dispersion theory point of view integral representations will certainly be possible but they must involve integration with complex contours.

Similar conclusions can probably be reached for any single-loop graph because the $n$-point loop possesses only the singularities of its contracted loops of order five and less (Brown 1961).

*References*

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