THE SHAPE OF SPECTRAL LINES : TABLES OF THE VOIGT PROFILE $\frac{a}{\pi}\int_{-\infty}^{\infty}\frac{e^{-y^2}dy}{a^2+(v-y)^2}$

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Summary

A brief discussion is given of some methods of computing the line shape, commonly known as the *Voigt profile*, resulting when a spectral line is broadened simultaneously by Gaussian and Lorentz effects, and tables and curves are presented for representative values of the variables.

I. Introduction

When the broadening of a spectral line is due to contributions from a Gaussian shape (e.g. Doppler effect) and from an independent Lorentz shape (e.g. natural line width), the resulting intensity distribution may be expressed by (Mitchell and Zemansky 1934, pp. 97, 160)

$$H(a,v)=\frac{a}{\pi}\int_{-\infty}^{\infty}\frac{e^{-y^2}dy}{a^2+(v-y)^2}. \qquad \ldots\ldots\ldots\ldots \quad (1a)$$

Here $v=(\nu-\nu_0)/\frac{1}{2}\Delta\nu'_G$ and is the distance from the line centre ν_0 in units of $\frac{1}{2}\Delta\nu'_G$; $\Delta\nu'_G=(\ln 2)^{-\frac{1}{2}}\Delta\nu_G$, where $\Delta\nu_G$ is the Gauss half-width (in frequency units) that the line would have if no Lorentz effect were present ; and $a=\Delta\nu_L/\Delta\nu'_G$, where $\Delta\nu_L$ is the Lorentz half-width.

Alternatively, $H(a,v)$ can be related to the error function of a complex variable through (Born 1933, p. 483 ; Harris 1948)

$$H(a,v)=2\pi^{-\frac{1}{2}}\mathrm{Re}\left\{e^{z^2}\int_z^{\infty}e^{-t^2}dt\right\}, \qquad \ldots\ldots\ldots\ldots \quad (1b)$$

where $z=a+iv$.

In the common case of a line broadened by Doppler and collision effects the two processes are not independent (Mitchell and Zemansky 1934, p. 160), but it appears that even in this event the use of equation (**1**) is a sufficiently good approximation for most purposes.

In some cases, $H(a,v)$ may also be used to approximate to the measured intensity distribution when the influence of the observing apparatus has to be taken into account (Dennison 1928 ; van de Hulst 1946 ; van de Hulst and Reesinck 1947).

The intensity distribution (**1**) is commonly called by astrophysicists a *Voigt profile* because of the original discussion of this effect by Voigt (1912). Its application to the solution of a number of practical problems has been well discussed in an important paper by van de Hulst and Reesinck (1947).

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II. Numerical Evaluation of the Integral

The numerical evaluation of $H(a,v)$ over the whole range of the variables presents some difficulties.

Zemansky (1930) has described a series expansion, due to T. H. Gronwall, which works well for small v and a, and he has tabulated $H(a,v)$ for $a=0\cdot5$, 1, and $1\cdot5$, for different ranges of v; these tables have been extended by Mitchell and Zemansky (1934, p. 328), and the values therein have now been checked and found to be substantially correct. Examination of the formula used shows that the largest term to contribute to the sum occurs when the summation index n equals v^2, so it is apparent that the calculation becomes rather unwieldy for large v. To illustrate, for small a and for $v=24$ the only terms which contributed to 1 part in 10^7 were found to be those for $n=483$ to 651, and the answer was then correct to only 1 part in 10^6 because of truncation errors. Further, for values of a greater than about 4, the recurrence formula for $I_n(a)$ given by Zemansky (1930) breaks down in practice after a very few terms because a small error in the initial term is rapidly propagated and magnified. Even if the recurrence relation is replaced by the equivalent series it is found that this is asymptotic (see, for example, Bromwich 1949, p. 332) and in fact becomes divergent for large enough n. Thus Zemansky's series is usable only for values of a less than about four.

An alternative expansion of equation (**1a**), due to Dr. D. G. Lampard (personal communication), runs as follows:

$$\begin{aligned}H(a,v)&=\frac{1}{\pi a}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-y^2}\mathrm{d}y}{1+\{(y-v)/a\}^2}\\&=\frac{1}{\pi a}\sum_{n=0}^{s-1}\frac{1}{a^{2n}}\int_{-\infty}^{\infty}(-\mathrm{i}v+\mathrm{i}y)^{2n}\,\mathrm{e}^{-y^2}\mathrm{d}y+R_s\\&=\frac{1}{\pi^{\frac{1}{2}}a}\sum_{n=0}^{s-1}\frac{H_{2n}(-\mathrm{i}v)}{(2a)^{2n}}+R_s, \qquad \ldots\ldots\ldots\ldots \quad (2)\end{aligned}$$

where the series is obtained on long division by the denominator, and the integral representation of the Hermite polynomials (Morse and Feshbach 1953, p. 786) has been used. The remainder R_s is given by

$$R_s=\frac{1}{\pi a}\int_{-\infty}^{\infty}\frac{[\mathrm{i}(y-v)/a]^{2s}\mathrm{e}^{-y^2}\mathrm{d}y}{1+\{(y-v)/a\}^2},$$

so that

$$|R_s|<\left|\frac{1}{\pi^{\frac{1}{2}}a}\cdot\frac{H_{2s}(-\mathrm{i}v)}{(2a)^{2s}}\right|.$$

However,

$$\frac{H_{2n}(-\mathrm{i}v)}{(2a)^{2n}}=\mathrm{i}^{2n}\left\{\left(\frac{v}{a}\right)^{2n}+\ldots+\frac{1\cdot3\ldots(2n-1)}{2^n}\cdot\frac{1}{a^{2n}}\right\},$$

so when the series (**2**) is not divergent it is asymptotic, and its usefulness depends on the values of v/a and a. It is found that sufficient accuracy is obtainable if $a\geqslant4$, provided $v/a\leqslant\frac{1}{2}$.

Born (1933, p. 482) has given series expansions of equation (**1b**) for large and small values of v/a, and has essentially tabulated $H(a,v)$ for $a=0\cdot5$, 1, 2, and

10, with $v/a=0(0\cdot2)4$. In the present work the range of applicability of his series has not been investigated, but the numerical values of his tables have been checked by other methods and found to be substantially correct, except for $a=0\cdot5$, where for $v/a\geqslant2\cdot0$ the tabulated values are systematically too large.

For astrophysical purposes only small values of a seem to have been of interest, and Hjerting (1938) has discussed calculations in this region and has given tables of $H(a,v)$ for $v=0(0\cdot25)5$ and $a=0(0\cdot01)0\cdot2(0\cdot1)0\cdot5$; his values for $a=0\cdot1$, $0\cdot2$, and $0\cdot5$ have now been checked by other methods and found correct.

Harris (1948) has given the coefficients in a Taylor series expansion of equation (**1b**) also for small a, but usable for larger values of v than available with Hjerting's formula.

The present study has been undertaken in connexion with another problem, for which it was found that published methods of computation were unsuitable and that existing tables of $H(a,v)$ were inadequate.

For the numerical integration of equation (**1a**), a standard method* of numerical analysis is the use of Gauss-Hermite quadrature (Kopal 1955):

$$\int_{-\infty}^{\infty} \mathrm{e}^{-y^2}f(y)\mathrm{d}y=\sum_{j=1}^{n} H_jf(a_j)+R_n, \qquad \ldots\ldots\ldots\ldots \quad (3)$$

with the quantities H_j and a_j given by Kopal (1955) for n up to 20. Essentially, the method involves approximating $f(y)$ by a suitable polynomial; difficulty in its use lies in the estimation of the remainder R_n in those cases, such as the present, where it is not known to vanish.

A contribution of the present work is to report that for $n=20$ the integral $H(a,v)$, as given by equation (**1a**), may be conveniently and accurately evaluated by Gauss-Hermite quadrature except for simultaneously small values of v and a. The error involved in using equation (**3**) has been determined by comparison of the results obtained by it with those computed by the Zemansky and Lampard series expansions in their ranges of usefulness, and with Born's tabulated values for $a=10$ and v up to 40. The general agreement is such as to justify extrapolation to those regions in which the above series expansions cannot be used.

Figure 1 shows the *maximum* percentage error in $H(a,v)$ computed by means of equation (**3**) when $n=20$; for very small a (e.g. $0\cdot01$) the error is particularly large for values of v at or very near to the Hermite polynomial roots a_j. For practical reasons the last two pairs of H_j and $|a_j|$ were not used; their omission does not affect the results of this work to the accuracies quoted, but causes the curves in Figure 1 to level off in the vicinity of $0\cdot0001$ per cent. Neglect of the next last two pairs introduces tails on these curves levelling off around $0\cdot001$ per cent.

Figure 2, derived from the curves of Figure 1, indicates the regions of usefulness of equation (**3**) for given error limits.

The quantity w, which is the effective half-width in units of $\Delta\nu'_G$ and is defined by $H(a,w)=\frac{1}{2}H(a,0)$, is shown in Figure 3; it can be roughly approximated

* The author is indebted to Dr. J. M. Bennett, of the Adolph Basser Computing Laboratory, for telling him of this method.

by $w \sim (1+a^2)^{\frac{1}{2}}$ over most of its range.* Rather better approximations have been given by Burger and van Cittert (1927) and graphically summarized by Minkowski and Bruck (1935), in fair agreement with the present results which were obtained by a method of successive approximations.

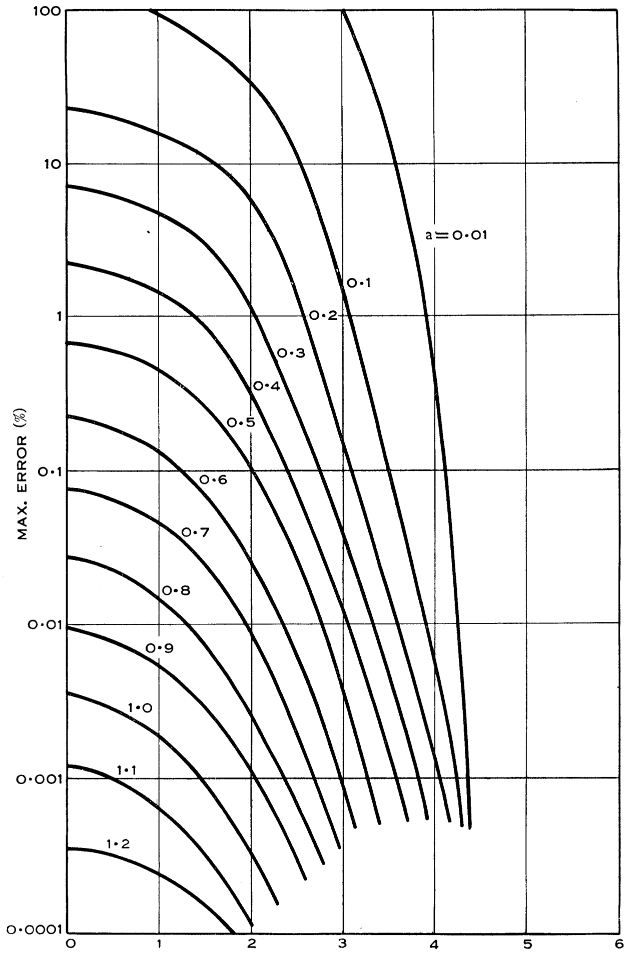

Fig. 1.—Maximum percentage error in $H(a,v)$ computed by Gauss-Hermite quadrature (equation (**3**)).

III. Tabulation

Apart from the cases $a=0$ and $a=\infty$, Table 1 was constructed using Zemansky's series for $v<5-2a$ if $a<1 \cdot 2$, and Gauss-Hermite quadrature elsewhere; for $a<1 \cdot 2$, $H(a,0)$ was calculated by a polynomial approximation (Hastings 1955, pp. 42, 169), although it can be obtained from tables of the error function. The values tabulated in Table 1 have errors of less than one unit

* Danos and Geschwind (1953) give $\Delta\nu_{\text{eff.}} \sim (\Delta\nu_G^2+\Delta\nu_L^2)^{\frac{1}{2}}$, or, in the present notation, $w \sim (\ln 2+a^2)^{\frac{1}{2}}$, an approximation which is adequate for many purposes.

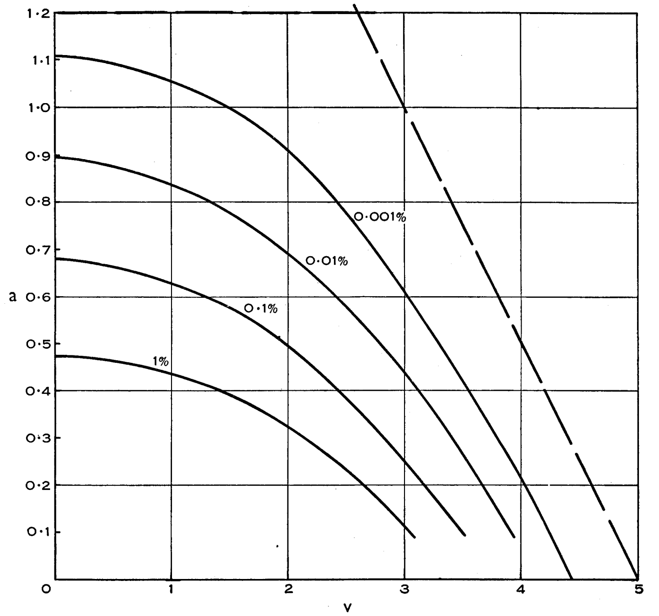

Fig. 2.—Regions of usefulness of Gauss-Hermite quadrature for given error limits. The dashed lines $v=5-2a$, $a=1\cdot2$, separate the regions in which different methods were used for the computation of Table 1.

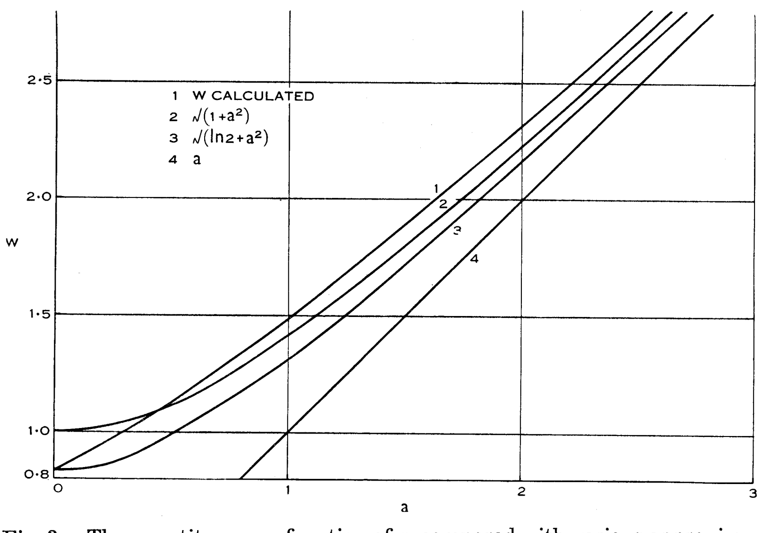

Fig. 3.—The quantity w as a function of a, compared with various approximations and with the line $w=a$.

in the last place (due mainly to round-off), and have been given to five decimal places—more than adequate for most line shape problems—because of the possibility that to this accuracy the integral may be of use in other fields.

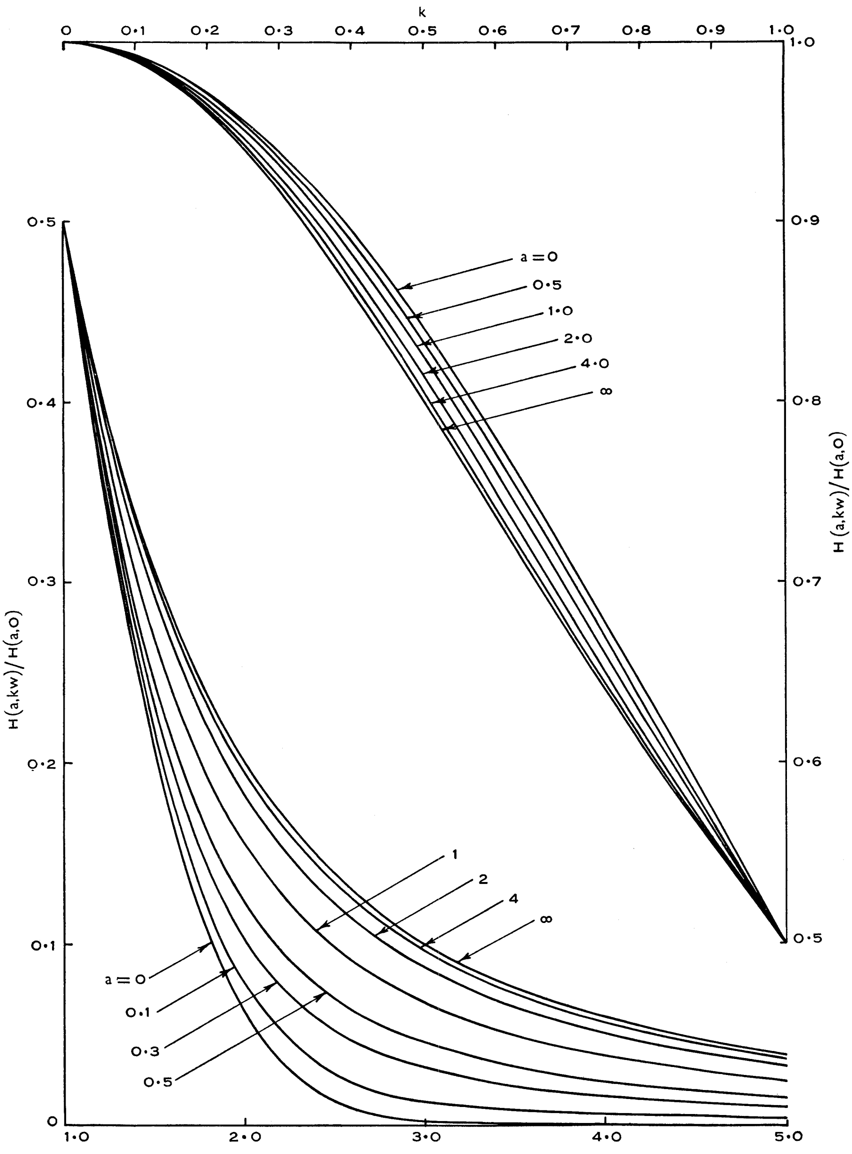

Fig. 4.—Values of the function $H(a,kw)/H(a,0)$ for representative values of a.

In preference to listing $H(a,v)$ directly, the table shows, for each value of a, the quantities $H(a,0)$ and w, and tabulates the normalized profiles at multiples k of the effective half-width w, i.e. lists $H(a,kw)/H(a,0)$ for selected values of k in the range 0–20. The values of a were chosen in an attempt to cover the region fairly evenly, but with a bias to small a (<1) where the calculations are rather

TABLE 1

VALUES OF $H(a,kw)/H(a,0)$

a	0·00	0·01	0·02	0·03	0·04	0·05	0·06	0·07	0·08	0·09
$H(a,0)$	1·00000	0·98882	0·97783	0·96703	0·95642	0·94599	0·93574	0·92567	0·91576	0·90603
w	0·83255	0·83789	0·84326	0·84865	0·85407	0·85952	0·86500	0·87050	0·87603	0·88158
k										
0·0	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000
0·1	0·99309	0·99308	0·99307	0·99306	0·99305	0·99304	0·99303	0·99302	0·99301	0·99300
0·2	0·97265	0·97262	0·97258	0·97254	0·97250	0·97246	0·97242	0·97238	0·97234	0·97230
0·3	0·93952	0·93944	0·93936	0·93928	0·93920	0·93911	0·93903	0·93895	0·93887	0·93878
0·4	0·89503	0·89490	0·89477	0·89464	0·89451	0·89438	0·89425	0·89412	0·89399	0·89386
0·5	0·84090	0·84073	0·84055	0·84038	0·84021	0·84004	0·83987	0·83970	0·83952	0·83935
0·6	0·77916	0·77897	0·77877	0·77857	0·77837	0·77817	0·77798	0·77778	0·77758	0·77738
0·7	0·71203	0·71183	0·71163	0·71143	0·71123	0·71103	0·71083	0·71063	0·71043	0·71023
0·8	0·64171	0·64154	0·64137	0·64120	0·64104	0·64087	0·64070	0·64053	0·64036	0·64019
0·9	0·57038	0·57028	0·57018	0·57007	0·56997	0·56987	0·56976	0·56966	0·56956	0·56946
1·0	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000
1·1	0·43227	0·43241	0·43254	0·43268	0·43281	0·43295	0·43309	0·43322	0·43336	0·43349
1·2	0·36857	0·36887	0·36917	0·36947	0·36977	0·37007	0·37036	0·37066	0·37095	0·37125
1·3	0·30993	0·31041	0·31090	0·31138	0·31186	0·31234	0·31281	0·31329	0·31376	0·31423
1·4	0·25703	0·25771	0·25839	0·25906	0·25973	0·26040	0·26106	0·26172	0·26238	0·26303
1·5	0·21022	0·21110	0·21197	0·21283	0·21369	0·21454	0·21539	0·21624	0·21707	0·21791
1·6	0·16958	0·17063	0·17168	0·17272	0·17376	0·17479	0·17581	0·17683	0·17784	0·17884
1·7	0·13490	0·13612	0·13734	0·13854	0·13973	0·14092	0·14210	0·14326	0·14442	0·14557
1·8	0·10584	0·10720	0·10855	0·10989	0·11122	0·11254	0·11384	0·11514	0·11642	0·11769
1·9	0·08190	0·08337	0·08483	0·08627	0·08770	0·08912	0·09053	0·09192	0·09330	0·09466
2·0	0·06250	0·06405	0·06559	0·06710	0·06861	0·07010	0·07157	0·07303	0·07448	0·07591
2·2	0·03492	0·03653	0·03813	0·03971	0·04127	0·04282	0·04434	0·04585	0·04734	0·04882
2·4	0·01845	0·02003	0·02159	0·02313	0·02465	0·02615	0·02763	0·02910	0·03054	0·03197
2·6	0·00923	0·01069	0·01214	0·01357	0·01499	0·01638	0·01776	0·01911	0·02045	0·02177
2·8	0·00436	0·00569	0·00699	0·00828	0·00955	0·01080	0·01204	0·01326	0·01446	0·01565
3·0	0·00195	0·00312	0·00427	0·00540	0·00652	0·00763	0·00872	0·00980	0·01086	0·01191
3·2	0·00083	0·00184	0·00285	0·00384	0·00481	0·00578	0·00674	0·00768	0·00861	0·00953
3·4	0·00033	0·00121	0·00209	0·00295	0·00380	0·00465	0·00548	0·00631	0·00712	0·00793
3·6	0·00013	0·00090	0·00166	0·00241	0·00316	0·00390	0·00463	0·00535	0·00607	0·00678
3·8	0·00004	0·00072	0·00139	0·00205	0·00271	0·00336	0·00401	0·00465	0·00528	0·00590
4·0	0·00002	0·00061	0·00121	0·00179	0·00238	0·00295	0·00353	0·00409	0·00466	0·00521
4·2	0·00000	0·00054	0·00107	0·00159	0·00211	0·00263	0·00314	0·00365	0·00415	0·00465
4·4	0·00000	0·00048	0·00095	0·00143	0·00189	0·00236	0·00282	0·00327	0·00373	0·00418
4·6	0·00000	0·00043	0·00086	0·00129	0·00171	0·00213	0·00255	0·00296	0·00337	0·00378
4·8	0·00000	0·00039	0·00078	0·00117	0·00155	0·00194	0·00231	0·00269	0·00307	0·00344
5·0	0·00000	0·00036	0·00071	0·00107	0·00142	0·00177	0·00211	0·00246	0·00280	0·00314
5·5	0·00000	0·00029	0·00058	0·00087	0·00115	0·00144	0·00172	0·00200	0·00228	0·00255
6·0	0·00000	0·00024	0·00048	0·00072	0·00096	0·00119	0·00143	0·00166	0·00189	0·00212
6·5	0·00000	0·00020	0·00041	0·00061	0·00081	0·00101	0·00120	0·00140	0·00160	0·00179
7·0	0·00000	0·00017	0·00035	0·00052	0·00069	0·00086	0·00103	0·00120	0·00137	0·00153
8·0	0·00000	0·00013	0·00026	0·00039	0·00052	0·00065	0·00078	0·00091	0·00104	0·00116
9·0	0·00000	0·00010	0·00021	0·00031	0·00041	0·00051	0·00061	0·00071	0·00081	0·00091
10·0	0·00000	0·00008	0·00017	0·00025	0·00033	0·00041	0·00049	0·00057	0·00066	0·00074
12·0	0·00000	0·00006	0·00011	0·00017	0·00023	0·00028	0·00034	0·00040	0·00045	0·00051
15·0	0·00000	0·00004	0·00007	0·00011	0·00015	0·00018	0·00022	0·00025	0·00029	0·00032
20·0	0·00000	0·00002	0·00004	0·00006	0·00008	0·00010	0·00012	0·00014	0·00016	0·00018

Table 1 (*Continued*)

a	0·10	0·12	0·14	0·16	0·18	0·20	0·22	0·24	0·26	0·28
$H(a,0)$	0·89646	0·87779	0·85974	0·84228	0·82538	0·80902	0·79318	0·77784	0·76297	0·74857
w	0·88716	0·89841	0·90976	0·92121	0·93278	0·94444	0·95621	0·96809	0·98007	0·99215
k										
0·0	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000
0·1	0·99299	0·99297	0·99295	0·99292	0·99290	0·99288	0·99286	0·99284	0·99282	0·99280
0·2	0·97226	0·97218	0·97210	0·97202	0·97194	0·97185	0·97177	0·97169	0·97161	0·97153
0·3	0·93870	0·93853	0·93837	0·93820	0·93803	0·93786	0·93770	0·93753	0·93736	0·93719
0·4	0·89373	0·89347	0·89320	0·89294	0·89268	0·89242	0·89215	0·89189	0·89162	0·89136
0·5	0·83918	0·83883	0·83849	0·83814	0·83779	0·83745	0·83710	0·83675	0·83640	0·83606
0·6	0·77718	0·77678	0·77639	0·77599	0·77559	0·77519	0·77479	0·77440	0·77400	0·77360
0·7	0·71003	0·70963	0·70923	0·70883	0·70843	0·70803	0·70764	0·70724	0·70685	0·70645
0·8	0·64002	0·63968	0·63935	0·63901	0·63868	0·63834	0·63801	0·63768	0·63735	0·63702
0·9	0·56936	0·56915	0·56895	0·56875	0·56855	0·56834	0·56814	0·56795	0·56775	0·56755
1·0	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000
1·1	0·43362	0·43389	0·43415	0·43442	0·43468	0·43494	0·43520	0·43545	0·43571	0·43596
1·2	0·37154	0·37212	0·37270	0·37328	0·37384	0·37441	0·37497	0·37552	0·37607	0·37661
1·3	0·31470	0·31563	0·31655	0·31746	0·31837	0·31926	0·32014	0·32102	0·32188	0·32273
1·4	0·26368	0·26497	0·26625	0·26751	0·26875	0·26998	0·27120	0·27239	0·27358	0·27474
1·5	0·21874	0·22038	0·22200	0·22359	0·22517	0·22672	0·22826	0·22977	0·23125	0·23272
1·6	0·17983	0·18180	0·18374	0·18564	0·18752	0·18937	0·19119	0·19298	0·19474	0·19647
1·7	0·14671	0·14896	0·15118	0·15336	0·15550	0·15760	0·15967	0·16170	0·16369	0·16564
1·8	0·11895	0·12144	0·12388	0·12628	0·12863	0·13094	0·13320	0·13542	0·13759	0·13972
1·9	0·09602	0·09868	0·10129	0·10385	0·10636	0·10882	0·11123	0·11358	0·11589	0·11814
2·0	0·07732	0·08011	0·08283	0·08550	0·08811	0·09067	0·09316	0·09560	0·09799	0·10032
2·2	0·05028	0·05314	0·05593	0·05866	0·06133	0·06393	0·06646	0·06893	0·07135	0·07370
2·4	0·03338	0·03615	0·03884	0·04147	0·04403	0·04652	0·04895	0·05132	0·05363	0·05587
2·6	0·02308	0·02564	0·02813	0·03055	0·03292	0·03522	0·03745	0·03963	0·04176	0·04382
2·8	0·01682	0·01912	0·02136	0·02354	0·02566	0·02773	0·02974	0·03170	0·03361	0·03547
3·0	0·01295	0·01498	0·01696	0·01889	0·02077	0·02261	0·02439	0·02613	0·02783	0·02948
3·2	0·01044	0·01223	0·01397	0·01567	0·01732	0·01894	0·02052	0·02206	0·02356	0·02502
3·4	0·00873	0·01029	0·01182	0·01332	0·01478	0·01620	0·01760	0·01896	0·02029	0·02159
3·6	0·00748	0·00886	0·01021	0·01153	0·01282	0·01408	0·01532	0·01653	0·01771	0·01887
3·8	0·00652	0·00774	0·00894	0·01012	0·01126	0·01239	0·01349	0·01457	0·01563	0·01667
4·0	0·00576	0·00685	0·00792	0·00897	0·01000	0·01101	0·01199	0·01296	0·01391	0·01484
4·2	0·00514	0·00612	0·00708	0·00802	0·00895	0·00985	0·01075	0·01162	0·01248	0·01332
4·4	0·00462	0·00550	0·00637	0·00722	0·00806	0·00888	0·00969	0·01048	0·01126	0·01203
4·6	0·00418	0·00498	0·00577	0·00654	0·00730	0·00805	0·00879	0·00951	0·01022	0·01092
4·8	0·00381	0·00453	0·00525	0·00596	0·00665	0·00734	0·00801	0·00867	0·00932	0·00996
5·0	0·00348	0·00415	0·00480	0·00545	0·00609	0·00672	0·00734	0·00794	0·00854	0·00913
5·5	0·00283	0·00337	0·00391	0·00444	0·00496	0·00548	0·00598	0·00648	0·00697	0·00746
6·0	0·00235	0·00280	0·00325	0·00369	0·00413	0·00456	0·00498	0·00540	0·00581	0·00621
6·5	0·00198	0·00237	0·00275	0·00312	0·00349	0·00385	0·00421	0·00457	0·00492	0·00526
7·0	0·00170	0·00203	0·00235	0·00267	0·00299	0·00330	0·00361	0·00392	0·00422	0·00451
8·0	0·00129	0·00154	0·00179	0·00203	0·00227	0·00251	0·00274	0·00298	0·00320	0·00343
9·0	0·00101	0·00121	0·00140	0·00159	0·00178	0·00197	0·00216	0·00234	0·00252	0·00270
10·0	0·00082	0·00097	0·00113	0·00129	0·00144	0·00159	0·00174	0·00189	0·00203	0·00218
12·0	0·00056	0·00067	0·00078	0·00089	0·00099	0·00110	0·00120	0·00130	0·00140	0·00150
15·0	0·00036	0·00043	0·00050	0·00057	0·00063	0·00070	0·00077	0·00083	0·00090	0·00096
20·0	0·00020	0·00024	0·00028	0·00032	0·00036	0·00039	0·00043	0·00047	0·00050	0·00054

TABLE 1 (*Continued*)

a	0·30	0·35	0·40	0·45	0·50	0·55	0·60	0·65	0·70	0·75
$H(a,0)$	0·73460	0·70150	0·67079	0·64225	0·61569	0·59093	0·56780	0·54618	0·52593	0·50694
w / k	1·00433	1·03523	1·06675	1·09887	1·13160	1·16491	1·19878	1·23321	1·26818	1·30367
0·0	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000
0·1	0·99277	0·99272	0·99267	0·99261	0·99256	0·99251	0·99245	0·99240	0·99235	0·99229
0·2	0·97145	0·97124	0·97104	0·97083	0·97063	0·97042	0·97022	0·97002	0·96982	0·96963
0·3	0 93702	0·93660	0·93618	0·93576	0·93534	0·93492	0·93451	0·93410	0·93370	0·93330
0·4	0·89110	0·89044	0·88978	0·88912	0·88847	0·88783	0·88719	0·88655	0·88593	0·88531
0·5	0·83571	0·83485	0·83399	0·83313	0·83229	0·83145	0·83062	0·82980	0·82899	0·82819
0·6	0·77321	0·77223	0·77125	0·77028	0·76932	0·76838	0·76744	0·76652	0·76562	0·76473
0·7	0·70606	0·70508	0·70412	0·70316	0·70222	0·70129	0·70038	0·69948	0·69860	0·69775
0·8	0·63669	0·63588	0·63508	0·63429	0·63351	0·63275	0·63200	0·63127	0·63056	0·62986
0·9	0·56736	0·56687	0·56639	0·56593	0·56547	0·56502	0·56459	0·56416	0·56375	0·56334
1·0	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000
1·1	0·43621	0·43682	0·43742	0·43800	0·43856	0·43911	0·43964	0·44016	0·44065	0·44113
1·2	0·37714	0·37846	0·37974	0·38098	0·38217	0·38333	0·38445	0·38553	0·38656	0·38756
1·3	0·32358	0·32564	0·32763	0·32956	0·33141	0·33320	0·33491	0·33656	0·33813	0·33964
1·4	0·27589	0·27870	0·28140	0·28400	0·28649	0·28888	0·29116	0·29335	0·29543	0·29741
1·5	0·23416	0·23766	0·24102	0·24424	0·24731	0·25024	0·25304	0·25570	0·25822	0·26063
1·6	0·19817	0·20230	0·20623	0·20998	0·21355	0·21695	0·22017	0·22323	0·22612	0·22887
1·7	0·16756	0·17219	0·17660	0·18079	0·18476	0·18852	0·19208	0·19544	0·19862	0·20162
1·8	0·14181	0·14684	0·15160	0·15612	0·16038	0·16441	0·16820	0·17178	0·17515	0·17832
1·9	0·12035	0·12565	0·13066	0·13539	0·13984	0·14404	0·14798	0·15169	0·15518	0·15845
2·0	0·10260	0·10806	0·11321	0·11805	0·12260	0·12687	0·13088	0·13464	0·13817	0·14148
2·2	0·07599	0·08147	0·08661	0·09143	0·09594	0·10016	0·10411	0·10781	0·11127	0·11450
2·4	0·05806	0·06327	0·06815	0·07271	0·07697	0·08096	0·08468	0·08816	0·09142	0·09446
2·6	0·04583	0·05062	0·05510	0·05929	0·06320	0·06686	0·07028	0·07348	0·07647	0·07927
2·8	0·03728	0·04160	0·04563	0·04941	0·05295	0·05626	0·05935	0·06225	0·06496	0·06751
3·0	0·03109	0·03494	0·03855	0·04193	0·04510	0·04807	0·05086	0·05347	0·05592	0·05822
3·2	0·02645	0·02987	0·03309	0·03611	0·03895	0·04161	0·04412	0·04647	0·04868	0·05075
3·4	0·02286	0·02590	0·02877	0·03147	0·03402	0·03641	0·03867	0·04079	0·04278	0·04466
3·6	0·02000	0·02272	0·02529	0·02771	0·03000	0·03216	0·03419	0·03611	0·03791	0·03961
3·8	0·01768	0·02012	0·02243	0·02461	0·02667	0·02862	0·03047	0·03220	0·03384	0·03538
4·0	0·01576	0·01796	0·02004	0·02202	0·02389	0·02566	0·02733	0·02891	0·03040	0·03181
4·2	0·01414	0·01614	0·01803	0·01982	0·02152	0·02313	0·02466	0·02610	0·02746	0·02875
4·4	0·01278	0·01459	0·01631	0·01795	0·01950	0·02097	0·02237	0·02369	0·02494	0·02612
4·6	0·01160	0·01326	0·01483	0·01633	0·01776	0·01911	0·02039	0·02160	0·02275	0·02384
4·8	0·01059	0·01210	0·01355	0·01493	0·01624	0·01748	0·01866	0·01978	0·02084	0·02184
5·0	0·00970	0·01110	0·01243	0·01370	0·01491	0·01606	0·01715	0·01818	0·01916	0·02009
5·5	0·00793	0·00908	0·01018	0·01123	0·01223	0·01318	0·01409	0·01495	0·01577	0·01654
6·0	0·00661	0·00757	0·00850	0·00938	0·01022	0·01102	0·01179	0·01251	0·01320	0·01386
6·5	0·00560	0·00642	0·00720	0·00795	0·00867	0·00936	0·01001	0·01063	0·01122	0·01178
7·0	0·00480	0·00551	0·00618	0·00683	0·00745	0·00804	0·00861	0·00914	0·00965	0·01014
8·0	0·00365	0·00419	0·00471	0·00520	0·00568	0·00613	0·00656	0·00698	0·00737	0·00774
9·0	0·00287	0·00330	0·00370	0·00410	0·00447	0·00483	0·00517	0·00550	0·00581	0·00610
10·0	0·00232	0·00266	0·00299	0·00331	0·00361	0·00390	0·00418	0·00445	0·00470	0·00494
12·0	0·00160	0·00184	0·00207	0·00229	0·00250	0·00270	0·00290	0·00308	0·00326	0·00342
15·0	0·00102	0·00117	0·00132	0·00146	0·00160	0·00173	0·00185	0·00197	0·00208	0·00219
20·0	0·00057	0·00066	0·00074	0·00082	0·00090	0·00097	0·00104	0·00111	0·00117	0·00123

TABLE 1 (*Continued*)

a	0·80	0·85	0·90	0·95	1·00	1·10	1·20	1·30	1·40	1·50
$H(a,0)$	0·48910	0·47233	0·45653	0·44164	0·42758	0·40173	0·37854	0·35764	0·33874	0·32159
w	1·33968	1·37618	1·41315	1·45059	1·48848	1·56553	1·64419	1·72434	1·80586	1·88866
k										
0·0	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000
0·1	0·99224	0·99219	0·99214	0·99209	0·99204	0·99195	0·99186	0·99177	0·99169	0·99161
0·2	0·96943	0·96924	0·96905	0·96887	0·96868	0·96833	0·96798	0·96766	0·96734	0·96704
0·3	0·93290	0·93251	0·93213	0·93175	0·93138	0·93066	0·92997	0·92931	0·92868	0·92808
0·4	0·88470	0·88410	0·88351	0·88294	0·88237	0·88127	0·88022	0·87923	0·87828	0·87738
0·5	0·82741	0·82664	0·82589	0·82515	0·82443	0·82304	0·82171	0·82046	0·81927	0·81816
0·6	0·76386	0·76301	0·76218	0·76136	0·76057	0·75904	0·75760	0·75624	0·75496	0·75376
0·7	0·69691	0·69609	0·69529	0·69451	0·69375	0·69230	0·69094	0·68966	0·68847	0·68736
0·8	0·62918	0·62852	0·62788	0·62726	0·62665	0·62550	0·62443	0·62343	0·62250	0·62164
0·9	0·56295	0·56257	0·56220	0·56185	0·56151	0·56086	0·56026	0·55970	0·55918	0·55871
1·0	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000
1·1	0·44160	0·44204	0·44246	0·44287	0·44327	0·44400	0·44467	0·44529	0·44585	0·44636
1·2	0·38851	0·38942	0·39030	0·39114	0·39193	0·39342	0·39478	0·39601	0·39712	0·39813
1·3	0·34108	0·34246	0·34377	0·34502	0·34621	0·34841	0·35041	0·35220	0·35382	0·35528
1·4	0·29930	0·30110	0·30281	0·30443	0·30597	0·30881	0·31136	0·31365	0·31570	0·31754
1·5	0·26290	0·26506	0·26711	0·26904	0·27087	0·27424	0·27725	0·27994	0·28234	0·28448
1·6	0·23146	0·23391	0·23622	0·23840	0·24046	0·24424	0·24760	0·25058	0·25324	0·25560
1·7	0·20444	0·20711	0·20961	0·21198	0·21420	0·21827	0·22187	0·22506	0·22790	0·23041
1·8	0·18131	0·18412	0·18675	0·18923	0·19156	0·19581	0·19956	0·20288	0·20582	0·20842
1·9	0·16152	0·16440	0·16711	0·16965	0·17203	0·17637	0·18020	0·18357	0·18655	0·18920
2·0	0·14458	0·14749	0·15021	0·15277	0·15516	0·15951	0·16334	0·16672	0·16970	0·17235
2·2	0·11753	0·12036	0·12302	0·12550	0·12782	0·13204	0·13575	0·13902	0·14191	0·14446
2·4	0·09730	0·09997	0·10246	0·10479	0·10697	0·11094	0·11442	0·11750	0·12021	0·12261
2·6	0·08188	0·08433	0·08663	0·08878	0·09079	0·09445	0·09767	0·10051	0·10303	0·10525
2·8	0·06989	0·07212	0·07421	0·07617	0·07801	0·08135	0·08431	0·08691	0·08922	0·09127
3·0	0·06037	0·06240	0·06429	0·06608	0·06775	0·07079	0·07349	0·07587	0·07798	0·07985
3·2	0·05270	0·05453	0·05625	0·05787	0·05939	0·06216	0·06461	0·06679	0·06871	0·07043
3·4	0·04642	0·04808	0·04964	0·05111	0·05249	0·05501	0·05725	0·05923	0·06099	0·06256
3·6	0·04121	0·04272	0·04414	0·04547	0·04673	0·04903	0·05107	0·05288	0·05450	0·05593
3·8	0·03684	0·03821	0·03951	0·04072	0·04187	0·04397	0·04584	0·04750	0·04898	0·05030
4·0	0·03314	0·03439	0·03557	0·03669	0·03774	0·03966	0·04137	0·04290	0·04426	0·04547
4·2	0·02997	0·03112	0·03220	0·03322	0·03419	0·03595	0·03753	0·03893	0·04018	0·04130
4·4	0·02724	0·02829	0·02929	0·03023	0·03111	0·03274	0·03419	0·03549	0·03664	0·03767
4·6	0·02487	0·02584	0·02676	0·02762	0·02844	0·02994	0·03128	0·03248	0·03355	0·03450
4·8	0·02279	0·02369	0·02454	0·02534	0·02610	0·02749	0·02873	0·02984	0·03083	0·03172
5·0	0·02097	0·02180	0·02259	0·02333	0·02403	0·02533	0·02648	0·02751	0·02843	0·02925
5·5	0·01727	0·01797	0·01862	0·01925	0·01983	0·02092	0·02188	0·02275	0·02352	0·02421
6·0	0·01448	0·01506	0·01562	0·01615	0·01665	0·01756	0·01839	0·01912	0·01978	0·02037
6·5	0·01231	0·01281	0·01329	0·01374	0·01417	0·01496	0·01567	0·01630	0·01686	0·01737
7·0	0·01060	0·01103	0·01145	0·01184	0·01221	0·01289	0·01351	0·01406	0·01455	0·01499
8·0	0·00809	0·00843	0·00875	0·00905	0·00934	0·00987	0·01034	0·01077	0·01115	0·01149
9·0	0·00639	0·00665	0·00691	0·00715	0·00737	0·00779	0·00817	0·00851	0·00881	0·00908
10·0	0·00517	0·00538	0·00559	0·00578	0·00597	0·00631	0·00662	0·00689	0·00714	0·00736
12·0	0·00358	0·00373	0·00388	0·00401	0·00414	0·00438	0·00460	0·00479	0·00496	0·00512
15·0	0·00229	0·00239	0·00248	0·00257	0·00265	0·00280	0·00294	0·00306	0·00318	0·00328
20·0	0·00129	0·00134	0·00139	0·00144	0·00149	0·00159	0·00165	0·00172	0·00179	0·00184

TABLE 1 (*Continued*)

a	1·60	1·70	1·80	1·90	2·00	2·20	2·40	2·60	2·80	3·00
$H(a,0)$	0·30595	0·29166	0·27856	0·26651	0·25540	0·23559	0·21850	0·20361	0·19055	0·17900
w	1·97262	2·05767	2·14370	2·23065	2·31843	2·49624	2·67667	2·85932	3·04386	3·23002
k										
0·0	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000
0·1	0·99153	0·99146	0·99139	0·99132	0·99126	0·99115	0·99105	0·99096	0·99089	0·99082
0·2	0·96676	0·96649	0·96624	0·96600	0·96577	0·96536	0·96499	0·96466	0·96438	0·96412
0·3	0·92751	0·92698	0·92648	0·92600	0·92555	0·92474	0·92402	0·92339	0·92283	0·92234
0·4	0·87654	0·87574	0·87499	0·87429	0·87363	0·87243	0·87138	0·87046	0·86965	0·86894
0·5	0·81711	0·81612	0·81520	0·81434	0·81353	0·81208	0·81081	0·80971	0·80874	0·80790
0·6	0·75264	0·75159	0·75062	0·74971	0·74886	0·74735	0·74604	0·74491	0·74392	0·74307
0·7	0·68632	0·68536	0·68447	0·68365	0·68288	0·68152	0·68035	0·67935	0·67849	0·67774
0·8	0·62085	0·62011	0·61944	0·61881	0·61824	0·61722	0·61635	0·61561	0·61498	0·61444
0·9	0·55828	0·55787	0·55751	0·55717	0·55686	0·55632	0·55586	0·55547	0·55514	0·55486
1·0	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000
1·1	0·44682	0·44724	0·44762	0·44797	0·44828	0·44883	0·44929	0·44967	0·45000	0·45027
1·2	0·39904	0·39986	0·40060	0·40128	0·40189	0·40295	0·40382	0·40455	0·40517	0·40568
1·3	0·35659	0·35777	0·35884	0·35980	0·36067	0·36216	0·36340	0·36442	0·36528	0·36600
1·4	0·31919	0·32067	0·32200	0·32319	0·32427	0·32613	0·32765	0·32891	0·32996	0·33085
1·5	0·28639	0·28811	0·28964	0·29102	0·29226	0·29439	0·29613	0·29757	0·29876	0·29977
1·6	0·25771	0·25960	0·26128	0·26279	0·26415	0·26647	0·26837	0·26993	0·27123	0·27232
1·7	0·23265	0·23465	0·23643	0·23803	0·23946	0·24191	0·24391	0·24556	0·24692	0·24807
1·8	0·21074	0·21280	0·21464	0·21629	0·21776	0·22028	0·22234	0·22403	0·22544	0·22661
1·9	0·19154	0·19363	0·19549	0·19716	0·19865	0·20120	0·20327	0·20498	0·20640	0·20759
2·0	0·17469	0·17677	0·17863	0·18029	0·18178	0·18432	0·18639	0·18809	0·18951	0·19070
2·2	0·14673	0·14874	0·15053	0·15214	0·15358	0·15603	0·15804	0·15969	0·16106	0·16221
2·4	0·12475	0·12664	0·12834	0·12985	0·13121	0·13353	0·13542	0·13698	0·13828	0·13936
2·6	0·10723	0·10899	0·11057	0·11197	0·11324	0·11540	0·11716	0·11862	0·11983	0·12084
2·8	0·09309	0·09471	0·09616	0·09746	0·09863	0·10062	0·10226	0·10360	0·10472	0·10567
3·0	0·08152	0·08301	0·08434	0·08554	0·08661	0·08845	0·08995	0·09119	0·09223	0·09310
3·2	0·07196	0·07332	0·07454	0·07564	0·07662	0·07831	0·07970	0·08084	0·08179	0·08259
3·4	0·06396	0·06521	0·06633	0·06734	0·06824	0·06979	0·07107	0·07212	0·07300	0·07374
3·6	0·05722	0·05836	0·05939	0·06032	0·06115	0·06257	0·06374	0·06471	0·06552	0·06620
3·8	0·05148	0·05253	0·05348	0·05432	0·05509	0·05640	0·05748	0·05837	0·05912	0·05975
4·0	0·04655	0·04752	0·04839	0·04917	0·04988	0·05109	0·05209	0·05291	0·05360	0·05418
4·2	0·04230	0·04319	0·04399	0·04472	0·04537	0·04649	0·04741	0·04817	0·04881	0·04934
4·4	0·03860	0·03942	0·04017	0·04083	0·04144	0·04247	0·04332	0·04403	0·04462	0·04512
4·6	0·03536	0·03612	0·03681	0·03743	0·03799	0·03895	0·03974	0·04040	0·04095	0·04141
4·8	0·03251	0·03322	0·03386	0·03444	0·03496	0·03585	0·03658	0·03719	0·03771	0·03814
5·0	0·02999	0·03065	0·03125	0·03178	0·03227	0·03310	0·03378	0·03435	0·03483	0·03523
5·5	0·02484	0·02539	0·02589	0·02635	0·02675	0·02746	0·02804	0·02852	0·02892	0·02926
6·0	0·02090	0·02138	0·02180	0·02219	0·02254	0·02314	0·02363	0·02404	0·02439	0·02468
6·5	0·01783	0·01824	0·01861	0·01894	0·01924	0·01976	0·02019	0·02054	0·02084	0·02109
7·0	0·01539	0·01574	0·01606	0·01635	0·01662	0·01707	0·01744	0·01775	0·01801	0·01823
8·0	0·01180	0·01207	0·01232	0·01255	0·01275	0·01310	0·01339	0·01363	0·01383	0·01400
9·0	0·00933	0·00955	0·00975	0·00993	0·01009	0·01037	0·01060	0·01079	0·01095	0·01109
10·0	0·00756	0·00774	0·00790	0·00805	0·00818	0·00841	0·00860	0·00876	0·00889	0·00900
12·0	0·00526	0·00538	0·00549	0·00560	0·00569	0·00585	0·00598	0·00609	0·00618	0·00626
15·0	0·00337	0·00345	0·00352	0·00359	0·00365	0·00375	0·00383	0·00391	0·00396	0·00402
20·0	0·00189	0·00194	0·00198	0·00202	0·00205	0·00211	0·00216	0·00220	0·00223	0·00226

TABLE 1 (*Continued*)

a	3·50	4·00	4·50	5·00	6·00	7·00	8·00	9·00	10·00	Infinity
$H(a,0)$	0·15529	0·13700	0·12248	0·11070	0·09278	0·07980	0·06999	0·06231	0·05614	Lorentz
w / k	3·70112	4·17840	4·66013	5·14515	6·12213	7·10531	8·09252	9·08245	10·07435	
0·0	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000
0·1	0·99067	0·99057	0·99049	0·99042	0·99034	0·99028	0·99024	0·99021	0·99019	0·99010
0·2	0·96361	0·96322	0·96293	0·96270	0·96238	0·96218	0·96204	0·96194	0·96187	0·96154
0·3	0·92134	0·92060	0·92004	0·91961	0·91901	0·91862	0·91836	0·91817	0·91804	0·91743
0·4	0·86752	0·86647	0·86568	0·86508	0·86424	0·86371	0·86334	0·86309	0·86290	0·86207
0·5	0·80623	0·80501	0·80410	0·80341	0·80245	0·80184	0·80143	0·80114	0·80093	0·80000
0·6	0·74139	0·74017	0·73927	0·73859	0·73766	0·73706	0·73667	0·73639	0·73619	0·73529
0·7	0·67628	0·67524	0·67447	0·67389	0·67311	0·67261	0·67228	0·67205	0·67188	0·67114
0·8	0·61338	0·61263	0·61209	0·61168	0·61113	0·61078	0·61055	0·61038	0·61027	0·60976
0·9	0·55432	0·55393	0·55366	0·55345	0·55317	0·55300	0·55288	0·55280	0·55274	0·55249
1·0	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000	0·50000
1·1	0·45079	0·45115	0·45141	0·45160	0·45186	0·45202	0·45213	0·45220	0·45226	0·45249
1·2	0·40667	0·40734	0·40783	0·40819	0·40867	0·40897	0·40917	0·40931	0·40941	0·40984
1·3	0·36737	0·36831	0·36899	0·36948	0·37015	0·37056	0·37083	0·37102	0·37116	0·37175
1·4	0·33252	0·33367	0·33449	0·33509	0·33590	0·33640	0·33673	0·33696	0·33712	0·33784
1·5	0·30167	0·30297	0·30390	0·30459	0·30550	0·30607	0·30644	0·30670	0·30688	0·30769
1·6	0·27438	0·27579	0·27680	0·27754	0·27853	0·27914	0·27954	0·27982	0·28003	0·28090
1·7	0·25023	0·25172	0·25277	0·25355	0·25459	0·25523	0·25565	0·25594	0·25615	0·25707
1·8	0·22883	0·23036	0·23144	0·23224	0·23330	0·23396	0·23439	0·23469	0·23491	0·23585
1·9	0·20984	0·21137	0·21247	0·21327	0·21435	0·21501	0·21545	0·21575	0·21597	0·21692
2·0	0·19293	0·19446	0·19556	0·19636	0·19743	0·19809	0·19853	0·19883	0·19905	0·20000
2·2	0·16437	0·16586	0·16692	0·16770	0·16874	0·16938	0·16981	0·17010	0·17031	0·17123
2·4	0·14142	0·14283	0·14383	0·14457	0·14556	0·14617	0·14657	0·14685	0·14705	0·14793
2·6	0·12276	0·12408	0·12502	0·12571	0·12664	0·12721	0·12759	0·12786	0·12804	0·12887
2·8	0·10745	0·10867	0·10954	0·11019	0·11105	0·11158	0·11194	0·11218	0·11236	0·11312
3·0	0·09474	0·09588	0·09669	0·09728	0·09808	0·09857	0·09890	0·09913	0·09930	0·10000
3·2	0·08411	0·08516	0·08590	0·08645	0·08719	0·08765	0·08795	0·08816	0·08831	0·08897
3·4	0·07513	0·07610	0·07679	0·07729	0·07798	0·07840	0·07868	0·07903	0·07901	0·07962
3·6	0·06749	0·06838	0·06902	0·06949	0·07012	0·07051	0·07076	0·07094	0·07107	0·07163
3·8	0·06094	0·06176	0·06235	0·06278	0·06336	0·06372	0·06402	0·06413	0·06425	0·06477
4·0	0·05528	0·05604	0·05658	0·05698	0·05752	0·05786	0·05808	0·05823	0·05834	0·05882
4·2	0·05036	0·05107	0·05157	0·05194	0·05244	0·05275	0·05296	0·05310	0·05320	0·05365
4·4	0·04607	0·04672	0·04719	0·04753	0·04800	0·04855	0·04847	0·04861	0·04870	0·04912
4·6	0·04229	0·04290	0·04333	0·04365	0·04408	0·04435	0·04453	0·04465	0·04474	0·04513
4·8	0·03895	0·03952	0·03992	0·04022	0·04063	0·04088	0·04104	0·04115	0·04124	0·04160
5·0	0·03599	0·03652	0·03690	0·03718	0·03758	0·03779	0·03794	0·03805	0·03813	0·03846
5·5	0·02991	0·03035	0·03067	0·03091	0·03123	0·03143	0·03156	0·03165	0·03171	0·03200
6·0	0·02523	0·02562	0·02589	0·02614	0·02637	0·02654	0·02665	0·02673	0·02678	0·02703
6·5	0·02157	0·02190	0·02214	0·02231	0·02255	0·02270	0·02279	0·02286	0·02291	0·02312
7·0	0·01864	0·01893	0·01914	0·01929	0·01950	0·01963	0·01971	0·01977	0·01981	0·02000
8·0	0·01433	0·01455	0·01471	0·01483	0·01499	0·01509	0·01516	0·01521	0·01524	0·01538
9·0	0·01135	0·01153	0·01166	0·01175	0·01188	0·01196	0·01202	0·01205	0·01208	0·01220
10·0	0·00921	0·00936	0·00946	0·00954	0·00965	0·00971	0·00975	0·00978	0·00981	0·00990
12·0	0·00641	0·00651	0·00659	0·00664	0·00672	0·00676	0·00679	0·00681	0·00683	0·00690
15·0	0·00411	0·00418	0·00423	0·00426	0·00431	0·00434	0·00436	0·00437	0·00438	0·00442
20·0	0·00232	0·00235	0·00238	0·00240	0·00243	0·00245	0·00246	0·00246	0·00247	0·00249

lengthy. Behaviour of the tabulated values is illustrated by the representative curves shown in Figure 4.

It should be observed that the useful tables of van de Hulst and Reesinck (1947) are in effect a short inverse tabulation of the present work, and give k as a function of a for selected values of $H(a,v)/H(a,0)$; their notation is related to that used here by

$$\beta_1,\ \beta_2,\ c,\ h,\ b/h \rightarrow \tfrac{1}{2}\Delta\nu_L,\ \tfrac{1}{2}\Delta\nu_G',\ H(a,0),\ w\Delta\nu_G',\ k.$$

The " Working graph of the parameters " given in their Figure 2 has been checked and is correct to one scale division except for the curve of $b_{0\cdot 1}/h$ which is in error by as much as 5 per cent. Since this curve is of some importance in data analysis the following values of $b_{0\cdot 1}/h$ may be used to sketch in a correction:

β_1/h ..	0·200	0·225	0·250	0·275	0·300	0·325	0·350	0·375	0·400	0·425	0·450	0·475	0·500
$b_{0\cdot 1}/h$ ..	2·11	2·17	2·23	2·30	2·37	2·45	2·52	2·60	2·68	2·76	2·84	2·92	3·00

The columns in their Table 1, headed 0·2 and 0·05, and corresponding to the fifth- and twentieth-widths respectively, have also been checked and found to be essentially correct.

Most of the numerical work reported here was carried out on SILLIAC, the electronic digital computer of the Adolph Basser Computing Laboratory, School of Physics, University of Sydney. Because of the large number of small values of a, the computing time was relatively long, and preparation of Table 1 required 35 minutes of machine time.

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