THE STRUCTURE OF A STREAM OF ELECTRONS AND IONS DRIFTING AND DIFFUSING IN A GAS WHEN IONIZATION BY COLLISION AND MOLECULAR ATTACHMENT ARE PRESENT

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Summary

The theory is developed of the structure of a stream of electrons and ions drifting under the action of a uniform field and diffusing in a gas when either or both ionization by collision and electron attachment are present. The cases considered include a point source and a line source, and in the latter case the influence of a magnetic field is discussed.

The investigation provides a theoretical basis for methods of measuring electron attachment, about to be put into practice.

I. Introduction

In the well-known method devised by Townsend for studying the motions of slow electrons in gases (Townsend 1916) the ratio W/D of the drift velocity $\vec{W}$ to the coefficient of diffusion D is found by measuring the distribution of electrons in a stream moving through the gas in a steady state of motion in a uniform electric field. When separate measurements are made of the drift velocity $\vec{W}$ for a range of values of the ratio Z/p, of the electric field strength to the pressure p of the gas, of which ratio W and W/pD are functions, then the dependence of pD upon Z/p may be deduced.

From the quantities W/D, W, D, and p the microscopic details of the motion, such as mean energy of agitation Q of the electrons, the collisional cross sections of the molecules, and the mean rate at which electrons lose energy in collisions with molecules, are found as functions of Z/p (e.g. Healey and Reed 1941; Townsend 1947; Loeb 1955; Huxley 1957).

The experimental procedure is essentially as follows: a stream of electrons, having already acquired a steady state of motion in a uniform field Z, enters the diffusion chamber through an aperture (circular or rectangular) and moves through the gas under the same uniform field Z to a receiving electrode which is divided into separately insulated sections. In one method of division there is a central disk surrounded by one or more concentric annuli; in another, there is a central strip flanked by other strips or the remaining portions of the electrode.

The ratio R of the currents to any two portions of the electrode can be calculated in terms of W/D for a given apparatus, consequently, when the ratio of the currents is measured, W/D may be immediately derived from it.

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If the centre of the aperture of entry of the stream is taken as the origin of coordinates and the direction of drift $\vec{W}$ as the axis Oz, then the differential equation for the distribution of the concentration n of the electrons in unit volume in the steady stream is

$$\nabla^2 n = (W/D)\partial n/\partial z, \qquad (1)$$

a solution of which, satisfying the boundary conditions, is found and from it the ratio R (dimensions of apparatus, W/D) is calculated. In Townsend's work, since the dimensions of the aperture of entry to the diffusion chamber could not usually be considered as mathematically small in comparison with the dimensions of the chamber, the appropriate solutions of (**1**) were in the form of infinite series which were not convenient for calculating the ratio R. However, if the aperture is made small it may be regarded as a point source and simple solutions are found in the form of pole or dipole sources which with suitable image sources satisfy the boundary condition $n=0$ over the receiving electrode. The calculation of R is then a simple matter (Huxley and Zaazou 1949 ; Crompton, Huxley, and Sutton 1953).

In practice the pole solution (Crompton and Huxley 1955)

$$\left.\begin{aligned} n &= (S/4\pi Dr) \exp\,[\lambda(z-r)], \\ 2\lambda &= W/D, \end{aligned}\right\} \qquad (2)$$

together with a suitable image source accurately describes the actual distribution of n in the diffusion chamber.

The quantity S is the rate at which the pole source emits electrons.

Equation (**1**) and its solutions refer to situations in which electrons are neither lost to the stream by attachment to molecules to form negative ions nor gained by ionization of the molecules by collision. When either or both these processes are present to an important degree equation (**1**) and its solutions no longer accurately describe the distribution of n in the stream and that of the current over the receiving electrode. When the process of electron attachment occurs the stream becomes enriched with negative ions whose distribution is not precisely known. Many years ago V. A. Bailey (Healey and Reed 1941 ; Loeb 1955) devised methods for measuring the coefficient of attachment of electrons in such streams in which the unknown functions representing the distribution of negative ions, both entering the chamber and formed in the chamber by attachment, were eliminated by special but somewhat complicated experimental procedures.

Another special situation of interest is that in which ionization by collision is present and it is required to measure the mean energy of agitation of the electrons.

In view of the practical importance of measurements of coefficients of attachment and of ionization, in what follows the theory of the diffusing stream of electrons in a gas is extended to include the effects of electron attachment and ionization by collision.

New experimental procedures for measuring attachment coefficients and agitational energies Q in the presence of ionization are also discussed.

II. Extension of the Theory to include the Effects of Electron Attachment and Ionization by Collision

Consider the distribution of the concentration n of the electrons in a steady stream in which W is drift speed and D the coefficient of diffusion of the electrons. Adopt a system of coordinates in which the axis Oz is parallel to the direction of $\vec{W}$.

Let ν be the collisional frequency of an electron moving with a velocity of agitation c and suppose that the probability of attachment in single collisions of such electrons is f. It follows that the number of attachments that occur in time dt in an element of volume $d\tau$ containing $n d\tau$ electrons is $(\overline{f\nu})n d\tau dt$, where $(\overline{f\nu})$ is the mean value of $f\nu$ averaged over all speeds c. Similarly, if g is the probability of ionization by collision in single collisions, the number of new electrons and positive ions produced in $d\tau$ in time dt is $(\overline{g\nu})n d\tau dt$. The equation of continuity for n is therefore

$$dn/dt = D\nabla^2 n - \vec{W}\cdot \text{grad}\, n - (\overline{f\nu} - \overline{g\nu})n. \quad \ldots\ldots\ldots \quad (3)$$

In a steady stream $dn/dt=0$, and when $W=W_z$ equation (3) reduces to

$$\nabla^2 n = 2\lambda \partial n/\partial z + 2\lambda\alpha n, \quad \ldots\ldots\ldots \quad (4)$$

where $2\lambda = W/D$ and $2\lambda\alpha = (\overline{f\nu} - \overline{g\nu})/D$. It follows from (4) that

$$\alpha = (\overline{f\nu} - \overline{g\nu})/W = \alpha_a - \alpha_i. \quad \ldots\ldots\ldots \quad (5)$$

The concentration N in unit volume of the negative ions in the stream satisfies the differential equation

$$D_1\nabla^2 N = W_1 \partial N/\partial z - \alpha_a W n, \quad \ldots\ldots\ldots \quad (6)$$

where D_1 and W_1 are respectively the coefficient of diffusion and speed of drift of negative ions.

In what follows it is assumed that electrons enter the diffusion chamber through a small aperture which may be regarded as a point source.

Place the origin of coordinates at the aperture and let the axis Oz be parallel to the direction of $\vec{W}$ (that is, antiparallel to the field Z). The receiving electrode coincides with the plane $z=h$. Radial distances from the origin are denoted by $r=(x^2+y^2+z^2)^{\frac{1}{2}}$.

(a) *Distribution of* n

Write $n = e^{\lambda z} V$ in equation (4); then after reduction this equation becomes

$$\nabla^2 V = (\lambda^2 + 2\lambda\alpha)V = \mu^2 V. \quad \ldots\ldots\ldots \quad (7)$$

The solution of this equation representing a simple pole source is

$$V = \text{constant} \times e^{-\mu r}/r,$$

and the corresponding solution of equation (4) is

$$n = (s/4\pi D)e^{\lambda z - \mu r}/r, \quad \ldots\ldots\ldots \quad (8)$$

in which the constant s is the number of electrons emitted by the source in unit time.

In order to make $n=0$ over the plane of the receiving electrode, $z=h$, when the electrode is absent, place an image source of suitable strength and negative sign at the point $(0,0,2h)$ to give the following distribution of n:

$$n=(s/4\pi D)e^{\lambda z}(e^{-\mu r}/r-e^{-\mu r'}/r'), \qquad \ldots\ldots\ldots\ldots \quad (9)$$

where

$$r^2=\rho^2+z^2,\ r'^2=\rho^2+(2h-z)^2,\ \rho^2=x^2+y^2.$$

Equation (9) gives the distribution of n within the diffusion chamber to a good approximation in practice, since n is small over the plane $z=0$ except near the origin (e.g. Crompton and Huxley 1955).

In order to calculate the current to an annular division $a\leqslant\rho\leqslant b$ of the receiving electrode ($z=h$; $n=0$) it is necessary first to find $(\partial n/\partial z)_\rho$ at the plane $z=h$.

When ρ is constant, $r\mathrm{d}r=z\mathrm{d}z$; $r'\mathrm{d}r'=-(2h-z)\mathrm{d}z$.

From equation (9)

$$\left(\frac{\partial n}{\partial z}\right)_\rho=\lambda n+\frac{s}{4\pi D}\left[\frac{z}{r}\cdot\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{e^{-\mu r}}{r}\right)+\left(\frac{2h-z}{r'}\right)\frac{\mathrm{d}}{\mathrm{d}r'}\left(\frac{e^{-\mu r'}}{r'}\right)\right]e^{\lambda z}.$$

On the plane $z=h$; $r=r'$; $n=0$; $\rho\mathrm{d}\rho=r\mathrm{d}r$,

$$\left(\frac{\partial n}{\partial z}\right)_{z=h}=\frac{s}{4\pi D}\cdot\frac{e^{\lambda h}2h}{r}\cdot\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{e^{-\mu r}}{r}\right).$$

The current carried by diffusion to the annulus $a\leqslant\rho\leqslant b$ is

$$\begin{aligned}(i)_{ab}&=-2\pi\varepsilon D\int_a^b\left(\frac{\partial n}{\partial z}\right)_{z-h}\rho\mathrm{d}\rho\\&=s\varepsilon h e^{\lambda h}\int_{d_a}^{d_b}\mathrm{d}\left(\frac{e^{-\mu r}}{r}\right)\\&=s\varepsilon h e^{\lambda h}[e^{-\mu d_a}/d_a-e^{-\mu d_b}/d_b],\end{aligned} \qquad \ldots\ldots\ldots\ldots \quad (10)$$

where ε is the electronic charge; $d_a^2=a^2+h^2$; $d_b^2=b^2+h^2$.

The total electronic current to the whole electrode ($a=0$; $b=\infty$) is

$$i=i_{0\infty}=\varepsilon s e^{-(\mu-\lambda)h}=i_0e^{(\lambda-\mu)h} \quad (i_0=\varepsilon s). \qquad \ldots\ldots\ldots \quad (11)$$

But from equation (7), $\mu^2-\lambda^2=2\lambda\alpha$; $(\mu-\lambda)=2\lambda\alpha/(\mu+\lambda)$, so that when $2\lambda\alpha\leqslant\lambda^2$, $\mu\simeq\lambda$, $\mu-\lambda\simeq\alpha$.

In this event,

$$i=i_{0\infty}=i_0e^{-\alpha h}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

(b) *Ionization by Collision*

When $\alpha_a=0$, then $\alpha=-\alpha_i$ and equations (11) and (12) become

$$i=i_0e^{(\lambda-\mu)h};\ i\simeq i_0e^{\alpha_i h}, \qquad \ldots\ldots\ldots\ldots\ldots \quad (13)$$

which is Townsend's well-known formula for the growth of current with increasing electrode separation h in a uniform field when ionization by collision is present.

It is seen that the quantity found from the exponential growth of currents between plane parallel electrodes is $\lambda-\mu$ and not α_i. In practice the difference between $\lambda-\mu$ and α_i would usually be unimportant.

(c) *Measurement of* μ *when* $\alpha_a=0$

The current to an annulus of the receiving electrode $a\leqslant\rho\leqslant b$ is (equation **10**)

$$i_{ab}=\varepsilon she^{\lambda h}[e^{-\mu d_a}/d_a-e^{-\mu d_b}/d_b], \qquad \textbf{(14)}$$

and that to another annulus $c\leqslant\rho\leqslant k$ $(c\geqslant b)$

$$i_{ck}=\varepsilon she^{\lambda h}[e^{-\mu d_c}/d_c-e^{-\mu d_k}/d_k]. \qquad \textbf{(15)}$$

The ratio R of these currents is independent of λ and εs and is a function of the dimensions h, a, b, c, k and of μ. Thus μ may be found by measuring R in an apparatus in which h, a, b, c, and k are known. Particular arrangements are: $a=0$, $b=c$, $k=\infty$; or $a\neq0$, $b=c$, $k=\infty$. In such cases the formula for R is relatively simple.

Thus by separate measurements (at the same values of Z/p) of $\lambda-\mu$ and of μ, both λ and μ and therefore $\alpha_i=(\lambda^2-\mu^2)/2\lambda$ can be found accurately. The mean agitational energy Q of the electrons in the stream is derived from $\lambda=W/2D$. When both α_a and α_i are zero $\alpha=0$ and $\lambda=\mu$, which is the case that is most frequently investigated.

III. The Distribution of Negative Ions

This distribution is to be obtained as a solution of equation (**6**) in which n is distributed within the diffusion chamber according to equation (**9**). A formal solution of equation (**6**) is readily obtained either by the standard methods of potential theory using Green's formulae or from physical considerations as follows.

Let $d\tau$ be an element of volume at the point $P(x,y,z)$ containing $n d\tau$ electrons. This element acts as a point source of negative ions whose strength (rate of emission) is $\alpha_a W n d\tau$. This source therefore contributes to the total concentration N_0 of negative ions at a point $Q(x_0,y_0,z_0)$ an amount

$$dN=(\alpha_a W n d\tau/4\pi D_1)\exp[-\lambda_1\{r_1-(z_0-z)\}]/r_1,$$

where $r_1=[(x_0-x)^2+(y_0-y)^2+(z_0-z)^2]^{\frac{1}{2}}=$distance PQ.

The total concentration N_0 is obtained by integrating over all elementary sources $\alpha_a W n d\tau$ throughout the whole of space, devoid of electrodes, into which the original source of electrons s and its image at $(0,0,h)$ are emitting. Whence,

$$N_0=\frac{\alpha_a W}{4\pi D_1}\int_\infty \frac{n\exp[-\lambda_1\{r_1-(z_0-z)\}]d\tau}{r_1}, \qquad \textbf{(16)}$$

in which n is given by equation (**9**).

Equation (**16**) is the particular solution of equation (**6**) and it gives the distribution of the concentration N_0 of negative ions formed by electron attachment.

To this solution may be added a complementary function of equation (**6**) to allow for a stream of negative ions that may enter the diffusion chamber through the aperture of entry of the electrons. This complementary function is

$$N_c = \frac{C}{4\pi D_1} \frac{\exp\{-\lambda_1(r_0 - z_0)\}}{r_0}, \qquad \textbf{(17)}$$

where C is the strength of the source and $r_0^2 = x_0^2 + y_0^2 + z_0^2$.

If the point $Q(x_0, y_0, z_0)$ lies on the receiving electrode, the flux of ions to an element of surface $\mathrm{d}S$ containing Q will be taken to be

$$\mathrm{d}I = (N_0 + N_c) W_1 \mathrm{d}S = N W_1 \mathrm{d}S.$$

When the point Q is sufficiently removed from the axis Oz (i.e. from the centre of the electrode) N_c becomes inappreciable and $N = N_0$.

In practice the receiving electrode would be divided into a central disk $\rho = a$ and two surrounding annular regions $a \leqslant \rho \leqslant b$, $b \leqslant \rho \leqslant \infty$, the radius a of the disk being chosen large enough to ensure that all but an unimportant proportion of the ions in the stream N_c reach the central disk. In order to appreciate the rough order of magnitude of the radius a consider a practical case in which the stream N_c is of the form described in equation (**17**).

According to equation (**14**) the proportion of the current carried by the ions N_c that impinge on the disk $0 \leqslant \rho \leqslant a$ is $(\lambda = \mu = \lambda_1)$

$$R_c = 1 - h \exp\{-\lambda_1(d_a - h)\}/d_a.$$

But for ions, $2\lambda_1 = W_1/D_1 = Z\varepsilon/kT$ (k is Boltzmann's constant, T is the absolute temperature, and ε is the ionic charge).

When $T = 288$ °K, and Z is expressed in volt/cm this relationship reduces to

$$\lambda_1 = 20 \cdot 15 Z. \qquad \textbf{(18)}$$

Since $d_a = h(1 + a^2/h^2)^{\frac{1}{2}}$, it follows that

$$\frac{\exp[-20 \cdot 15 Zh\{(1 + a^2/h^2)^{\frac{1}{2}} - 1\}]}{(1 + a^2/h^2)^{\frac{1}{2}}} = 1 - R.$$

Suppose, as would be so in practice, that $(a/h)^2 \ll 1$, then the condition becomes

$$a \simeq \{(h/10Z) \ln [1/(1-R)]\}^{\frac{1}{2}}. \qquad \textbf{(19)}$$

Suppose $R = 90$ per cent., then $a = 0 \cdot 48 (h/Z)^{\frac{1}{2}}$; if $R = 95$ per cent., then $a = 0 \cdot 3 (h/Z)^{\frac{1}{2}}$; if $R = 99$ per cent., $a = 0 \cdot 68 (h/Z)^{\frac{1}{2}}$.
Take, for instance, $h = 5$ cm, $Z = 10$ V/cm, then for $R = 90$ per cent., $a = 0 \cdot 33$ cm; for $R = 95$ per cent., $a = 0 \cdot 39$ cm.

The stream of electrons spreads, in general much more widely, since for electrons $\lambda = 20 \cdot 15 Z(Q_0/Q)$, where Q/Q_0 is the ratio of the agitational energy of an electron to that of the molecules. The formula for a in terms of R now becomes

$$a \simeq \{(hQ/10ZQ_0) \ln [1/(1-R)]\}^{\frac{1}{2}}. \qquad \textbf{(20)}$$

Consequently

$$R=90 \text{ per cent.}, \quad a=0\cdot 48(hQ/ZQ_0)^{\frac{1}{2}};$$
$$R=95 \text{ per cent.}, \quad a=0\cdot 3(hQ/ZQ_0)^{\frac{1}{2}}.$$

Let $R=90$ per cent., $h=5$ cm, $Z=10$ V/cm, $Q/Q_0=10$, then $a=1\cdot 05$ cm, whereas for ions $a=0\cdot 33$ cm.

It is therefore possible to examine the distribution of electrons n and the ions N in the outer regions of the mixed stream which are virtually uncontaminated by the ions N_c that have entered through the aperture.

Although equation (**16**) provides a formal solution of equation (**6**) it does not lead to a simple formula for the distribution of the ions N formed by the attachment of electrons to molecules from which computations can be made for the measurement of α_a. It is therefore expedient to seek an approximate solution. To this end, advantage is taken of the fact of the disparity between λ for electrons and λ_1 for ions unless $Q/Q_0 \to 1$, as explained above. In many practical situations of interest, the actual spread of the elementary stream of ions arising from a source $\alpha_a Wn\mathrm{d}\tau$ is small and these ions impinge on a small area of the electrode about a point whose x,y coordinates are the same as that of the elementary volume $\mathrm{d}\tau$.

Consider a hypothetical case in which the elementary sources are distributed in a plane layer $z=z_1$ with thickness $\mathrm{d}z$ parallel to and at a distance l from the receiving electrode $z=h$ and suppose that n is constant throughout the layer.

Let N be the concentration of ions at a point x,y,h on the electrode and consider what portion of the plane of the sources contributes a specified proportion, say X of the concentration N of ions. The point $O(x,y,z)$ is the point of intersection of the normal to the electrode at (x,y,h) and the layer $z=z_1$.

The contribution of an annular source with centre O, radius ρ, and width $\mathrm{d}\rho$ to N is

$$\frac{2\pi\alpha_a Wn\mathrm{d}z}{4\pi D_1} \cdot \frac{\exp\{-\lambda_1(r-l)\}}{r} \cdot \rho\mathrm{d}\rho,$$

where $r^2=\rho^2+l^2$.

Thus the contribution of the circular disk with centre O and radius a is ($\rho\mathrm{d}\rho=r\mathrm{d}r$, $2\lambda_1=W_1/D_1$)

$$\mathrm{d}N=XN=\frac{\alpha_a Wn\mathrm{d}z}{2D_1}\exp(\lambda_1 l)\int_l^{(l^2+a^2)^{\frac{1}{2}}}\exp(-\lambda_1 r)\mathrm{d}r$$
$$=\frac{\alpha_a Wn\mathrm{d}z}{W_1}\Big\{1-\exp\left[-\lambda_1\{(l^2+a^2)^{\frac{1}{2}}-l\}\right]\Big\}.$$

The proportion of the total ionic current $I=\varepsilon W_1 N\mathrm{d}S$ received by an element of area $\mathrm{d}S$ of the electrode that is contributed by the disk with radius a is

$$XI=\varepsilon\alpha_a Wn\mathrm{d}z\Big\{1-\exp\left[-\lambda_1\{(l^2+a^2)^{\frac{1}{2}}-l\}\right]\Big\}.$$

as $a\to\infty$, $X\to 1$, consequently $I=\varepsilon\alpha Wn\mathrm{d}z$. Whence

$$X=1-\exp\left[-20\cdot 15Z\{(l^2+a^2)^{\frac{1}{2}}-l\}\right]. \quad \ldots\ldots\ldots \quad (\mathbf{21})$$

When $a^2/l^2 \ll 1$, the following expression for a is derivable from equation (**21**)

$$a \simeq \{(l/10Z) \ln [1/(1-X)]\}^{\frac{1}{2}}. \quad \ldots\ldots\ldots\ldots \quad (\mathbf{22})$$

Since equations (**19**) and (**22**) are identical it follows that:

$X=90$ per cent., $a=0\cdot48(l/Z)^{\frac{1}{2}}$;
$X=95$ per cent., $a=0\cdot3(l/Z)^{\frac{1}{2}}$;
$X=99$ per cent., $a=0\cdot68(l/Z)^{\frac{1}{2}}$.

For instance, if $l=2$ cm, $Z=10$ V/cm, and $X=99$ per cent., then $a=0\cdot30$ cm. The total ionic current to a surface element dS of the electrode is carried in effect by ions that are formed in a small cylinder of the source layer centred on the normal to dS. When n is not uniform, but varies with radial distance, a good approximation to the current to dS is obtained in many cases by supposing that n is distributed uniformly throughout the layer with a value equal to that it possesses on the normal to dS.

It follows that an approximation to the total ionic current to dS when n is distributed throughout the whole of space above the receiving electrode is to be found by integration of n along the normal to dS. The current is JdS, where

$$J=\varepsilon\alpha_a W\int_{-\infty}^{h} n\mathrm{d}z. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{23})$$

This is the solution that is derived from equation (**6**) when the term $\nabla^2 N$ is omitted, that is to say, when ionic diffusion is neglected. In what follows it will be supposed that measurements are made under conditions in which the approximation implied by equation (**23**) is justified.

IV. Calculation of the Total Current to an Annulus of the Electrode

The total current to an annulus $a \leqslant \rho \leqslant b$ of the electrode is the sum of the electronic current i_{ab} given by equation (**14**) and the ionic current $I_{ab}=\int J\mathrm{d}S$ (equation (**23**)) taken over the surface of the annulus. To this may be added the contribution, if necessary, of the ions N_c.

(a) *Ionic Current*

$$I_{ab}=\int J\mathrm{d}S=2\pi\varepsilon\alpha_a W\int_{-\infty}^{h}\int_{a}^{b} n\rho\mathrm{d}\rho\mathrm{d}z. \quad \ldots\ldots\ldots \quad (\mathbf{24})$$

When the expression for n given in equation (**9**) is used and the integration is carried out with respect to ρ the following expression is obtained:

$$I_{ab}=\frac{\varepsilon\alpha_a s\lambda}{\mu}\int_{-\infty}^{h} \exp(\lambda z)[\exp\{-\mu(a^2+z^2)^{\frac{1}{2}}\}-\exp\{-\mu(b^2+z^2)^{\frac{1}{2}}\}$$
$$-\exp\{-\mu[a^2+(2h-z)^2]^{\frac{1}{2}}\}+\exp\{-\mu[b^2+(2h-z)^2]^{\frac{1}{2}}\}]\mathrm{d}z, \quad \ldots \quad (\mathbf{25})$$

whereas, from equation (**14**) the electronic current is

$$i_{ab}=s\varepsilon h e^{\lambda h}[e^{-\mu d_a}/d_a-e^{-\mu d_b}/d_b]. \quad \ldots\ldots\ldots\ldots \quad (\mathbf{26})$$

The total current εs to the whole electrode should, as a check, be given as the sum, $i_{\text{total}} = i_{0\infty} + I_{0\infty}$ by equations (25) and (26).

It is easily shown that, when $a=0$, $b=\infty$,

$$i_{\text{total}} = i_{0\infty} + I_{0\infty} = \varepsilon s\left[e^{-(\mu-\lambda)h} + 1 - \frac{(\mu+\lambda)}{2\mu}e^{-(\mu-\lambda)h} - \frac{(\mu-\lambda)}{2\mu}e^{-(\mu+\lambda)h}\right].$$

Since $\alpha_a \ll \mu$ and λ; $\mu-\lambda = 2\lambda\alpha_a/(\mu+\lambda) \simeq \alpha_a$; $\mu \simeq \lambda$; $(\mu+\lambda)h \gg 1$, the quantity in the bracket differs inappreciably from unity and $i_{\text{total}} = \varepsilon s$.

V. Procedures for Measuring α_a

(a) *Measurement of α_a—First Procedure*

The ratio of the currents to the annulus $a \leqslant \rho \leqslant b$ and to the portions of the electrode $a \leqslant \rho \leqslant \infty$ is (a large enough to eliminate current due to N_c)

$$\begin{aligned}
R_1 &= (i_{ab} + I_{ab})/(i_{a\infty} + I_{a\infty}) \\
&= 1 - (i_{b\infty} + I_{b\infty})/(i_{a\infty} + I_{a\infty}) \\
&= 1 - \frac{(h/d_b)\exp\{-\mu h(d_b/h) + \lambda h\} + \lambda h(\alpha_a/\mu)\int_{-\infty}^{1}\exp(\lambda h s)\,.\,B\,\mathrm{d}s}{(h/d_a)\exp\{-\mu h(d_a/h) + \lambda h\} + \lambda h(\alpha_a/\mu)\int_{-\infty}^{1}\exp(\lambda h s)\,.\,A\,\mathrm{d}s} \\
&= F_1(b/h,\ a/h,\ \lambda h,\ \alpha_a h,\ \lambda/\mu)\ ;\ \mu \simeq \lambda + \alpha_a\ ;\ d_b^2 = b^2 + h^2,\ d_a^2 = a^2 + h^2\ ;\ \lambda/\mu \simeq 1, \qquad \ldots\ldots\ldots\ldots\ldots\ (27)
\end{aligned}$$

where $B = \exp\left[-\mu h(b^2/h^2 + s^2)^{\frac{1}{2}}\right] - \exp\left[-\mu h\{b^2/h^2 + (2-s)^2\}^{\frac{1}{2}}\right]$,

$A = \exp\left[-\mu h(a^2/h^2 + s^2)^{\frac{1}{2}}\right] - \exp\left[-\mu h\{a^2/h^2 - (2-s)^2\}^{\frac{1}{2}}\right]$.

Thus, if a set of curves is prepared for an apparatus in which a/h and b/h are given, showing R_1 as a function of λh for a range of values of the parameter α_a, then it is possible to measure α_a by measuring R_1 and λ.

An extensive set of such curves has been prepared by Dr. Barbara I. H. Hall and measurements are of α_a to be undertaken in this department.

(b) *Measurement of α_a—Second Procedure*

A second procedure by which the unknown contribution I_c to the current of the ions N_c may be eliminated, is the following.

The current to the central disk $\rho \leqslant a$ is

$(i_{0a})_{\text{total}} = i_{0a} + I_{0a} + I_c$, which, from equations (25) and (26), is

$$\begin{aligned}
(i_{0a})_{\text{total}} = \varepsilon s\Big[&\exp(\lambda h)\{1 - (h/d_a)\exp(-\mu d_a)\} \\
&+ (\lambda/\mu)\alpha_a h\Big\{\int_0^1 \exp[-(\mu-\lambda)hs]\mathrm{d}s + \int_0^\infty \exp[-(\mu+s)hs]\mathrm{d}s \\
&- \exp(-2\mu h)\int_{-\infty}^{1}\exp[(\mu+\lambda)hs]\mathrm{d}s - \int_{-\infty}^{1}\exp[\lambda hs - \mu h(a^2/h^2 + s^2)^{\frac{1}{2}}] \\
&+ \int_{-\infty}^{1}\exp[\lambda hs - \mu h\{a^2/h^2 + (2-s)^2\}^{\frac{1}{2}}]\mathrm{d}s\Big\}\Big] + I_c. \qquad \ldots\ldots\ldots\ (28)
\end{aligned}$$

Consider three distances h_1, h_2, and h_3 and let i_1, i_2, i_3 be the corresponding values of $(i_{0a})_{\text{total}}$ in an apparatus with a movable electrode.

The values of (i_1-I_c), (i_2-I_c), and (i_3-I_c) can then be calculated for given values of α_a, λ, and a.

Each typical current (i_k-I_c) can be expressed as a function $f_k(a/h_k, \lambda h_k, \alpha_a h, \lambda-\mu)$ which can be represented by a family of curves in which f_k is shown as a function of λh_k for a range of values of the parameter α_a.

The ratio

$$R_2=(i_1-i_2)/(i_2-i_3)=(f_1-f_2)/(f_2-f_3)$$

is independent of I_c and can be measured.

The ratio R_2, in an apparatus in which h_1, h_2, h_3 and a are given, is a function $F_2(\lambda,\alpha_a)$. Thus, from the curves for f_1, f_2, f_3, those showing R as a function of λ for a range of values of α_a can be constructed, and λ and α_a can be measured.

VI. The Effect of a Transverse Magnetic Field

Studies of the behaviour of a stream of electrons moving under the combined action of an electric field Z and a magnetic field B provide additional information about the motions of electrons in gases.

It is therefore of interest to attempt an extension of the theory to include the effects of ionization by collision and of electron attachment.

Let the source of electrons be a small aperture (or a short slot) at the origin of coordinates. Let the axis Oz be antiparallel to Z and suppose that the magnetic field B is transverse to Z and parallel to Oy. Let the components of the drift speed of electrons be W_z and W_x, that of the ions being $W_{1z}=W_1$ as before.

The coefficients of diffusion of electrons are D parallel to Oy and D_B in directions perpendicular to Oy.

The equation (3) of continuity for the electronic concentration in a steady stream now becomes

$$\frac{\mathrm{d}n}{\mathrm{d}t}=D_B\frac{\partial^2 n}{\partial x^2}+\frac{\partial^2 n}{\partial z^2}+D\frac{\partial^2 n}{\partial y^2}-\vec{W}\cdot\operatorname{grad} n-(\overline{f\nu}-\overline{g\nu})n=0. \quad \ldots (29)$$

The variable y is eliminated by introducing in place of n the quantity $q=\int_{-\infty}^{\infty} n\mathrm{d}y$. Integrate equation (29) with respect to y between the limits $+\infty$ and $-\infty$ and suppose that n and its derivatives have vanished at the lateral boundaries of the diffusion chamber. It follows that

$$D_B\frac{\partial^2 q}{\partial x^2}+\frac{d^2 q}{\partial z^2}-W_x\frac{\partial q}{\partial x}-W_z\frac{\partial q}{\partial z}-\alpha W_z q=0,$$

that is,

$$\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial z^2}=2\beta\frac{\partial q}{\partial x}+2\gamma\frac{\partial q}{\partial z}+2\gamma\alpha q, \quad \ldots\ldots\ldots\ldots (30)$$

where

$$2\beta = W_x/D_B,\ 2\gamma = W_z/D_B,\ \text{and}\ \alpha W_z = (\overline{f\nu} - \overline{g\nu}) = (\alpha_a - \alpha_i)W_z.$$

In equation (**30**) write $q = e^{\beta x + \gamma z}V$ to obtain

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial z^2} = (\beta^2 + \gamma^2 + 2\gamma\alpha)V = \chi^2 V. \quad \ldots\ldots\ldots\ldots (\mathbf{31})$$

Solutions of equation (**30**) are sought in which q is the two-dimensional concentration of electrons per unit length from an equivalent simple line source along the axis Oy. The appropriate solution of equation (**31**) is $V = AK_0(\chi r)$ where A is a constant and K_0 is the modified Bessel function of the second kind and of zero order.

If s is the rate of emission of electrons per unit length of the equivalent line source it may be shown from the property $K_0(v) \to -\log v,\ v \to 0$, that the solution of equation (**31**) that is sought is

$$q = (s/2\pi D_B)e^{\beta x + \gamma z}K_0(\chi r), \quad \ldots\ldots\ldots\ldots (\mathbf{32})$$

where $r^2 = x^2 + y^2$.

The solution that gives $q = 0$ over the receiving electrode $z = h$ is obtained by adding an image source at $(0,0,2h)$, that is,

$$q = (s/2\pi D_B)e^{\beta x + \gamma z}[K_0(r) - K_0(r')], \quad \ldots\ldots\ldots\ldots (\mathbf{33})$$

where $r'^2 = x^2 + (2h - z)^2$.

It follows from equation (**33**) that

$$\frac{\partial q}{\partial z} = \frac{s}{2\pi D_B}\left[\gamma q + e^{\beta x + \gamma z}\left\{\frac{z}{r} \cdot \frac{d}{dr}K_0(\chi r) + \frac{(2h - z)}{r'} \cdot \frac{d}{dr'}K_0(\chi r')\right\}\right],$$

whence

$$(\partial q/\partial z)_{z=h} = -(s/2\pi D_B)e^{\beta x + \gamma h}2\chi h K_1\{\chi(h^2 + x^2)^{\frac{1}{2}}\}/(h^2 + x^2)^{\frac{1}{2}}.$$

The current received by a strip of the electrode with width $a \leqslant x \leqslant b$ whose edges are parallel to Oy is

$$\begin{aligned} i_{ab} &= -\varepsilon D_B \int_a^b \left(\frac{\partial q}{\partial z}\right)_{z=h} dx \\ &= \frac{\varepsilon s \chi h e^{\gamma h}}{\pi} \int_a^b \frac{e^{\beta x}K_1\{\chi(h^2 + x^2)^{\frac{1}{2}}\}dx}{(h^2 + x^2)^{\frac{1}{2}}} \\ &= \frac{\varepsilon s \chi h e^{\gamma h}}{\pi} \int_{a/h}^{b/h} \frac{e^{\beta h s}K_1\{\chi h(1 + s^2)^{\frac{1}{2}}\}ds}{(1 + s^2)^{\frac{1}{2}}}. \quad \ldots\ldots\ldots (\mathbf{34}) \end{aligned}$$

The total current carried by electrons to the electrode is obtained by extending the limits of integration in equation (**34**) to $\pm\infty$. Thus,

$$\begin{aligned} i_{\text{total}} = i_{-\infty,\,\infty} &= \frac{2\varepsilon s \chi h \exp(\gamma h)}{\pi} \int_0^\infty \frac{\cosh \beta x \,.\, K_1\{\chi(h^2 + x^2)^{\frac{1}{2}}\}dx}{(h^2 + x^2)^{\frac{1}{2}}} \\ &= \frac{\varepsilon s \chi h}{\pi} \cdot \frac{\pi}{\chi h} \exp(\gamma h) \,.\, \exp\{-h(\chi^2 + \beta^2)^{\frac{1}{2}}\} \\ &= \varepsilon s \exp\{\gamma - (\gamma^2 + 2\gamma\alpha)^{\frac{1}{2}}h\}. \end{aligned}$$

But $\alpha \ll \gamma$, consequently $\gamma^2+2\gamma\alpha=(\gamma+\alpha)^2-\alpha^2 \simeq (\gamma+\alpha)^2$ and

$$i_{\text{total}}=\varepsilon s \exp(-\alpha h)=i_0 \exp(-\alpha h)=i_0 \exp\{-(\alpha_a-\alpha_i)h\}, \quad \ldots (35)$$

in agreement with equation (**12**) except that here $\alpha=(\overline{f\nu}-\overline{g\nu})/W_z$ whereas in equation (**12**) $\alpha=(\overline{f\nu}-\overline{g\nu})/W$. However, W_z does not usually differ significantly from W in practice.

(a) Distribution of Negative Ions

Equation (**6**) now becomes

$$\frac{\partial^2 N}{\partial x^2}+\frac{\partial^2 N}{\partial y^2}+\frac{\partial^2 N}{\partial z^2}=2\lambda_1\frac{\partial N}{\partial z}-(\alpha_a W_z/D_1)n.$$

Put $Q=\int_{-\infty}^{\infty} N\mathrm{d}y$ and integrate N in the equation with respect to y to obtain, as before,

$$\frac{\partial^2 Q}{\partial x^2}+\frac{\partial^2 Q}{\partial z^2}=2\lambda_1\frac{\partial Q}{\partial z}-(\alpha_a W_z/D_1)q,$$

with the formal particular solution, for the value Q_0 of Q at (x_0,y_0,z_0),

$$Q_0=(\alpha_a W_z/2\pi D_1)\int_{\infty} q\mathrm{e}^{\lambda_1(z_0-z)}\mathrm{K}_0(\lambda_1 r_1)\mathrm{d}\tau, \quad \ldots\ldots (36)$$

in which q is given in equation (**33**) and $r_1^2=(x_0-x)^2+(y_0-y)^2+(z_0-z)^2$.

As before, the general solution is intractable and the approximate solution

$$(N)_{z=h}=(\alpha_a W_z/2\lambda_1 D_1)\int_{-\infty}^{h} q\mathrm{d}z=(\alpha_a W_z/W_1)\int_{-\infty}^{h} q\mathrm{d}z$$

is adopted.

The ionic current to the strip $a \leqslant x \leqslant b$ is therefore

$$\begin{aligned} I_{ab}&=\varepsilon\alpha_a W_z\int_{-\infty}^{h}\int_a^b q\mathrm{d}z\mathrm{d}x \\ &=\frac{\varepsilon s\alpha_a W_z}{2\pi D_B}\int_{-\infty}^{h}\int_a^b \mathrm{e}^{\beta x+\gamma z}[\mathrm{K}_0(\chi r)-\mathrm{K}_0(\chi r')]\mathrm{d}z\mathrm{d}x, \end{aligned}$$

where $r^2=x^2+z^2$, $r'^2=x^2+(2h-z)^2$.

Thus, the total current to the strip is

$$\begin{aligned}(i_{ab})_{\text{total}}=\varepsilon s\Bigg[&\frac{h\chi\mathrm{e}^{\gamma h}}{\pi}\int_a^b \frac{\mathrm{e}^{\beta x}\mathrm{K}_1\{\chi(h^2+x^2)^{\frac{1}{2}}\}\mathrm{d}x}{(h^2+x^2)^{\frac{1}{2}}} \\ &+I_c+\frac{\alpha_a W_z}{2\pi D_B}\int_{-\infty}^{h}\int_a^b \mathrm{e}^{\beta x+\gamma z}\{\mathrm{K}_0(\chi r)-\mathrm{K}_0(\chi r')\}\mathrm{d}z\mathrm{d}x\Bigg], \end{aligned}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots (37)$$

where, as before, I_c is the current from ions entering through the aperture, and $\chi^2=\beta^2+\gamma^2+2\gamma\alpha$, $\gamma=W_z/2D_B$.

The procedures for measuring α_a described in Section IV become inaccurate or inapplicable in conditions where the spread of the stream of electrons does not differ greatly from that of the ions N_c, that is to say when $\lambda \rightarrow \lambda_1$. However, by the application of a transverse magnetic field the stream of electrons is deflected from that of the ions N_c, consequently, methods for measuring α_a based upon the analyses of this section should prove useful under conditions where $\lambda \rightarrow \lambda_1$.

VII. Approximations

When the receiving annulus or strip is narrow then approximate values of integrals such as those appearing in equations (**27**) and (**34**) may be found as the product of the range of integration and the value of the integrand at the centre of the range. Also the lower limit $-\infty$ of integrals such as those in equation (**28**) may be replaced by the physically more correct value of zero, with little error. These changes greatly simplify the computations.

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IX. References

Crompton, R. W., and Huxley, L. G. H. (1955).—*Proc. Phys. Soc.* B **68** : 381.

Crompton, R. W., Huxley, L. G. H., and Sutton, D. J. (1953).—*Proc. Roy. Soc.* A **218** : 507.

Healey, R. H., and Reed, J. W. (1941).—" The Behaviour of Slow Electrons in Gases." (Amalgamated Wireless (Australasia) Ltd. : Sydney.)

Huxley, L. G. H. (1957).—*Aust. J. Phys.* **10** : 118.

Huxley, L. G. H., and Zaazou, A. A. (1949).—*Proc. Roy. Soc.* A **196** : 402.

Loeb, L. B. (1955).—" Basic Processes of Gaseous Electronics." (University of California Press.)

Townsend, J. S. (1947).—" Electrons in Gases." (Hutchinson : London.)