DISPERSAL OF DUST PARTICLES FROM ELEVATED SOURCES

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Summary

The equations of Sutton for the atmospheric diffusion of gaseous pollutants are extended to the case of particles of appreciable free falling velocity with the boundary condition that dust fall rate equals ground level concentration times free falling velocity. It is shown that the effect of gravitational settling can be allowed for by (*a*) substituting the height above ground of the downward deflected plume axis for chimney height and (*b*) by introducing a factor $\alpha(x,z)$ as a multiplier to the " reflection " term in Sutton's equation.

I. Introduction

Owing to a large extent to the excellent work of Sutton there now exists a rational theory of diffusion of gaseous and finely divided particles from industrial stacks (Sutton 1932, 1947*a*, 1947*b*). Although the theory rests on certain simplifying assumptions, it is sufficiently accurate for practical applications, taking into account, among other factors, the influence of atmospheric stability.

An alternative treatment has been given by Bosanquet and Pearson (1936) and later applied to larger size particles by Bosanquet, Carey, and Halton (1950). This theory, however, fails to account for atmospheric stability and is, as regards the extension to finite size particles at least, less rigorous than Sutton's.

An extension of Sutton's formulae to cover gravitational settling has been attempted by Baron, Gerhard, and Johnstone (1949) ; an explicit solution has however, not been obtained.

From an engineering point of view rates of dust deposition are probably at least as important as maximum ground level concentrations of gaseous pollutants, the latter constituting a " public nuisance " relatively rarely while communal protests on grit fall have become increasingly numerous in the vicinity of major industries. The present article is an attempt to apply Sutton's theories to the problem of dust deposition.

II. Sutton's Equations

Using a theorem due to Taylor (1922), Sutton (1932) has obtained an expression for the standard deviation of diffusing particles from their mean position

$$\sigma^2=\tfrac{1}{2}C^2x^{2-n},$$

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where σ is the root-mean-square distance of particles, e.g. in a cloud of smoke from the centre of the cloud, C is the (constant) virtual eddy diffusion coefficient, and $x=ut$ the distance travelled down-wind by the cloud in a wind of mean velocity u in time interval t. The index " n " is a parameter affected by atmospheric stability, having the value 0·25 in a neutral atmosphere, 0–0·25 for high lapse rates, and 0·25–1·0 for inversions.

With the aid of this result Sutton (1947*a*, 1947*b*) has derived a good approximate expression for eddy diffusion of gases emitted by a chimney :

$$c=\frac{W}{\pi C_y C_z u x^{2-n}} \exp\left(-\frac{y^2}{C_y^2 x^{2-n}}\right)\left[\exp\left\{-\frac{(z-h)^2}{C_z^2 x^{2-n}}\right\}+\exp\left\{-\frac{(z+h)^2}{C_z^2 x^{2-n}}\right\}\right],$$

where

c = concentration of pollutant at some point at a distance x down-wind from the source, a perpendicular horizontal distance y from the plume axis, and an elevation z above ground level ;

W = strength of source, emitted matter per unit time ;

C_y, C_z = virtual diffusion coefficients in the y and z directions respectively ;

h = chimney height ;

u = mean wind velocity ;

n = parameter of stability.

On close examination the expression in the square brackets is seen to consist of two terms with a definite physical significance : the first one gives the diffusion from a plume at a mean distance h above ground as if the ground were not present. The second one is the exact mirror image of the first and has the effect of exactly replacing that part of the plume " cut off " by the ground ; it may be regarded as the " reflected " plume. By this device the condition of continuity, namely, that all emitted particles remain airborne, is satisfied.

III. Effect of Gravity on Plume

If now the emitted particles possess a groundward velocity not negligible compared to the wind velocity, it may be assumed with confidence that their mean motion will be the geometrical sum of windward travel and free fall, i.e. that the axis of the plume will have an inclination to the horizontal equal to f/u where f is the velocity of free fall for the particles concerned. Or more accurately, if z^* denotes the vertical distance of the plume axis above ground level at a distance x down-wind from the chimney, then

$$z^*=h(1-x/x_0),$$

where x_0 is the point at which the inclined plume axis reaches the ground. Also (Fig. 1) :

$$x_0=(u/f)h,$$

therefore

$$z^*=h\{1-(f/u)\cdot(x/h)\}.$$

In Sutton's equation the mean height of the particles above ground was h=const; replacing this by the variable z^*,

$$c=\frac{W}{\pi C_y C_z u x^{2-n}} \exp\left(-\frac{y^2}{C_y^2 x^{2-n}}\right)$$
$$\times\left[\exp\left(-\frac{\{z+(f/u)x-h\}^2}{C_z^2 x^{2-n}}\right)+\exp\left(-\frac{\{z-(f/u)x+h\}^2}{C_z^2 x^{2-n}}\right)\right].$$

This equation now gives a deflected plume path but the presence of the mirror image term still guarantees that all particles are "reflected" by the ground. Physically this is incorrect because in the stagnant air layer immediately adjacent to the ground the motion of the particles consists solely

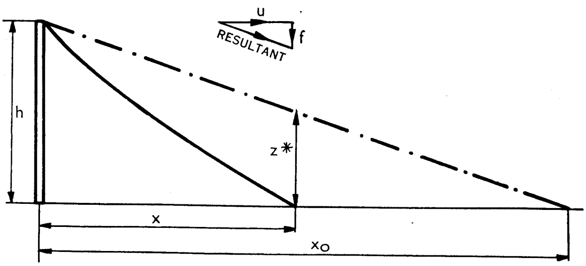

Fig. 1.—Groundward deflection of plume due to free fall of particles; mean velocity is given by geometrical sum of wind velocity and fall velocity. " Streamlines " in random diffusion are given by the condition that distance from the mean position is a constant multiple of the local standard deviation.

of free fall; and, as the concentration in this layer must be substantially the same as in the bordering turbulent mass, a transport of particles given by $c(z=0)\,.\,f$ settles through the stagnant layer to the ground. Thus the continuity condition, one of the boundary conditions for Sutton's solution, is replaced by the prescription that the rate of dust deposition at any point on the ground should be $c_0 f$ and that the airborne particles should be diminished by a corresponding amount.

IV. Allowance for Deposition

If the mirror image term in the last equation were neglected altogether, that is, if it were assumed that all particles reaching the ground by diffusion would be retained there, none reflected, the effective settling velocity of particles would be given by the sum of the mean vertical velocity of particles, i.e. f, and the instantaneous rate of increase of their mean distance from the plume axis. Let this latter term be called Δw; then from the foregoing it is clear that, the vertical distribution of particles remaining " normal " with increase of x, Δw will be proportional to $d\sigma_z/dt$, moreover, the factor of proportionality will be z^*/σ_z, i.e. the vertical distance of the plume axis above ground expressed in terms of standard deviations. To see this, consider that a constant fraction of particles is enclosed by the surface $\rho=$const. σ (where $\rho=$distance from plume axis) and that therefore mean flow is along this surface. Consequently,

$$\Delta w=\frac{z^*}{\sigma_z}\cdot\frac{\mathrm{d}\sigma_z}{\mathrm{d}t}.$$

σ_z may be found with Sutton,

$$\sigma_z=\tfrac{1}{2}\sqrt{2C_z}x^{1-\frac{1}{2}n},$$

from which it follows that

$$\Delta w=(1-\tfrac{1}{2}n)(hu/x-f).$$

The significance of this result is that if the mirror image term is neglected the condition of continuity requires a dust fall rate of $D=c_0(f+\Delta w)$ to be postulated instead of the physically correct c_0f.

On the other hand, it is not necessary to neglect the whole of the mirror image term: a fraction may be retained so that the dust fall rate is exactly as required.

It should be noted now that in computing ground level concentrations (c_0) the mirror image term reduces to a form identical to the simple diffusion term, i.e. that the mirror image term doubles the ground level concentration obtained by the simple term alone. Also, the upward directed "settling velocity" contributed by the mirror image term is numerically equal but opposite in sense

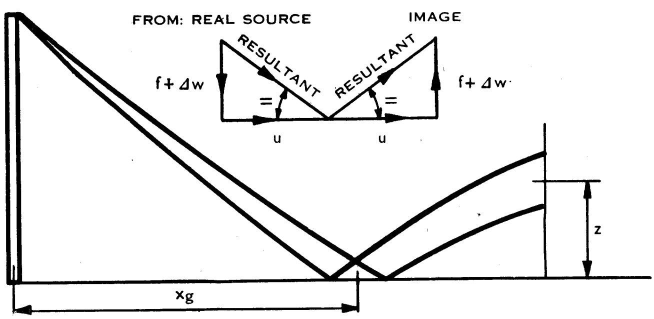

Fig. 2.—"Reflection" of plume by ground. Strength of "reflected" beam has to be adjusted to satisfy boundary condition that settling rate equals ground level concentration multiplied by free fall velocity.

to the velocity of settling from the simple plume (Fig. 2). If now the reflection term is multiplied by a factor $\alpha_0(x)$ the net rate of settling may be postulated to be the physically correct one

$$c_{so}(1+\alpha_0)f=c_{so}(f+\Delta w)-\alpha_0c_{so}(f+\Delta w),$$

where c_{so} is the ground level concentration given by the simple diffusion term alone and α_0c_{so} the contribution from the retained fraction of the reflection term.

From the above

$$\alpha_0(x)=\Delta w/(2f+\Delta w),$$

which, with the previously obtained value of Δw, reduces to

$$\alpha_0(x)=1-\frac{2}{(1-n/2)(hu/xf-1)+2}.$$

The value of α_0 thus computed refers to the degree of reflection on the ground which is seen to vary with distance from the source. Above ground level the strength of the reflected beam will be governed by the value of α_0 at the point of reflection (Fig. 2),

$$\alpha(z, x)=\alpha_0(x_g).$$

The computation of $\alpha(z, x)$ unfortunately presents a certain amount of difficulty owing to the fractional powers of x that have to be dealt with. As previously remarked, the equation for " streamlines " becomes :

$$z=z^*-k\sigma, \qquad k=\text{const.}$$

or, using Sutton's expression for σ and the equation of z^* derived above,

$$z=h-(f/u)x-\tfrac{1}{2}k\sqrt{2C_z}x^{1-\frac{1}{2}n}$$

for $z\to 0$, $x\to x_g$ (Fig. 2).

Therefore

$$\tfrac{1}{2}k\sqrt{2C_z}=\{h-(f/u)x_g\}/x_g^{1-\frac{1}{2}n},$$

and with this

$$z=h-(f/u)x-\{h-(f/u)x_g\}(x/x_g)^{1-\frac{1}{2}n}.$$

The significance of the last equation is that, given an arbitrary point (x_g, z), the point of " reflection "—which by a previously derived relationship determines the strength of the " reflected " beam—may be found by solving this equation for x_g. A general explicit solution is not readily obtained so that $\alpha(x, z)\equiv\alpha_0(x_g)$ cannot be given in an explicit form. This fact should not be of very great practical disadvantage as concentrations above ground level are mostly of academic interest only. If, in experimental checks on the theory, measurements above ground level are conducted, predictions of the theory have to be found by the somewhat cumbersome method of first finding x_g numerically. In such cases, however, the complication can be tolerated.

Modifying now Sutton's expressions by inclusion of the multiplier α the solution for concentration of particles at any point (x, y, z) becomes

$$c=\frac{W}{\pi C_y C_z u x^{2-n}}\exp\left(-\frac{y^2}{C_y^2x^{2-n}}\right)$$
$$\times\left[\exp\left(-\frac{\{z+(f/u)x-h\}^2}{C_z^2x^{2-n}}\right)+\alpha\exp\left(-\frac{\{z-(f/u)x+h\}^2}{C_z^2x^{2-n}}\right)\right],$$

where

$$\alpha=1-\frac{2}{(1-\frac{1}{2}n)(uh/x_gf-1)+2},$$

$x_g=x_g(z, x)$ to be found from

$$\{h-(f/u)x_g\}(x/x_g)^{1-\frac{1}{2}n}+(f/u)x+z-h=0.$$

Ground level concentrations can be expressed explicitly as $x_g = x$; dust fall rates are then $c_0 f$, or

$$D = \frac{Wf}{\pi C_y C_z u x^{2-n}} \exp\left(-\frac{y^2}{C_y^2 x^{2-n}} - \frac{\{(f/u)x - h\}^2}{C_z^2 x^{2-n}}\right)\left[2 - \frac{2}{(1-\frac{1}{2}n)(uh/xf - 1) + 2}\right].$$

Regarding units it should be noted that, as pointed out by Sutton (1947*b*), C_y and C_z have the dimension of $L^{\frac{1}{2}n}$. The dimension of D will then be $ML^{-2}T^{-1}$.

The manner in which an expression for α at $z \neq 0$ was obtained should guarantee that continuity is preserved, i.e. that the rate of disappearance of particles from suspension equals the rate of their deposition. Unfortunately, the mathematical expression of this condition yields a rather intractable equation

$$\frac{\partial}{\partial x}\int_0^\infty \int_{-\infty}^{+\infty} cu\,\mathrm{d}y\,\mathrm{d}z = \int_{-\infty}^{+\infty} c_0 f\,\mathrm{d}y,$$

which after integration with respect to y reduces to

$$\frac{\partial}{\partial x}\int_0^\infty \frac{u}{x^{3-3n/2}}[\varphi_1(x, z) + \varphi_2(x, z)]\,\mathrm{d}z = [\varphi_1(x, 0) + \varphi_2(x, 0)]f,$$

with

$$\varphi_1 = \exp\left(-\frac{\{z + (f/u)x - h\}^2}{C_z^2 x^{2-n}}\right),$$

$$\varphi_2 = \left[1 - \frac{2}{(1-\frac{1}{2}n)(uh/x_g f - 1) + 2}\right] \exp\left[-\frac{\{z - (f/u)x + h\}^2}{C_z^2 x^{2-n}}\right],$$

$$x_g = x_g(x, z)$$

as before. Verification of this equation would appear impracticable by analytical methods.

V. Conclusion

Application of Sutton's theories to particles of appreciable size is thus seen to be possible without introducing restricting assumptions, explicit expressions being, however, only obtained for ground level concentrations and dust fall rates, not for concentrations above ground level. The reliability of the formulae arrived at in this paper should not differ very greatly from that of the Sutton equation; they are only intended, however, to be practically applicable approximations.

VI. References

Baron, T., Gerhard, E. R., and Johnstone, H. F. (1949).—Dissemination of aerosol particles from stacks. *Ind. Eng. Chem.* **41**: 2403–8.

Bosanquet, C. H., and Pearson, J. L. (1936).—*Trans. Faraday Soc.* **32**: 1249–63.

Bosanquet, C. H., Carey, W. F., and Halton, E. M. (1950).—Dust deposition from chimney stacks. *Proc. Inst. Mech. Eng.* **162**: 355–65.

Sutton, O. G. (1932).—The theory of eddy diffusion in the atmosphere. *Proc. Roy. Soc.* A **135**: 143–65.

Sutton, O. G. (1947*a*).—*Quart. J. R. Met. Soc.* **73**: 257.

Sutton, O. G. (1947*b*).—The theoretical distribution of airborne pollution from factory chimneys. *Quart. J. R. Met. Soc.* **73**: 426–36.

Taylor, G. I. (1922).—On the theory of diffusion by continuous movements. *Proc. Lond. Math. Soc.* **20**: 196–212.