# THE THEORY OF DISPERSAL DURING LAMINAR FLOW IN TUBES. I

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[*Manuscript received February* 15, 1963]

*Summary*

This is the first of a series on dispersal under the combined influence of molecular diffusion and convection in a fluid subject to steady Poiseuille flow in a straight circular tube.

This paper clears the ground for later papers, in which the detailed results of the analysis are presented. The earlier studies of Taylor and Aris are reviewed, and the failure of their work to apply to problems involving periodic and stationary random boundary conditions is indicated.

The equations governing the dispersal are

$$\left.\begin{aligned} &\frac{\partial\theta}{\partial\tau}+2Y(1-\rho^2)\frac{\partial\theta}{\partial\xi}=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\theta}{\partial\rho}\right)+\frac{\partial^2\theta}{\partial\xi^2};\\ &\rho=1,\qquad \partial\theta/\partial\rho=0. \end{aligned}\right\}\tag{A}$$

$\theta$ is concentration, $\tau$, $\xi$, and $\rho$, time and space coordinates in appropriate reduced form, and $Y$ is the "diffusion Péclet number". It is shown that a diffusion description (in terms of $\theta$ averaged over the cross section) of the phenomenon is valid for a periodic system, when $\theta$ is of the form

$$\theta=f(\rho)\exp(\Omega i\tau+\alpha\xi/Y).\tag{B}$$

$\Omega$ represents frequency in reduced form.

The search for solutions of (A) of form (B) reduces to an eigenvalue problem, each eigenvalue $\alpha_s$ being a function of $\Omega$ and $Y$. The problem may be solved formally in terms of confluent hypergeometric functions, but this is not practically feasible. It is shown that a knowledge of $\alpha_0$ is sufficient to enable the apparent longitudinal diffusivity to be found, whilst $\alpha_0$ and $\alpha_1$ are sufficient to establish the limit of validity of the diffusion approximation.

A Galerkin-type process of finding the eigenvalues, suggested by Professor John W. Miles, is presented. The process is apparently rapidly convergent, and leads to a practical method of solution to be developed in later papers.

An appendix outlines an approximate method of analysis, which is applied to dispersal both in the tube and between plane parallel walls. In the limit as $\Omega\to 0$, the results are consistent, for both cases, with those of the earlier, aperiodic, Taylor-Aris approach.

## I. Introduction

The dispersal of dissolved material in a moving liquid (or of a particular component of a moving gas mixture) is, in general, a complicated process, involving the interacting effects of convection and molecular diffusion. The scope of the present series of papers is limited to the case where the fluid flow is steady and laminar, and the flow passage is a straight circular tube. It is hoped to set out briefly in a separate,

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later, paper, the extension of the results of the present series to the related problem of dispersal during flow between plane parallel walls. We shall be concerned only with the case where the flow is longitudinally uniform (i.e. we shall not consider the influence of any disturbances to the flow pattern associated with the tube exit or entrance), and we shall take the molecular diffusivity of the dispersing substance to be independent of concentration.

The problems to be studied are relevant to the question of dispersal of soluble materials in blood vessels and in the water-conducting organs of plants, to a proposed method of measuring molecular diffusivity (Taylor 1953, 1954), and to a theory of dispersal during flow in porous media (Saffman 1960). An apparently new application is the question of the damping of a fluctuating concentration (in, say, a turbulent flow field) by continuous sampling through a tube, a technique now commonly used in micrometeorological studies of carbon dioxide and, occasionally, water-vapour, concentrations. It was a study of this application which revealed the limitations of the existing theories of longitudinal dispersal during laminar flow in tubes (Taylor 1953, 1954; Aris 1956) and prompted the developments set out in this series of papers. The application of the present improved theory to the continuous tube sampling problem will be presented in a separate paper.

In the present Part I, the general nature of the problem will be explored, Part II (Philip 1963) presents the detailed solution for the case of large Péclet number, Part III (Philip, in preparation) will present that for moderate, small, and zero Péclet number, and Part IV (Philip, in preparation) that for the special case where the molecular diffusivity is zero. Part V (Philip, in preparation) will discuss the interesting question of the influence of molecular diffusivity when the other parameters of the system are held constant. To facilitate cross-reference between the papers, a decimal system of numbering equations is employed, the first numeral signifying the part number.

## II. Statement of the Problem. Symbolism

We are concerned, in this series of papers, with dispersal during steady Poiseuille flow with mean velocity $U$ in a straight circular tube of radius $a$. The equation governing the dispersal of material of concentration $\theta$ and molecular diffusivity $D$ is then

$$\frac{\partial\theta}{\partial t}+\frac{2U(a^2-r^2)}{a^2}\cdot\frac{\partial\theta}{\partial x}=\frac{D}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)+D\frac{\partial^2\theta}{\partial x^2}. \tag{1.2.1}$$

Here $t$ denotes time, $x$ the axial coordinate, positive in the flow direction, and $r$ the radial coordinate. (1.2.1) involves the assumption that the dispersing material in the tube is distributed symmetrically about the tube axis. In view of the fact that the flux density of dispersing material is zero normal to the tube wall, we have the condition on (1.2.1)

$$r=a, \qquad \partial\theta/\partial r=0. \tag{1.2.2}$$

We introduce the following dimensionless quantities:

$$\left.\begin{aligned} \tau &= Dt/a^2, \\ \xi &= x/a, \\ \rho &= r/a, \\ \Upsilon &= Ua/D. \end{aligned}\right\} \tag{1.2.3}$$

Then (1.2.1), (1.2.2) become

$$\frac{\partial \theta}{\partial \tau}+2Y(1-\rho^2)\frac{\partial \theta}{\partial \xi} = \frac{1}{\rho}\cdot\frac{\partial}{\partial \rho}\left(\rho\,\frac{\partial \theta}{\partial \rho}\right)+\frac{\partial^2 \theta}{\partial \xi^2}, \tag{1.2.4}$$

$$\rho = 1, \qquad \partial\theta/\partial\rho = 0. \tag{1.2.5}$$

We shall have occasion to refer to the average value of $\theta$ over any cross section, $x$ = constant. We shall denote this quantity by $c$. Evidently

$$c = \frac{2}{a^2}\int_0^a r\theta\,\mathrm{d}r = 2\int_0^1 \rho\theta\,\mathrm{d}\rho. \tag{1.2.6}$$

In the course of the study, we shall introduce the frequency in radians per unit time, $\omega$. The corresponding dimensionless variable is evidently

$$\Omega = \omega a^2/D. \tag{1.2.7}$$

When $D$ tends to zero (see Part IV), the quantities $Y$ and $\Omega$ both tend to infinity and fail to characterize a periodic dispersing system. However, the ratio $\Omega/Y$ remains, in general, determinate as $D \to 0$. We therefore introduce the quantity W, which is useful in this particular case,

$$W = \Omega/Y = \omega a/U. \tag{1.2.8}$$

We note that $Y$ is, essentially *a diffusion Péclet number*. It is fully analogous to the Péclet number of heat transfer terminology, thermal diffusivity being replaced here by molecular diffusivity.* It is, of course, closely akin to the Reynolds number, $D$ here taking the place of kinematic viscosity. In the loose sense in which the Reynolds number is spoken of as "the ratio of the inertial forces to the viscous forces", we may here speak of the Péclet number (as we shall call it, for short), $Y$, as "the ratio of convective dispersal processes to dispersal processes due to molecular diffusion".

The quantity $\Omega$ can be considered as the ratio of a characteristic molecular diffusion time $a^2/D$ to the oscillation period $\omega^{-1}$. Similarly, the quantity $W$ may be regarded as the ratio of a characteristic convection time $a/U$ to $\omega^{-1}$.

Before proceeding to the problem of obtaining appropriate solutions of (1.2.4), (1.2.5), we shall first give a brief account of earlier studies of the problem of longitudinal dispersal during laminar flow in a tube. It will be understood that, where necessary, we adjust the symbolism of the early work to conform to that employed in this series.

## III. Earlier Studies

Sir Geoffrey Taylor (1953, 1954) appears to have been the first to recognize clearly that the dispersal of soluble matter during laminar flow in a tube arises from the subtle interaction of convection and molecular diffusion. He set up the exact equation to describe the dispersal, (1.2.1), but introduced certain limitations, assumptions, and approximations in the course of his analysis of the problem.

* It will be evident to the reader that the dispersing entity treated in this series may be taken to be heat, *provided the tube wall is a perfect insulator.*

In both papers, he considered only the case of aperiodic dispersal and he neglected longitudinal molecular diffusion. In the 1953 paper, he assumed, essentially, that a quasi-equilibrium between the longitudinal mean concentration gradient $\partial c/\partial x$ and the radial deviations from $c$, was established by the interaction of convection and radial molecular diffusion. Then, by neglecting $\partial\theta/\partial t$ (which implied $\partial c/\partial x$ constant and independent of $x$) and assuming that $\partial\theta/\partial x$ was independent of $r$ (i.e. equal to $\partial c/\partial x$), he obtained the result that $K$, the apparent longitudinal diffusivity down a gradient of $c$ in the frame of reference moving with the mean flow velocity $U$, is given by (1.3.1)

$$K/D = \Upsilon^2/48. \tag{1.3.1}$$

Taylor suggested that a suitable criterion for the establishment of the quasi-equilibrium in the radial profiles of $\theta$ might be

$$L/a \gg \Upsilon/7\cdot 3, \tag{1.3.2}$$

where $L$ is the "longitudinal extent of the region in which $\partial\theta/\partial x$ is appreciable". By implication (1.3.2) was the criterion for the validity of the result (1.3.1).

In the 1954 paper, Taylor returned to the problem and attempted to provide a more satisfactory analysis. This second paper investigated the effect of a small spatial variation of $\partial c/\partial x$ (thus, in some fashion, retaining time-dependence in the problem), and again arrived at (1.3.1), now subject to the criterion

$$L/a \gg \Upsilon/4. \tag{1.3.3}$$

This criterion is slightly more stringent than (1.3.2), and Taylor did not seek further for a criterion governing the establishment of a quasi-equilibrium radial concentration variation.

Taylor stated a further criterion for the validity of (1.3.1), namely,

$$\Upsilon \gg 6\cdot 9. \tag{1.3.4}$$

When (1.3.1) holds, (1.3.4) is equivalent to $K \gg D$, an inequality which asserts simply that the apparent longitudinal diffusivity is to be greatly in excess of the longitudinal molecular diffusivity, which was neglected by Taylor.

Two years later, Aris (1956) showed how the aperiodic dispersal of a slug of material (with initially finite moments) may be studied (in principle) exactly in terms of its moments with respect to the axial coordinate. He obtained the result, asymptotically valid for large times, that the rate of increase of the dispersion is equal to

$$2D(1+\Upsilon^2/48), \tag{1.3.5}$$

when the total mass of dispersing material is unity. Aris showed further that the longitudinal distribution of the slug asymptotically tends to normality as the dispersal time increases indefinitely, whatever the initial distribution. He then argued from the well-known relation between diffusivity and rate of increase of dispersion of a finite instantaneous release of material that it is reasonable to take

$$K/D = 1+\Upsilon^2/48. \tag{1.3.6}$$

This modification of Taylor's result (1.3.1) removed the necessity for the criterion (1.3.4) (at least for the case of aperiodic dispersal).

Aris's analysis enables an estimate to be made of the minimum dispersal time at which (1.3.5), and hence (1.3.6), can be taken as valid. For the case where the initial radial gradients of concentration are zero, the criterion may be shown to be

$$\tau \gg 1/15. \tag{1.3.7}$$

Under the assumption that the $L$ of (1.3.2) and (1.3.3) may be identified with the product $Ut$, this becomes

$$L/a \gg Y/15, \tag{1.3.8}$$

a somewhat less stringent condition than those proposed by Taylor.

It will be observed that Taylor and Aris discuss only the special (though not unimportant) case of aperiodic dispersal, Aris's more rigorous study being restricted to a slug of material with initially finite moments with respect to the axial coordinate. Their papers, however, contain no explicit warning that the theory has been worked out for a restricted class of dispersal problem only. It has, indeed, often been assumed tacitly that the results apply with greater generality.

The existence of serious limitations to the Taylor-Aris theory was first demonstrated by a preliminary, approximate, analysis, which is presented as the Appendix to the present paper.

In this preliminary work it was found that the apparent diffusivity $K$ was, in general, complex* and markedly dependent on the frequency in the reduced form, $\Omega$. A fairly good approximation to the dependence of $K$ on $D$, $Y$, and $\Omega$ was given by

$$\frac{K}{D} \approx 1+\frac{Y^2}{48}\cdot\frac{15}{15+\Omega i}. \tag{1.3.9}$$

In the limit as $\Omega \to 0$, (1.3.9) agrees with (1.3.6) suggested by Aris.

It was hoped that the analysis of the Appendix might have yielded a reasonably accurate picture of dispersal under these more general conditions. Further work has shown, however, that, although (1.3.9) gives a qualitative picture of the deviations from the Taylor-Aris theory, the more exact theory to be presented in this series should be employed if accurate quantitative results are desired.

We shall make brief reference to the connexions between these earlier studies and the present work as occasion arises in the later papers of this series.

## IV. The Treatment of the Dispersal as Fickian Diffusion

The exact solution of (1.2.1), (1.2.2) is, in general, an onerous undertaking. For many purposes all that is required is an understanding of the gross features of the dispersal, and a full solution would be needlessly burdensome. Thus, the variation

* The diffusivity in physical problems is normally real. In the preliminary study (Appendix), and in the present series, $K$ is found, in general, to be complex, so that it is somewhat stretching the concept of diffusivity to apply the term to $K$. However, so long as $K$ is associated with a definite frequency $\omega$, the complex diffusivity has a definite physical meaning related to the amplitude damping and the phase shift arising from the dispersal.

of $c(x, t)$ frequently provides all the information desired, whereas the full solution would give the variation of $\theta(x, r, t)$. In this connexion, the formulation in terms of apparent Fickian diffusion down a gradient of $c$ relative to the mean motion is highly convenient, provided it is valid.

The strategy of the present series is to seek, primarily, exact solutions of (1.2.1), (1.2.2), and to use the properties of these solutions to develop simpler and more convenient formulations, and to establish the limits of validity of such formulations. As we shall see, the diffusion approximation can be shown to arise from the exact solution very naturally under a wide range of circumstances.

We can set down at once one form of the criterion under which the Fickian diffusion description of the dispersal is valid.

Integrating (1.2.1), over the whole tube cross section, we obtain, with the use of (1.2.2) and (1.2.6),

$$\frac{\partial c}{\partial t}+U\frac{\partial c}{\partial x}-\frac{4U}{a^2}\cdot\frac{\partial}{\partial x}\left[\int_0^a r^3(\theta-c)\,\mathrm{d}r\right]=D\frac{\partial^2 c}{\partial x^2}. \tag{1.4.1}$$

Now the form of the Fickian equation for longitudinal diffusion with diffusivity $K$ down a gradient of $c$ in the frame of reference moving with the mean flow velocity $U$ is

$$\frac{\partial c}{\partial t}+U\frac{\partial c}{\partial x}=K\frac{\partial^2 c}{\partial x^2}. \tag{1.4.2}$$

(1.4.1) and (1.4.2) are identical, so long as

$$K\frac{\partial^2 c}{\partial x^2}=D\frac{\partial^2 c}{\partial x^2}+\frac{4U}{a^2}\frac{\partial}{\partial x}\left[\int_0^a r^3(\theta-c)\,\mathrm{d}r\right]. \tag{1.4.3}$$

We now integrate (1.4.3) with respect to $x$. By considering the special case where $c=\theta=$ constant, we see that the constant of integration is zero, so that the result is

$$K\frac{\partial c}{\partial x}=D\frac{\partial c}{\partial x}+\frac{4U}{a^2}\int_0^a r^3(\theta-c)\,\mathrm{d}r, \tag{1.4.4}$$

which may be rewritten

$$K=D+\frac{4U}{a^2\partial c/\partial x}\int_0^a r^3(\theta-c)\,\mathrm{d}r,$$

that is,

$$\frac{K}{D}=1+4Y\cdot\frac{\int_0^1 \rho^3(\theta-c)\,\mathrm{d}\rho}{\partial c/\partial\xi}. \tag{1.4.5}$$

It follows that the criterion for the dispersal to be exactly describable as diffusion is that $K$, as given by (1.4.5), must be independent of $x$. That is, the ratio

$$\int_0^1 \rho^3(\theta-c)\,\mathrm{d}\rho\bigg/\frac{\partial c}{\partial\xi} \tag{1.4.6}$$

must be independent of $x$ (or $\xi$). For the apparent diffusion coefficient to be constant in time, the ratio (1.4.6) must also be independent of $t$ (or $\tau$). We use this result in later discussion of the relationship between various approximations and the exact solution.

Further, it is convenient to put (1.4.2) into the form

$$\frac{\partial c}{\partial \tau}+\Upsilon\frac{\partial c}{\partial \xi} = \frac{K}{D}\frac{\partial^2 c}{\partial \xi^2}, \tag{1.4.7}$$

and to note that solutions of (1.4.7) periodic in $\tau$ are of the form

$$c = c_0 \exp(\Omega \mathrm{i}\tau+\alpha\xi/\Upsilon). \tag{1.4.8}$$

$\alpha$ is a root of the quadratic

$$\frac{K}{D}\cdot\frac{\alpha^2}{\Upsilon^2}-\alpha-\Omega\mathrm{i} = 0. \tag{1.4.9}$$

We shall be concerned with solutions for which $c\to 0$ as $\xi\to\infty$, so that the relevant root of (1.4.9) is that with negative real part.

When we later require to deduce $K$ from $\alpha$, we shall use (1.4.9) in the form

$$\frac{K}{D} = \Upsilon^2\frac{\alpha+\Omega\mathrm{i}}{\alpha^2}. \tag{1.4.10}$$

Finally, we have from (1.4.5) and (1.4.8) that, if the diffusion description is valid for a periodic system,

$$\frac{K}{D} = 1+\frac{\Upsilon^2}{\alpha}\left[\frac{4}{c}\int_0^1 \rho^3\theta\,\mathrm{d}\rho-1\right]. \tag{1.4.11}$$

Also, a sufficient condition for $K$ to be independent of $\xi$ and $\tau$ is that $\theta/c$ is independent of $\xi$ and $\tau$. It follows that a diffusion description of the phenomenon is valid when $\theta$ is expressible in the form

$$\theta = f(\rho)\exp(\Omega\mathrm{i}\tau+\alpha\xi/\Upsilon). \tag{1.4.12}$$

## V. Formal Solution of the System (1.2.4), (1.2.5).

We are concerned with the behaviour of (1.2.4), (1.2.5) for periodic and stationary random boundary conditions at $\xi = 0$. We therefore seek solutions of the form (1.4.12). Substituting (1.4.12) in (1.2.4), (1.2.5), we obtain

$$\frac{1}{\rho}\cdot\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}f}{\mathrm{d}\rho}\right)+\left[\frac{\alpha^2}{\Upsilon^2}-2\alpha(1-\rho^2)-\Omega\mathrm{i}\right]f = 0, \tag{1.5.1}$$

$$\rho = 1, \qquad \mathrm{d}f/\mathrm{d}\rho = 0. \tag{1.5.2}$$

A further condition on $f(\rho)$ is that we exclude singularities from the interval $0 \leqslant \rho \leqslant 1$.

Equation (1.5.1) is reducible to the confluent hypergeometric equation and may be shown to have two linearly independent solutions, the "logarithmic" solution being singular at $\rho = 0$. The problem thus reduces, for both $\Omega$ and $\Upsilon$ fixed, to the determination of the set of eigenfunctions $\alpha_{\Omega,\Upsilon,s}$ ($s = 0, 1, 2, \ldots$), which permit

(1.5.2) to be satisfied. (The $\alpha$'s are ordered in ascending magnitude of their (negative) real parts.)

The solution could thus be carried through, in principle, at least, in terms of the confluent hypergeometric function ${}_1F_1\ [a;\ b;\ x]$. A minimum requirement for such a procedure however, would seem to be a knowledge of ${}_1F_1\ [a;\ b;\ x]$ for both $a$ and $x$ complex. It is a trivial mitigation of the problem that $b$ need assume only the real values 1 and 2.

Methods for the determination of the eigenvalues, $\alpha_{\Omega,Y,s}$, and the associated eigenfunctions, $f_{\Omega,Y,s}\ (\rho)$, are developed later in this paper and in the sequels.

For the moment we shall suppose the eigenvalues and eigenfunctions determined, and indicate how the exact solution of the problem of dispersal with periodic boundary condition

$$\xi = 0, \qquad \theta = \Theta_0(\rho) \cdot e^{\Omega i \tau}, \tag{1.5.3}$$

and $Y$ specified, would proceed. In the following discussion, we drop the suffixes $\Omega$, $Y$ to the $\alpha$'s and $f$'s where there is no danger of ambiguity.

Then, under the assumption that the set of eigenfunctions $f_s(\rho)$ comprises a complete orthogonal system, we may write

$$\Theta_0(\rho) = \sum_{s=0}^{\infty} A_s \cdot f_s(\rho), \tag{1.5.4}$$

where the set of coefficients $A_s$ may be readily determined. The required exact solution is then:

$$\theta = \sum_{s=0}^{\infty} A_s \cdot f_s(\rho) \cdot \exp(\Omega i\tau + \alpha_s \xi / Y). \tag{1.5.5}$$

So long as $A_0 \neq 0$, the right-hand side of (1.5.5) will be dominated by the zeroth term, when $\xi$ is sufficiently great. We may then replace (1.5.5) by

$$\theta = A_0 f_0(\rho) \exp(\Omega i\tau + \alpha_0 \xi / Y), \tag{1.5.6}$$

and

$$c = 2A_0 \int_0^1 \rho f_0(\rho)\, d\rho \cdot \exp(\Omega i\tau + \alpha_0 \xi / Y). \tag{1.5.7}$$

If the constant quantity

$$2A_0 \int_0^1 \rho\, f_0(\rho)\, d\rho$$

is identified with the $c_0$ of (1.4.8), (1.5.7) becomes identical with (1.4.8). We infer that the periodic dispersal is describable as apparent longitudinal Fickian diffusion for $\xi$ sufficiently large. The apparent diffusivity $K$ is, from (1.4.10), given by

$$\frac{K}{D} = Y^2 \cdot \frac{\alpha_0 + \Omega i}{\alpha_0^2}. \tag{1.5.8}$$

A criterion for the validity of (1.5.8) is, evidently, that

$$\left| A_0 \int_0^1 \rho\, f_0(\rho)\, d\rho \cdot \exp \frac{\alpha_0 \xi}{Y} \right| \gg \left| A_1 \int_0^1 \rho f_1\,(\rho)\, d\rho \cdot \exp \frac{\alpha_1 \xi}{Y} \right|. \tag{1.5.9}$$

It follows that, so long as

$$\left| A_0 \int_0^1 \rho\, f_0(\rho)\, \mathrm{d}\rho \right|$$

is at least of the same order of magnitude as

$$\left| A_s \int_0^1 \rho\, f_s(\rho)\, \mathrm{d}\rho \right|$$

for all $s \geqslant 1$ (and this is apparently so for almost all $\Theta_0(\rho)$), (1.5.9) reduces to

$$\left| \exp \frac{\alpha_0 \xi}{\Upsilon} \right| \gg \left| \exp \frac{\alpha_1 \xi}{\Upsilon} \right|,$$

or

$$\xi \gg \frac{\Upsilon}{\mathscr{R}[\alpha_0 - \alpha_1]}, \tag{1.5.10}$$

where $\mathscr{R}[\quad]$ signifies "real part of".

If $L$ now designates the minimum distance along the tube from $x = 0$ at which (1.5.8) may be taken as valid (i.e. at which the initial deviations of the shape of $\Theta_0(\rho)$ from the zeroth radial eigenfunction $f_0(\rho)$ become unimportant), we therefore have

$$\frac{L}{a} \gg \frac{\Upsilon}{\mathscr{R}[\alpha_0 - \alpha_1]}. \tag{1.5.11}$$

This criterion should be compared with the criteria (1.3.2), (1.3.3), (1.3.8) of Taylor and Aris. In later papers we are able to assign numerical values to $\mathscr{R}[\alpha_0 - \alpha_1]$, which is, of course, a function of $\Omega$ and $\Upsilon$.

The developments of this section reveal the intimate connexion between the heuristic idea of an apparent diffusion coefficient and the eigenvalue character of the system (1.5.1), (1.5.2).

## VI. Practical Method of Solution of (1.5.1), (1.5.2)

We have observed in the previous section that the system (1.5.1), (1.5.2) leads to an eigenvalue problem. Formal solution in terms of the function ${}_1\mathrm{F}_1$ does not lead to useful results because of our ignorance of the behaviour of ${}_1\mathrm{F}_1[a; b; x]$ for both $a$ and $x$ complex.

In this section we outline a method of solution (essentially that of Galerkin (1915)), which was suggested to the author by Professor John W. Miles. This method has proved highly practical.

We begin by introducing an expansion of $f(\rho)$ in the interval $0 \leqslant \rho \leqslant 1$ based on an appropriate system of orthogonal functions. In view of the "cylindrical" nature of the problem, a Fourier-Bessel series is adopted. Further, boundary condition (1.5.2) is satisfied by all

$$f(\rho) = \mathrm{J}_0(\lambda_m \rho), \tag{1.6.1}$$

where $\lambda_m$ is the $m$th root of $\mathrm{J}_1(x)$, with $\lambda_0 = 0$. The expansion indicated is therefore

$$f(\rho) = \sum_0^\infty B_m\, \mathrm{J}_0(\lambda_m \rho). \tag{1.6.2}$$

Substitution of (1.6.2) in (1.5.1) yields

$$\sum_{0}^{\infty} B_m \, \mathrm{J}_0(\lambda_m \rho) \left[\frac{a^2}{Y^2} - 2a(1-\rho^2) - (\lambda_m^2 + \Omega \mathrm{i})\right] = 0. \tag{1.6.3}$$

We now take the appropriate finite Hankel transform by integrating the product of (1.6.3) and $\rho \, \mathrm{J}_0(\lambda_n \rho)$ with respect to $\rho$ between the limits 0 and 1. The result may be written

$$B_n \left[\frac{a^2}{Y^2} - (2a + \lambda_n^2 + \Omega \mathrm{i})\right] + a \sum_{0}^{\infty} B_m \cdot a_{mn} = 0. \tag{1.6.4}$$

It will be understood that $n$, like $m$, assumes all non-negative integral values. The $a_{mn}$ are specified by the equation

$$a_{mn} = \frac{4 \int_0^1 \rho^3 \mathrm{J}_0(\lambda_m \rho) \cdot \mathrm{J}_0(\lambda_n \rho) \, \mathrm{d}\rho}{\mathrm{J}_0^2(\lambda_n)}. \tag{1.6.5}$$

(1.6.5) leads to simple results in the following cases:

$$\left.\begin{aligned} a_{00} &= 1, & & \\ a_{mm} &= \tfrac{2}{3}, & & m \neq 0, \\ a_{0n} &= \frac{8}{\lambda_n^2 \, \mathrm{J}_0(\lambda_n)}, & & n \neq 0, \\ a_{m0} &= \frac{8 \, \mathrm{J}_0(\lambda_m)}{\lambda_m^2}, & & m \neq 0. \end{aligned}\right\} \tag{1.6.6}$$

In other cases, $a_{mn}$ does not seem to be calculable in closed form, and numerical methods of quadrature are apparently needed.

Equations(1.6.4) form an infinite set of simultaneous equations, quadratic in $a$, and linear in the $B_m$. Elimination of the $B_m$ yields the infinite determinantal equation

$$\begin{vmatrix} \frac{a^2}{Y^2} - (a + \Omega \mathrm{i}) & a_{10} a & a_{20} a & \cdots & a_{m0} a & \cdot\cdot \\ a_{01} a & \frac{a^2}{Y^2} - \left(\frac{4a}{3} + \lambda_1^2 + \Omega \mathrm{i}\right) & a_{21} a & \cdots & a_{m1} a & \cdot\cdot \\ a_{02} a & a_{12} a & \frac{a^2}{Y^2} - \left(\frac{4a}{3} + \lambda_2^2 + \Omega \mathrm{i}\right) & \cdots & a_{m2} a & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot & \cdot\cdot \\ a_{0m} a & a_{1m} a & a_{2m} a & \cdots & \frac{a^2}{Y^2} - \left(\frac{4a}{3} + \lambda_m^2 + \Omega \mathrm{i}\right) & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot & \cdot\cdot \end{vmatrix} = 0. \tag{1.6.7}$$

(1.6.7) has a doubly infinite set of roots, which contains the set of required eigenvalues, $a_s$. (We anticipate later results by remarking that only half the roots of (1.6.7) possess negative real parts and are physically admissible. It is this half which constitutes the $a_s$.)

We proceed by obtaining approximations to the roots of (1.6.7) by replacing the infinite determinant by finite determinants derived from it which contain, in turn, 1, 2, 3, etc. elements of the leading diagonal. The determinant of order 1 yields a first approximation to $\alpha_0$; that of order 2 a second approximation to $\alpha_0$, and a first approximation to $\alpha_1$; that of order 3 a third approximation to $\alpha_0$, a second approximation to $\alpha_1$, and a first approximation to $\alpha_2$; and so on. The detailed computations described in the subsequent papers indicate that successive approximations to the eigenvalues converge rapidly, so that we may expect the method to be tolerably accurate.*

It will be clear from Section V above that the evaluation of $\alpha_0$ and $\alpha_1$ is essentially all that is needed to explore the diffusion description of the longitudinal dispersal and to establish the limits of its validity. Since these are the primary aims of the present work, it has therefore been found sufficient to establish $\alpha_0$, $\alpha_1$, $\alpha_2$ from the modified determinant of order 3. It is fortunate that so much information can be gained from the first few eigenvalues, since the labour of computation increases rapidly with the order of the modified determinant.

It is a considerable simplification to limit ourselves to the modified determinantal equation of order 3. We are still left, however, with the task of finding the three relevant roots of a sextic equation with complex coefficients, which are themselves functions of the two parameters $\Omega$ and $\Upsilon$. (Both these parameters may, of course, assume any positive real value.) An appropriate method is developed in the succeeding Parts II and III.

## VII. Acknowledgment

The author gratefully acknowledges the most helpful advice of Professor John W. Miles, Australian National University, who suggested the use of the Galerkin method. More generally, discussion with Professor Miles has clarified greatly the ideas presented in this paper and its sequels.

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* I am indebted to Professor Miles for the remark that the presence of the self-adjoint operator $(1/\rho)\,\mathrm{d}(\rho\,\mathrm{d}/\mathrm{d}\rho)/\mathrm{d}\rho$ in (1.5.1) suggests the existence of a variational principle associated with (1.5.1), (1.5.2); and that fairly rapid convergence of the Galerkin method might be expected in consequence.

## Appendix

*Approximate General Theory of Dispersal in Tubes and between Parallel Walls*

We use the symbolism of the main body of the paper. We seek solutions of (1.2.4) subject to (1.2.5) of the form

$$\theta = \Theta(\xi,\rho)\, \mathrm{e}^{\Omega \mathrm{i}\tau}. \tag{1.A.1}$$

Substituting (1.A.1) in (1.2.4) yields

$$\Omega \mathrm{i}\Theta+2Y(1-\rho^2)\frac{\partial \Theta}{\partial \xi} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Theta}{\partial \rho}\right)+\frac{\partial^2\Theta}{\partial \xi^2}. \tag{1.A.2}$$

$\Theta$ may be written in the form

$$\Theta = \overline{\Theta}\,(\xi)+\vartheta(\xi,\rho), \tag{1.A.3}$$

where

$$\overline{\Theta} = 2\int_0^1 \rho\Theta\, \mathrm{d}\rho.$$

We now assume that $\partial\vartheta/\partial\xi$ and $\partial^2\vartheta/\partial\xi^2$ are negligibly small. This amounts to asserting that $\partial\Theta/\partial\xi$ and $\partial^2\Theta/\partial\xi^2$ are adequately represented for all $\rho$ by $\partial\overline{\Theta}/\partial\xi$ and $\partial^2\overline{\Theta}/\partial\xi^2$ and is equivalent to one of the assumptions of Taylor (1953). (1.A.2) may then be reduced to an ordinary equation in $\rho$ and $\vartheta$:

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}\vartheta}{\mathrm{d}\rho}\right)-\Omega \mathrm{i}\vartheta = P(1-\rho^2)+Q, \tag{1.A.4}$$

with

$$P = 2Y\mathrm{d}\overline{\Theta}/\mathrm{d}\xi,$$
$$Q = \Omega \mathrm{i}\overline{\Theta}-\mathrm{d}^2\overline{\Theta}/\mathrm{d}\xi^2.$$

(1.A.4) is subject to the conditions

$$\rho = 1, \qquad \mathrm{d}\vartheta/\mathrm{d}\rho = 0, \qquad \text{and} \int_0^1 \rho\vartheta\, \mathrm{d}\rho = 0. \tag{1.A.5}$$

The solution of (1.A.4), (1.A.5), is found to be

$$\vartheta = 4P\sum_{m=1}^{\infty}\frac{1}{\lambda_m^2\,(\lambda_m^2+\Omega \mathrm{i})}\cdot\frac{\mathrm{J}_0(\lambda_m\rho)}{\mathrm{J}_0(\lambda_m)}. \tag{1.A.6}$$

We have the further result

$$\int_0^1 \rho^3\vartheta\, \mathrm{d}\rho = 8P\sum_{m=1}^{\infty}\frac{1}{\lambda_m^4\,(\lambda_m^2+\Omega \mathrm{i})}. \tag{1.A.7}$$

Now (1.4.5) may be rewritten as

$$\frac{K}{D} = 1+\frac{4Y}{\mathrm{d}\overline{\Theta}/\mathrm{d}\xi}\cdot\int_0^1 \rho^3\vartheta\, \mathrm{d}\rho. \tag{1.A.8}$$

In view of (1.A.7), this reduces to

$$\frac{K}{D} = 1+64\Upsilon^2 \sum_{m=1}^{\infty} \frac{1}{\lambda_m^4\,(\lambda_m^4+\Omega\mathrm{i})}. \tag{1.A.9}$$

We observe that $K$ is independent of $\xi$, and that, therefore, the interpretation of the dispersal as apparent longitudinal diffusion is valid in the present approximation.

We have the result from (1.A.9) that, in the limit as $\Omega \to 0$,

$$K/D \to 1+\Upsilon^2/48, \tag{1.A.10}$$

in agreement with Aris's improvement, (1.3.6), on Taylor's (1.3.1).

(1.A.9) may be approximated accurately by taking the leading terms in appropriate expansions in powers of $\Omega$. For the present purpose, however, it suffices to note that the expression

$$\frac{K}{D} = 1+\frac{\Upsilon^2}{48}\cdot\frac{15}{15+\Omega\mathrm{i}} \tag{1.A.11}$$

is a reasonably good approximation to (1.A.9) for all $\Omega \geqslant 0$. The number 15 here is the value of

$$\sum_{m=1}^{\infty} \lambda_m^{-6} \Big/ \sum_{m=1}^{\infty} \lambda_m^{-8}$$

(cf. Rayleigh 1874).

The plot of $k(\Omega) = (K(\Omega)-D)/(K(0)-D))$ on the complex plane, based on (1.A.11), is then a semicircle in the fourth quadrant with centre $k = \frac{1}{2}$ and radius $\frac{1}{2}$.

Some interest attaches to the question of the effect of the shape of the cross section of the flow passage on dispersal. The study was therefore extended to the problem of dispersal during laminar flow with mean velocity $U$ between parallel plane walls with separation $2h$. The analysis followed, essentially, that outlined above for the tube. It yields the result

$$\frac{K}{D} = 1+18\Upsilon_1^2 \sum_{m=1}^{\infty} \frac{1}{m^4\pi^4(m^2\pi^2+\Omega_1\mathrm{i})}, \tag{1.A.12}$$

where

$$\Omega_1 = \omega h^2/D$$

and

$$\Upsilon_1 = Uh/D.$$

As $\Omega \to 0$,

$$K/D \to 1+2\Upsilon_1^2/105. \tag{1.A.13}$$

This result is consistent with that of Wooding (1960), who applied the Taylor-Aris approach to the problem of dispersal between parallel walls and obtained

$$K/D = 1+2\Upsilon_1^2/105. \tag{1.A.14}$$

An approximation to (1.A.12) which is reasonably good for all $\Omega \geqslant 0$ is

$$\frac{K}{D} = 1+\frac{2\Upsilon_1^2}{105}\cdot\frac{10}{10+\Omega\mathrm{i}}. \tag{1.A.15}$$