RAYLEIGH WAVE DISPERSION FOR A SINGLE LAYER ON AN ELASTIC HALF SPACE

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Summary

Numerical solutions of the period equation for Rayleigh waves in a single surface layer were calculated using the SILLIAC computer at the University of Sydney. Values of the phase and group velocities for both the fundamental and first higher mode are tabulated against period for eleven models. These related models allow a sensitivity analysis of the effect of variation in the seismic parameters.

I. Introduction

The complicated algebraic form of the period equation for Rayleigh wave propagation in a semi-infinite layered solid elastic medium discourages manual computation and only a few solutions for fairly simple cases have been calculated in this way (Ewing, Jardetsky, and Press 1957). Recently, solutions for the fundamental mode in three cases, each with three layers, have been obtained on EDSAC (Stoneley 1954); and IBM 650 machines have been used to evaluate the fundamental, first, and second higher mode velocities for a 3-layered model, and two 4-layered models (Oliver, Dorman, and Sutton 1959).

In the present paper solutions for the fundamental and first higher mode have been obtained for 11 related 2-layered models. Values of the elastic constants were chosen to fit the seismic velocities determined for the Western Australian crust (Bolt, Doyle, and Sutton 1958), so that the solutions are relevant to studies of the crustal structure of Australia. Computations were carried out on SILLIAC at the Basser Computing Department, the University of Sydney.

II. The Period Equation

Consider plane waves of length $2\pi/\varkappa$, phase velocity c, propagated in a homogeneous perfectly elastic medium, 1, of uniform thickness H, welded to the plane boundary of a semi-infinite similar medium, 2.

In such media, the velocities α, β of compressional and shear waves are

$$\alpha^2=(\lambda+2\mu)/\rho,\quad \beta^2=\mu/\rho, \qquad \text{(1)}$$

where ρ is the density and λ, μ the Lamé parameters.

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As a convenient notation (Stoneley, loc. cit.) we define

$$r_i^2=\varkappa^2(1-c^2/\alpha_i^2),$$
$$s_i^2=\varkappa^2(1-c^2/\beta_i^2),$$
$$b_i=(\varkappa^2+\rho_i^2)/\varkappa^2=2-c^2/\beta_i^2,\ (i=1,2),$$

$\cosh r_1H\equiv Cr_1$, $\sinh r_1H\equiv Sr_1$, $\cosh s_1H\equiv Cs_1$, $\sinh s_1H\equiv Ss_1$, $\mu=\mu_2/\mu_1$, where subscripts indicate the particular medium.

From the equations of motion and boundary conditions it may be shown (Sezawa 1927) that Rayleigh surface waves exist so long as

$$\Delta\equiv\begin{vmatrix} (\varkappa b_1/r_1)Sr_1+(\varkappa b_1/r_2)Cr_1 & 2Cr_1+(2r_1/r_2)Sr_1 & s_2/\varkappa-\varkappa/r_2 \\ \quad-(2s_1/\varkappa)Ss_1-(2\varkappa/r_2)Cs_1 & \quad-b_1Cs_1-(\varkappa^2b_1/r_2s_1)Ss_1 & \\ 2b_1(1-\mu)Cr_1+(4\mu-2b_1)Cs_1 & (4r_1/\varkappa)(1-\mu)Sr_1 & \mu(2-b_2) \\ & \quad+(\varkappa b_1/s_1)(2\mu-b_1)Ss_1 & \\ (\varkappa b_1{}^2/r_1)Sr_1+(\varkappa\mu b_1b_2/r_2)Cr_1 & 2b_1(Cr_1-Cs_1) & \mu(2s_2/\varkappa-b_2\varkappa/r_2) \\ \quad-(4s_1/\varkappa)Ss_1-(2\varkappa\mu b_2/r_2)Cs_1 & \quad+(\mu b_2/r_2)[2r_1Sr_1-(\varkappa^2b_1/s_1)Ss_1] & \end{vmatrix}$$
$$=0. \qquad \ldots\ldots\ldots\ldots (2)$$

The derivation implies that $\varkappa\geqslant0$, $c<\beta_1<\beta_2<\alpha_1<\alpha_2$, but real solutions also exist for $\beta_1<c<\beta_2$. In this case Cs_1 is real and Ss_1 imaginary; but in (2) s_1 always occurs as s_1Ss_1 or Ss_1/s_1, so that Δ remains real. When $c=\beta_1$, $s_1=0$ and there is a limiting case but no analytical difficulty arises.

III. Numerical Calculations

For each given value of $\varkappa H$ equation (2) was solved for c using the Newton-Raphson iteration process. That is to say, if c_n is the nth iterate,

$$c_{n+1}=c_n-\Delta\Big/\frac{\partial\Delta}{\partial c_n}.$$

The process was terminated as soon as two successive values of c agreed to five decimal places, and generally only two or three iterations were necessary. This speed of convergence is comparable with that for more complicated matrix iterative methods (e.g. Haskell 1953).

In (2) c is a function of $\varkappa H$ so that the waves are dispersive. The group velocity C, given by

$$C=c+\varkappa\mathrm{d}c/\mathrm{d}\varkappa,$$

was calculated directly using

$$\varkappa\frac{\mathrm{d}c}{\mathrm{d}\varkappa}=-\varkappa H\frac{\partial\Delta}{\partial(\varkappa H)}\Big/\frac{\partial\Delta}{\partial c},$$

and the calculation time for each step was about 5 sec.

For each new value of $\varkappa H$ the first iterate to be used was found from the formula

$$c_0(\varkappa H+\delta(\varkappa H))=c(\varkappa H)-\delta(\varkappa H)\frac{\partial\Delta}{\partial(\varkappa H)}\bigg/\frac{\partial\Delta}{\partial c},$$

where $\delta(\varkappa H)$ is the difference between the two values of $\varkappa H$. The partial derivatives required here and in the calculation of C were found by a numerical differentiation.

The value of c_0 for the first value of $\varkappa H$ used in any case was found by a subsidiary program which examined Δ for a set of values of c in a specified arithmetical progression for specified values of $\varkappa H$, printing whenever a change of sign occurred in Δ as c was increased.

In the main program $\varkappa H$ was printed at each step, followed by c, C, and the number of iterations required. All printing was in floating decimal form and the calculation time for each step was about 5 sec. Program tapes are available at the Department of Applied Mathematics, University of Sydney, for future calculations. The elastic parameters can be varied to suit different crustal models and still higher modes may be calculated if required.

IV. Discussion of the Solutions

Velocities for the fundamental and first higher mode (M_{21} or Sezawa mode) for the models A to K defined in Table 1 are tabulated in Tables 2 and 3.

A solution for the last case in Table 1, previously obtained using a desk machine (Haskell 1953) was used as a program check. Typical period-group velocity curves for the case $H=35$ km are shown in Figure 1.

Table 1

VELOCITIES (IN KM/SEC) AND THE DENSITY RATIOS DEFINING THE ELEVEN MODELS

Model	A	B	C	D	E	F	G	H	I	J	K	Haskell
α_1	*5·8*	6·0	*6·2*	6·0	6·0	6·0	6·0	6·0	6·0	6·0	6·0	6·14
β_1	3·6	3·6	3·6	*3·4*	*3·8*	3·6	3·6	3·6	3·6	3·6	3·6	3·39
α_2	8·2	8·2	8·2	8·2	8·2	8·2	8·2	*8·0*	*8·4*	8·2	8·2	8·26
β_2	4·8	4·8	4·8	4·8	4·8	4·8	4·8	4·8	4·8	*4·6*	*5·0*	4·65
ρ_2/ρ_1	1·296	1·296	1·296	1·296	1·296	*1·400*	*1·207*	1·296	1·296	1·296	1·296	1·111

Fundamental mode solutions exist for $0<\varkappa H<\infty$ and as $\varkappa H\to 0$, $c\to\beta_2$. The higher mode M_{21} has a cut-off period corresponding to $c=\beta_2$ and as $\varkappa H\to\infty$, $c\to\beta_1$.

There is always a well-defined minimum to the group velocity for both modes; waves associated with these minima are designated Airy phases (Pekeris 1948).

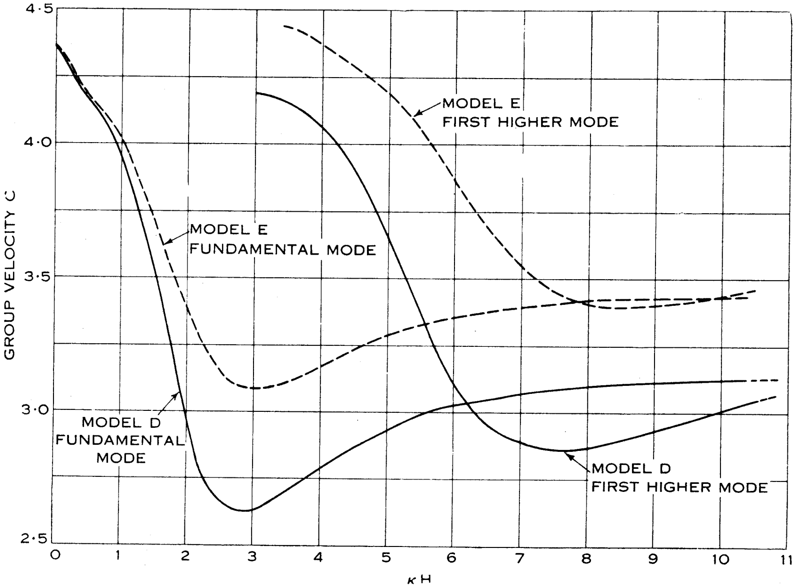

Fig. 1

The models indicate that the Airy wave velocity is sensitive mainly to β_1, the shear velocity in the upper layer. Of the 11 models, D has the minimum and E the maximum Airy wave velocity for both models. In general M_{21} is affected more by changes in β_1 than is the fundamental mode.

TABLE 2

PHASE AND GROUP VELOCITIES (IN KM/SEC) FOR THE ELEVEN MODELS : FUNDAMENTAL MODE

$\varkappa H$	Model A		Model B		Model C		Model D	
	c	C	c	C	c	C	c	C
0·10	4·36925	4·33570	4·37124	4·33995	4·37303	4·34376	4·36905	4·33573
0·20	4·33757	4·27786	4·34205	4·28772	4·34603	4·29643	4·33799	4·28025
0·40	4·28428	4·18898	4·29465	4·21185	4·30374	4·23155	4·28780	4·20051
0·60	4·24051	4·11804	4·25717	4·15332	4·27151	4·18304	4·24841	4·13952
0·80	4·20082	4·04330	4·22346	4·08879	4·24270	4·12649	4·21299	4·07070
1·00	4·16026	3·94839	4·18829	4·00162	4·21185	4·04537	4·17544	3·97379
1·50	4·03376	3·58409	4·07180	3·64295	4·10344	3·69182	4·04407	3·54222
2·00	3·86631	3·15691	3·90744	3·19540	3·94174	3·22796	3·84666	2·98114
2·50	3·69583	2·91036	3·73392	2·92586	3·76571	2·93846	3·63735	2·67553
3·00	3·55877	2·86055	3·59217	2·86729	3·61998	2·87221	3·47116	2·63439
3·50	3·46159	2·90469	3·49105	2·91063	3·51551	2·91493	3·35576	2·70060
4·00	3·39628	2·97465	3·42290	2·98242	3·44494	2·98834	3·27938	2·78922
4·50	3·35327	3·04225	3·37791	3·05217	3·39828	3·06021	3·22953	2·87016
6·00	3·29462	3·17795	3·31625	3·19259	3·33404	3·20454	3·16169	3·02553
7·00	3·28146	3·22227	3·30223	3·23867	3·31929	3·25238	3·14619	3·07506
8·00	3·27573	3·24613	3·29605	3·26435	3·31271	3·27934	3·13926	3·10288
10·00	3·271	3·271	3·292	3·283	3·308	3·298	3·135	3·123

TABLE 2 (*Continued*)

$\varkappa H$	Model E		Model F		Model G		Model H	
	c	C	c	C	c	C	c	C
0·10	4·37287	4·34302	4·37381	4·34502	4·36865	4·33483	4·35471	4·32365
0·20	4·34492	4·29268	4·34705	4·29744	4·33698	4·27781	4·32589	4·27255
0·40	4·29901	4·21820	4·30415	4·22995	4·28495	4·19322	4·27984	4·20023
0·60	4·26230	4·16058	4·27085	4·17933	4·24311	4·12648	4·24405	4·14543
0·80	4·22951	4·00978	4·24122	4·12273	4·20521	4·05402	4·21206	4·08405
1·00	4·19615	4·02208	4·21006	4·04316	4·16601	3·95981	4·17849	3·99952
1·50	4·09273	3·72703	4·10196	3·69011	4·04177	3·59920	4·06552	3·64642
2·00	3·95656	3·37662	3·93835	3·20921	3·87758	3·18431	3·90405	3·20160
2·50	3·81521	3·15273	3·75765	2·90646	3·71114	2·94398	3·73234	2·93056
3·00	3·69775	3·08952	3·60786	2·84067	3·57702	2·89221	3·59147	2·87000
3·50	3·61207	3·11527	3·50093	2·88809	3·48143	2·93213	3·49074	2·91196
4·00	3·55334	3·17110	3·42907	2·96536	3·41684	2·99865	3·42275	2·98304
4·50	3·51416	3·22942	3·38180	3·04015	3·37408	3·06412	3·37784	3·05251
6·00	3·46014	3·35244	3·31728	3·18843	3·31523	3·19682	3·31624	3·19291
7·00	3·44796	3·39308	3·30267	3·23694	3·30180	3·24109	3·30223	3·23855
8·00	3·44269	3·41526	3·29625	3·26243	3·29586	3·26470	3·29605	3·26449
10·00	3·439	3·432	3·292	3·285	3·292	3·286	3·292	3·283

$\varkappa H$	Model I		Model J		Model K	
	c	C	c	C	c	C
0·10	4·38563	4·35422	4·21602	4·18953	4·51961	4·48313
0·20	4·35618	4·30110	4·19144	4·14599	4·48544	4·42160
0·40	4·30771	4·22227	4·15220	4·08436	4·42939	4·33089
0·60	4·26880	4·16050	4·12169	4·03758	4·38469	4·26072
0·80	4·23363	4·09317	4·09441	3·98522	4·34454	4·18460
1·00	4·19705	4·00364	4·06580	3·91355	4·30292	4·08273
1·50	4·07744	3·63987	3·97039	3·61853	4·16585	3·66131
2·00	3·91049	3·18984	3·83482	3·24242	3·97327	3·14549
2·50	3·73534	2·92143	3·68830	2·99467	3·77421	2·86045
3·00	3·59281	2·86488	3·56491	2·92496	3·61585	2·81537
3·50	3·49134	2·90944	3·47475	2·95220	3·50507	2·87404
4·00	3·42304	2·98161	3·41300	3·01126	3·43136	2·95729
4·50	3·37798	3·05213	3·37179	3·07212	3·38313	3·03522
6·00	3·31626	3·19289	3·31466	3·19934	3·31759	3·18703
7·00	3·30224	3·23951	3·30156	3·24254	3·30280	3·23606
8·00	3·296	3·263	3·29576	3·26585	3·29630	3·26285
10·00	3·292	3·284	3·292	3·285	3·292	3·284

Variation in the density contrast between the media (as in Models F and G) has only a slight effect on the M_{21} curve but for $3 \cdot 7 < c < 4 \cdot 1$ km/sec a change in ρ_2/ρ_1 of $0 \cdot 2$ causes 15 per cent. displacement in the fundamental mode curve relative to M_{21}. The seismological implications of this point have been discussed by Oliver, Dorman, and Sutton (1959).

An important further inference from the solutions is that different combinations of elastic parameter values give very similar dispersion curves in both modes (e.g. models A and G). The main conclusion from a comparison is that observed Rayleigh wave dispersion alone cannot be used to give precise values to individual elastic parameters.

TABLE 3

PHASE AND GROUP VELOCITIES (IN KM/SEC) FOR THE ELEVEN MODELS : FIRST HIGHER MODE

$\varkappa H$	Model A		Model B		Model C		Model D	
	c	C	c	C	c	C	c	C
3·00	4·73368	4·33063	4·73399	4·33471	4·73427	4·33834	4·65065	4·19180
3·50	4·67200	4·27672	4·67321	4·28524	4·67426	4·29263	4·58331	4·15835
4·00	4·61882	4·21087	4·62110	4·22184	4·62309	4·23134	4·52516	4·06725
4·50	4·56798	4·10360	4·57133	4·11615	4·57423	4·12704	4·46595	3·90390
5·00	4·51395	3·94197	4·51824	3·95487	4·52198	3·96610	4·39814	3·65893
5·50	4·45254	3·72817	4·45755	3·73920	4·46191	3·74861	4·31771	3·36665
6·00	4·38253	3·49883	4·38787	3·50566	4·39251	3·51138	4·22733	3·11285
6·50	4·30691	3·31110	4·31220	3·31401	4·31678	3·31590	4·13487	2·95405
7·00	4·23119	3·19455	4·23619	3·19442	4·24049	3·19406	4·04753	2·88196
7·50	4·15999	3·14068	4·16459	3·13878	4·16854	3·13750	3·96901	2·86468
8·00	4·09571	3·12762	4·09990	3·12515	4·10350	3·12336	3·90025	2·87681
8·50	4·03901	3·13851	4·04281	3·13635	4·04607	3·13334	3·84082	2·90484
9·00	3·98959	3·16128	3·99303	3·15841	3·99598	3·15726	3·78972	2·93813
9·50	3·94677	3·19045	3·94987	3·18651	3·95254	3·18601	3·74585	2·97296
10·00	3·909	3·225	3·912	3·219	3·914	3·220	3·708	3·009
15·00	3·721	3·4	3·722	3·5	3·724	3·4	3·519	3·2

$\varkappa H$	Model E		Model F		Model G		Model H	
	c	C	c	C	c	C	c	C
3·00			4·72245	4·31980	4·74448	4·35152	4·73147	4·32609
3·50	4·74510	4·43259	4·66313	4·29467	4·68249	4·27572	4·67004	4·27940
4·00	4·70198	4·36900	4·61445	4·24651	4·62697	4·19597	4·61772	4·21799
4·50	4·66109	4·29493	4·56892	4·15446	4·57289	4·07662	4·56802	4·11449
5·00	4·61956	4·19071	4·52051	4·00431	4·51517	3·90599	4·51523	3·95598
5·50	4·57439	4·04808	4·46427	3·78974	4·45028	3·69187	4·45505	3·74317
6·00	4·52340	3·87328	4·39773	3·54141	4·37783	3·47400	4·38600	3·51136
6·50	4·46642	3·69478	4·32320	3·32741	4·30129	3·30201	4·31091	3·31930
7·00	4·40583	3·54929	4·24674	3·19175	4·22578	3·19715	4·23533	3·19863
7·50	4·34534	3·45600	4·17397	3·12796	4·15533	3·14823	4·16401	3·14120
8·00	4·28810	3·40963	4·10795	3·11294	4·09192	3·13726	4·09951	3·12704
8·50	4·23587	3·39548	4·04962	3·12319	4·03602	3·14822	4·04253	3·13763
9·00	4·18927	3·40119	3·99878	3·14714	3·98728	3·16862	3·99283	3·15998
9·50	4·14820	3·41636	3·95473	3·17593	3·94500	3·19656	3·94972	3·18784
10·00	4·111	3·443	3·916	3·215	3·908	3·232	3·912	3·223
15·00	3·924	3·6	3·722	3·4	3·721	3·5	3·722	3·5

TABLE 3 (*Continued*)

$\varkappa H$	Model I		Model J		Model K	
	c	C	c	C	c	C
3·00	4·73621	4·34247	4·58280	4·33042	4·86340	4·37757
3·50	4·67602	4·29048	4·53799	4·22874	4·79202	4·34387
4·00	4·62410	4·22530	4·49579	4·17135	4·73148	4·26216
4·50	4·57428	4·11765	4·45582	4·09594	4·67191	4·11676
5·00	4·52093	3·95383	4·41464	3·98539	4·60604	3·89748
5·50	4·45978	3·73546	4·36897	3·83207	4·52942	3·62456
6·00	4·38953	3·50057	4·31661	3·64609	4·44289	3·36702
6·50	4·31335	3·30895	4·25783	3·46294	4·35269	3·18887
7·00	4·23695	3·19085	4·19572	3·32289	4·26584	3·09738
7·50	4·16510	3·13632	4·13442	3·23896	4·18668	3·06728
8·00	4·10025	3·12350	4·07703	3·19963	4·11674	3·07172
8·50	4·04305	3·13463	4·02511	3·19233	4·05593	3·09614
9·00	3·99321	3·15767	3·97907	3·20278	4·00346	3·12762
9·50	3·95001	3·18687	3·93866	3·22118	3·95830	3·16210
10·00	3·912	3·223	3·903	3·244	3·918	3·205
15·00	3·722	3·5	3·720	3·4	3·724	3·4

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