THE GRAVITATIONAL COMPRESSION OF AN ELASTIC SPHERE

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Summary

On considering a sphere in hydrostatic gravitational equilibrium, composed of a homogeneous elastic material for which the variation of incompressibility $\varkappa$ with pressure p is given by $\mathrm{d}\varkappa/\mathrm{d}p=n$, a constant, we find that there is an upper bound to the radius R of the sphere provided $n\leqslant 2$, and that for all values of n there is a lower bound to the value of I/MR^2, where I is the moment of inertia about a diameter and M is the mass of the sphere.

For a sphere composed of material for which $\mathrm{d}\varkappa/\mathrm{d}p$ is a decreasing function of pressure and $\mathrm{d}\varkappa/\mathrm{d}p\rightarrow 5/3$ as $p\rightarrow\infty$, it emerges that $0\cdot 40>I/MR^2>0\cdot 23$ and that the maximum radius is of the order of 10^4 km.

I. Introduction

Seventy years ago the major planets were either regarded as " water planets " or else they were thought to consist of a core of terrestrial material surrounded by a gaseous atmosphere. The acceptance of these hypotheses was due to the lack of knowledge on the compressibility of solid and liquid materials, and, although both hypotheses are now known to be incorrect, the effect of compression is still not completely known.

The theories of quantum mechanics and finite elastic strain, together with the high pressure experiments of Bridgman, have yielded results which should ultimately lead to the exact solution of the density distribution throughout a self-compressed elastic sphere. The analytical solution of the problem is forestalled by the lack of an exact solution of the Lane-Emden equation, but certain indicative results will be obtained in this paper by making use of the few cases in which the Lane-Emden equation has an analytical solution.

Using the results of quantum mechanical calculations and solving the Lane-Emden equation by numerical integration, Ramsey (1950) has solved the problem of the density distribution throughout a hydrogen planet. Earlier Birch (1939) used finite elasticity theory combined with numerical integration to determine the density distribution throughout a homogeneous layer of the Earth.

The model with which this paper is concerned is a homogeneous sphere composed of material for which $\mathrm{d}\varkappa/\mathrm{d}p$ is constant, and emphasis is placed on determining the lower bound of the ratio I/MR^2 and the upper bound of the radius. In the discussion at the end of this paper it is pointed out that the results can be applied to a sphere of material for which $\mathrm{d}\varkappa/\mathrm{d}p$ is a monotonic decreasing function of pressure.

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II. The Differential Equation for Density

Using the equation defining incompressibility, namely, $\varkappa=\rho(\mathrm{d}p/\mathrm{d}\rho)$, where ρ is the density at pressure p, together with the relation $\varkappa=\varkappa_0+np$, where $\varkappa_0$ is the incompressibility at zero pressure, we obtain the pressure-density relation

$$p=\frac{\varkappa_0}{n}\left\{\left(\frac{\rho}{\rho_0}\right)^n-1\right\}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where we have used ρ_0 to denote the density at zero pressure.

Combining (1) with the equation for hydrostatic equilibrium of a sphere, namely,

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r}\right)=-4\pi G\rho, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

where G denotes the gravitation constant and r is distance from the centre of the sphere, we obtain the differential equation

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r^2\frac{\mathrm{d}}{\mathrm{d}r}\rho^{n-1}\right\}=-\frac{4\pi G\rho_0^n(n-1)}{\varkappa_0}\cdot\rho. \qquad \ldots\ldots \quad (3)$$

III. The Density Distribution for $n=2$

On denoting $4\pi G\rho_0^2/\varkappa_0$ by β^2 and making the substitution $\rho=\chi/r$ we find that the general solution of (3) is $\rho=C\sin(\beta r-\delta)/r$ where C and δ are arbitrary constants of integration. Since density is to be finite at the origin we have $\delta=0$, and, on considering the surface values of a sphere of radius R,

$$\frac{\rho}{\rho_0}=\frac{R}{r}\frac{\sin\beta r}{\sin\beta R}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Since the mass of the sphere $M=\int_0^R 4\pi r^2\rho(r)\mathrm{d}r$ we have on substituting from equation (4) and integrating, that

$$\frac{1}{3}\frac{\bar{\rho}}{\rho_0}=\frac{1}{\beta^2R^2}-\frac{\cot\beta R}{\beta R}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

where $\bar{\rho}$ is the mean density given by $M=4\pi R^3\bar{\rho}/3$. Similarly, from the formula

$$I=\frac{8\pi}{3}\int_0^R r^4\rho(r)\mathrm{d}r,$$

where I is the moment of inertia of the sphere about a diameter, combined with equation (4), we deduce that

$$\frac{1}{2}a\frac{\bar{\rho}}{\rho_0}=-\frac{\cot\beta R}{\beta R}+\frac{3}{\beta^2R^2}+\frac{6\cot\beta R}{\beta^3R^3}-\frac{6}{\beta^4R^4},$$

where a is given by $I=aMR^2$, and on substituting in this equation from (5) we obtain

$$a=\frac{2}{3}+\frac{4}{3(1-\beta R\cot\beta R)}-\frac{4}{\beta^2R^2}. \qquad \ldots\ldots\ldots\ldots \quad (6)$$

IV. Approximate Solution of Equation (3) for Arbitrary $d\varkappa/dp$

As a first approximation to the density throughout the sphere we take $\rho=\bar{\rho}$, where $\bar{\rho}$ is the mean density of the self-compressed sphere. On solving equation (3) and remembering that ρ must be finite at the origin, we obtain as a second approximation to the density

$$\rho=\rho_0(A-Br^2)^\alpha, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

where $A=1+BR^2$, $B=\frac{2}{3}\pi G\rho_0\bar{\rho}/\varkappa_0\alpha$, and $\alpha=1/(n-1)$.

On determining the mass and moment of inertia as in the previous section, we obtain the results

$$\frac{\bar{\rho}}{\rho_0}=A^\alpha F\left\{-\alpha,\ \frac{3}{2};\ \frac{5}{2};\ \frac{BR^2}{A}\right\}, \quad \ldots\ldots\ldots\ldots \quad (8)$$

$$a\frac{\bar{\rho}}{\rho_0}=\frac{2}{5}A^\alpha F\left\{-\alpha,\ \frac{5}{2};\ \frac{7}{2};\ \frac{BR^2}{A}\right\}, \quad \ldots\ldots\ldots\ldots \quad (9)$$

where $F\{a,\ b\ ;\ c\ ;\ z\}$ is the hypergeometric function.

It is necessary at this stage to give some idea of the range of validity of the equations (7), (8), and (9).

On substituting the approximate solution (7) into the differential equation (3) we find that the left side is constant while the right side varies by the fraction $(A-BR^2/A)^\alpha$ of itself as r varies from 0 to R. Since $(A-BR^2/A)^\alpha=1/A^\alpha=\rho_0/\rho_c$, it follows that the solution (7) will be a valid approximation only if the density variation throughout the sphere is not large, so that ρ_0/ρ_c is approximately unity. We use ρ_c to denote central density.

From this argument it appears to follow that equations (8) and (9) are not valid for large compression because of their dependence on (7). This, however, is not the case since equations (8) and (9) are obtained from (7) by a further use of the equations of hydrostatic equilibrium and are therefore true to a higher order of approximation.

Let us consider the derivation of (8) as follows.

The equations of hydrostatic equilibrium are

$$\frac{dp}{dr}=-\frac{GM(r)}{r^2}\rho, \qquad \frac{dM(r)}{dr}=4\pi r^2\rho,$$

where $M(r)$ is the mass within the sphere of radius r, and on using (1) we can derive the differential equation for the mass $M(r)$ as

$$\left\{\frac{1}{4\pi r^2}\ \frac{dM(r)}{dr}\right\}^{n-2}\frac{d}{dr}\left\{\frac{1}{4\pi r^2}\ \frac{dM(r)}{dr}\right\}=-\frac{G\rho_0^n}{\varkappa_0}\cdot\frac{M(r)}{r^2}. \quad \ldots\ldots \quad (10)$$

To solve this differential equation we take as a first approximation that $M(r)=4\pi r^3\bar{\rho}/3$ where $\bar{\rho}$ is the mean density of the whole sphere of radius R. Substituting this value of $M(r)$ in the second member and integrating we obtain as a second approximation that

$$M(r)=\frac{4}{3}\pi r^3\rho_0 A^\alpha F\left\{-\alpha,\ \frac{3}{2};\ \frac{5}{2};\ \frac{Br^2}{A}\right\},$$

which gives equation (8) and no longer depends on the validity of (7).

Substituting back into the differential equation (**10**) we find that the left side is constant while the right side varies by a fraction $F\{-\alpha, 3/2\,;\, 5/2\,;\, BR^2/A\}$ of itself as r varies from 0 to R. This variation is seen to be equal to $\bar{\rho}/\rho_c$.

Now

$$1 \geqslant F\left\{-\alpha, \frac{3}{2}; \frac{5}{2}; \frac{BR^2}{A}\right\} \geqslant F\left\{-\alpha, \frac{3}{2}; \frac{5}{2}; 1\right\},$$

the lower bound being attained in the limit as $\bar{\rho}\to\infty$. Thus we see that, provided $F\{-\alpha, 3/2\,;\, 5/2\,;\, 1\}$ does not differ greatly from unity, the approximate solution (**8**) is valid for all values of $\bar{\rho}$. But

$$F\{-\alpha,\ 3/2\,;\ 5/2\,;\ 1\} = \frac{\Gamma(5/2)\Gamma(1+\alpha)}{\Gamma(5/2+\alpha)},$$

which is precisely unity when $\alpha=0$ and then decreases as α increases. Thus for α not too large the equation (**8**) is valid no matter how great is the variation of density throughout the sphere.

Combining (**10**) with the equation

$$\frac{3}{2r^2}\,\frac{\mathrm{d}I(r)}{\mathrm{d}r} = \frac{\mathrm{d}M(r)}{\mathrm{d}r},$$

where $I(r)$ is the moment of inertia of the sphere of radius r, we easily derive the integro-differential equation

$$\left\{\frac{3}{8\pi r^4}\,\frac{\mathrm{d}I(r)}{\mathrm{d}r}\right\}^{n-2}\frac{\mathrm{d}}{\mathrm{d}r}\left\{\frac{3}{8\pi r^4}\,\frac{\mathrm{d}I(r)}{\mathrm{d}r}\right\} = -\frac{G\rho_0^n}{\varkappa_0}\,\frac{1}{r^2}\int_0^r \frac{3}{2r^2}\,\frac{\mathrm{d}I(r)}{\mathrm{d}r}\,\mathrm{d}r,$$

and, on solving approximately by taking $I(r)=8\pi r^5\bar{\rho}/15$ in the second member, we find that

$$I(r) = \frac{8}{15}\pi\rho_0 r^5 A^\alpha F\left\{-\alpha, \frac{5}{2}; \frac{7}{2}; \frac{Br^2}{A}\right\},$$

which gives equation (**9**). On substituting back into the integro-differential equation we find that the left side is constant while the right side varies by the fraction $F\{-\alpha, 3/2\,;\, 5/2\,;\, BR^2/A\}$ of itself as r varies from 0 to R and so (**9**) is valid under the same conditions as (**8**).

V. An Alternate Method of Solution of Equation (3)

As an alternative to the approximate method of Section IV we may take $\rho=H-Kr^2$ as a first approximation to the density in the second member of equation (**3**). On integrating and inserting the boundary conditions that ρ is finite at the origin, and $\rho=\rho_0$ when $r=R$ we have

$$\left(\frac{\rho}{\rho_0}\right)^{n-1} = 1 + \frac{4\pi G\rho_0(n-1)}{\varkappa_0}\left\{\frac{H}{6}(R^2-r^2) - \frac{K}{20}(R^4-r^4)\right\}. \quad \text{.. } (\mathbf{11})$$

If we take

$$H-Kr^2 = \rho_c - \frac{\rho_c-\rho_0}{R^2}\,r^2, \quad \text{.............. } (\mathbf{12})$$

and substitute (**11**) back into the differential equation (**3**) we find that both sides of the equation vary from ρ_c to ρ_0 as r varies from 0 to R so that the solution (**11**) with the values of H and K given by (**12**) should be a valid approximation for compression of any magnitude.

Substituting the values of H and K given by equation (**12**) into (**11**) we see that, if ρ_c is large, then the density throughout the sphere is given asymptotically by

$$\frac{\rho}{\rho_0}\simeq\left\{\frac{4\pi G\rho_0}{\varkappa_0\alpha}\cdot\rho_c\left(\frac{R^2-r^2}{6}-\frac{R^4-r^4}{20R^2}\right)\right\}^{\alpha}. \qquad \ldots\ldots\ldots (13)$$

VI. The Exact Solution of Equation (3) for $d\varkappa/dp=6/5$

When $n=6/5$ the equation (**3**) permits of an exact solution (Chandrasekhar 1939), namely,

$$\frac{4}{5}\pi G\frac{\rho_0{}^{6/5}\rho^{4/5}}{\varkappa_0}=\frac{3C^2}{(1+C^2r^2)^2},$$

where C is an arbitrary constant. Inserting the boundary conditions $\rho=\rho_0$ when $r=R$ and $\rho=\rho_c$ when $r=0$ we obtain the relation between ρ_c and R

$$1+\frac{4\pi G\rho_0{}^{6/5}\rho_c{}^{4/5}}{15\varkappa_0}R^2=\left(\frac{\rho_c}{\rho_0}\right)^{2/5}. \qquad \ldots\ldots\ldots\ldots\ldots (14)$$

VII. Deductions

(*a*) From the equation $A=1+BR^2$ combined with (**8**) we have

$$R^2=\frac{A-1}{B}=\frac{1}{B}\left[\left(\frac{\bar{\rho}}{\rho_0F\{-\alpha,\ 3/2\,;\ 5/2\,;\ BR^2/A\}}\right)^{1/\alpha}-1\right], \qquad \ldots\ldots (15)$$

so that in the limit as $\bar{\rho}$ increases indefinitely, we see that R^2 behaves like $\bar{\rho}^{n-2}$. Hence in the limit as $\bar{\rho}\to\infty$ we have that for $n>2$, $R\to\infty$ and for $n<2$, $R\to 0$. Thus the case $n=2$ divides discontinuously the two types of behaviour.

When $n=2$, equation (**15**) reduces to

$$R^2=\frac{1}{B}\left[\frac{\bar{\rho}}{\rho_0F\{-1,\ 3/2\,;\ 5/2\,;\ BR^2/A\}}-1\right],$$

so that on taking the limit as $\bar{\rho}\to\infty$ we find that

$$R_{\text{max.}}=\frac{1}{2\rho_0}\sqrt{\frac{15\varkappa_0}{\pi G}}.$$

On considering the exact solution (**5**) for $n=2$, we see that $\bar{\rho}\to\infty$ as $\beta R\to\pi$, so that the exact value for the upper bound of the radius is given by

$$R_{\text{max.}}=\frac{\pi}{\beta}=\frac{1}{2\rho_0}\sqrt{\frac{\pi\varkappa_0}{G}}.$$

In Section IV the validity of the approximate equation (**8**) in the case of large $\bar{\rho}$ was shown to depend on α, so that the discrepancy between the exact and approximate values of $R_{\text{max.}}$ for $\alpha=1$, found in this section, gives an idea of the error involved.

As an alternative to the above we see from (**13**) that as $\rho_c\to\infty$, ρ_c behaves like $\rho_c^{\alpha}R^{2\alpha}$, or $R^{2\alpha}$ behaves like $\rho_c^{1-\alpha}$. Hence the results deduced from Section IV are supported by the solution of Section V.

(*b*) On combining equations (**8**) and (**9**) we obtain

$$a=\frac{2}{5}\,\frac{F\{-\alpha,\ 5/2\ ;\ 7/2\ ;\ BR^2/A\}}{F\{-\alpha,\ 3/2\ ;\ 5/2\ ;\ BR^2/A\}},$$

so that as $\bar{\rho}\to\infty$ we see that

$$a_{\text{min.}}=\frac{2}{2\alpha+5}. \qquad\qquad (\mathbf{16})$$

From the exact solution (**6**) in the case $\mathrm{d}\varkappa/\mathrm{d}p=2$ we see that, since $\beta R\to\pi$ as $\bar{\rho}\to\infty$,

$$a_{\text{min.}}=\frac{2}{3}\left(\frac{\pi^2-6}{\pi^2}\right). \qquad\qquad (\mathbf{17})$$

It emerges therefore that the ratio I/MR^2 has a lower limit as the mean density of the sphere increases indefinitely and the approximate value $a_{\text{min.}}=2/(2\alpha+5)$ is not far in error.

Using equation (**13**) and the integrals for mass and moment of inertia

$$a=\frac{I}{MR^2}=\frac{2}{3R^2}\,\frac{\int_0^R r^4\rho(r)\mathrm{d}r}{\int_0^R r^2\rho(r)\mathrm{d}r},$$

or

$$\lim_{\rho_c\to\infty} a=\frac{2}{3R^2}\,\frac{\int_0^R r^4\{(R^2-r^2)/6-(R^4-r^4)/20R^2\}^{\alpha}\mathrm{d}r}{\int_0^R r^2\{(R^2-r^2)/6-(R^4-r^4)/20R^2\}^{\alpha}\mathrm{d}r}$$

$$=\frac{2}{3}\,\frac{\int_0^1 u^4\{(1-u^2)(7-3u^2)\}^{\alpha}\mathrm{d}u}{\int_0^1 u^2\{(1-u^2)(7-3u^2)\}^{\alpha}\mathrm{d}u},$$

where we have made the substitution $u=r/R$.

Towards solving the integral

$$\int_0^1 u^4(1-u^2)^{\alpha}(7-3u^2)^{\alpha}\mathrm{d}u$$

we make the substitution $u^2=x$, so that it reduces to

$$\frac{1}{2}7^{\alpha}\int_0^1 x^{3/2}(1-x)^{\alpha}\left(1-\frac{3}{7}x\right)^{\alpha}\mathrm{d}x.$$

which can be evaluated by expanding the last term in the integrand by the binomial theorem. On integrating term by term, the resulting series can be shown to be equal to

$$\tfrac{1}{2}7^{\alpha}\frac{\Gamma(\alpha+1)\Gamma(5/2)}{\Gamma(7/2+\alpha)}\cdot F\{-\alpha,\ 5/2\ ;\ 7/2+\alpha\ ;\ 3/7\}.$$

Similarly the integral

$$\int_0^1 u^2(1-u^2)^{\alpha}(7-3u^2)^{\alpha}\mathrm{d}u$$

can be shown to equal

$$\tfrac{1}{2}7^{\alpha}\frac{\Gamma(\alpha+1)\Gamma(3/2)}{\Gamma(5/2+\alpha)}\cdot F\{-\alpha,\ 3/2\ ;\ 5/2+\alpha\ ;\ 3/7\}.$$

Hence

$$\lim_{\rho_c\to\infty} a=\frac{2}{2\alpha+5}\,\frac{F\{-\alpha,\ 5/2\ ;\ 7/2+\alpha\ ;\ 3/7\}}{F\{-\alpha,\ 3/2\ ;\ 5/2+\alpha\ ;\ 3/7\}}, \qquad \ldots\ldots \quad (\mathbf{18})$$

which, when $\alpha=1$, is seen to be very close to the exact value of equation (**17**). It would appear from this agreement that the equation (**18**) gives a slightly better approximation to the value of $\lim_{\rho_c\to\infty} a$, for large α, than does equation (**16**).

In the case $\alpha=3/2$, corresponding to $\mathrm{d}\varkappa/\mathrm{d}p=5/3$, which is of importance, the equation (**16**) gives $\lim_{\rho_c\to\infty} a=0\cdot25$, while the equation (**18**) gives $\lim_{\rho_c\to\infty} a=0\cdot23$. We thus take $a=0\cdot23$ to be the lower bound of the ratio I/MR^2 for a homogeneous sphere when $\mathrm{d}\varkappa/\mathrm{d}p=5/3$, but note that the simple equation (**16**) gives a reasonably accurate approximation.

(*c*) Using equation (**15**) we can express the mass of the sphere in the form

$$M=\frac{4}{3}\pi\bar{\rho}\frac{1}{B^{3/2}}\left[\left(\frac{\bar{\rho}}{\rho_0 F\{-\alpha,\ 3/2\ ;\ 5/2\ ;\ BR^2/A\}}\right)^{1/\alpha}-1\right]^{3/2},$$

so that in the limit as $\bar{\rho}$ increases indefinitely we see that M behaves like $\bar{\rho}^{\,3n/2-2}$, and thus M tends to a finite limit as $\bar{\rho}\to\infty$ if $n=4/3$.

This behaviour of M can also be deduced from equation (**13**) since for large values of the central density

$$\lim_{\rho_c\to\infty} M\rho_c^{-\alpha}=4\pi\rho_0\int_0^R r^2\left[\frac{4\pi G\rho_0}{\varkappa_0\alpha}\left\{\frac{R^2-r^2}{6}-\frac{R^4-r^4}{20R^2}\right\}\right]^{\alpha}\mathrm{d}r,$$

or, on making the substitution $u=r/R$,

$$\lim_{\rho_c\to\infty} M\rho_c^{-\alpha}=\frac{\pi\rho_0}{15}\left(\frac{4\pi G\rho_0}{\varkappa_0\alpha}\right)^{\alpha}R^{2\alpha+3}\int_0^1 u^2\{(1-u^2)(7-3u^2)\}^{\alpha}\mathrm{d}u$$

$$=\frac{\pi\rho_0}{15}\left(\frac{4\pi G\rho_0}{\varkappa_0\alpha}\right)^{\alpha}R^{2\alpha+3}\tfrac{1}{2}7^{\alpha}\frac{\Gamma(\alpha+1)\Gamma(3/2)}{\Gamma(5/2+\alpha)}F\{-\alpha,3/2\ ;\ 5/2+\alpha\ ;\ 3/7\}.$$

Hence as $\rho_c\to\infty$, M behaves like $\rho_c^{\alpha}R^{2\alpha+3}$.

On making use of the results of deduction (*a*) we see that for large central density M behaves like $\rho_c^{3n/2-2}$ and so M is finite if $n=4/3$.

This result agrees with that of Chandrasekhar (1939) who found an upper limit to the mass of a polytrope in the case where the pressure-density relation is of the form $p=A\rho^{4/3}$ corresponding to a completely degenerate electron gas when relativistic effects are taken into account.

It also follows from Chandrasekhar's treatment that values of $n<4/3$ are physically impossible ; and it would appear that on the Thomas-Fermi theory no sphere could have an infinite mass since the ultimate pressure density relation for all substances is $p=A\rho^{4/3}$.

(*d*) The exact solution (**14**) for the case $\mathrm{d}\varkappa/\mathrm{d}p=6/5$ bears out the previous result that $R\to 0$ as $\rho_c\to\infty$ if $\mathrm{d}\varkappa/\mathrm{d}p<2$. In this case we can find the maximum radius which, since it corresponds to $\mathrm{d}R/\mathrm{d}\rho_c=0$, is

$$R_{\text{max.}}=\sqrt{\left(\frac{15\varkappa_0}{16\pi G\rho_0^2}\right)},$$

which occurs when $(\rho_c/\rho_0)^{2/5}=2$.

(*e*) Taking R as half its limiting value, i.e. $\beta R=\frac{1}{2}\pi$ in the exact solution for $\mathrm{d}\varkappa/\mathrm{d}p=2$, we obtain from equations (**5**), (**6**), (**4**), and (**1**) that

$$\frac{\bar{\rho}}{\rho_0}=\frac{12}{\pi^2},$$

$$a=2-\frac{16}{\pi^2}.$$

$$\frac{\rho_c}{\rho_0}=\frac{\pi}{2},$$

$$p_c=\frac{\varkappa_0}{2}\left\{\frac{\pi^2}{4}-1\right\},$$

where ρ_c and p_c are the central values of the density and pressure respectively. It is to be noted that a sphere of this radius shows very little compression.

(*f*) Birch (1952) and Keane (1954) have shown that, at least for the alkali metals, $\mathrm{d}\varkappa/\mathrm{d}p$ is a decreasing function of pressure, so that, on assuming its universal applicability and in view of the quantum mechanical result for a completely degenerate electron gas (neglecting relativistic effects) that $\mathrm{d}\varkappa/\mathrm{d}p\to 5/3$ as $p\to\infty$, it emerges from the foregoing results that no homogeneous elastic sphere can have the ratio I/MR^2 less than approximately $0\cdot 23$, this value being approached as the mean density tends to infinity.

It is also evident from the previous results that, since $\mathrm{d}\varkappa/\mathrm{d}p\to 5/3$ as $p\to\infty$, then there is an upper limit to the radius of the sphere of the order of $\sqrt{(\varkappa_0/G\rho_0^2)}$. On taking $\varkappa_0=10^{12}$ dyn/cm² and $\rho_0=1$ g/cm³ this upper limit is of the order of 10^4 km.

(*g*) Considering $\bar{\rho}\to\infty$ is a mathematical idealism and overlooks possible large-scale changes of interaction between the individual atoms. Clearly an infinite density is impossible and so some mechanism of destruction must be brought into operation before the limit is reached. It follows that the lower bound for the ratio I/MR^2 is unattainable.

(*h*) For the major planets the empirical values of I/MR^2 are all close to the minimum, $0 \cdot 23$, being, according to Ramsey (1951), for Jupiter $0 \cdot 25$, Saturn $0 \cdot 22$, Uranus $0 \cdot 24$, and Neptune $0 \cdot 27$. These low values suggest a certain amount of central concentration of heavy material, especially in Saturn, and also indicate that the radii of the major planets, all of the order of 10^4 km, are near the maximum for the particular materials of which they are composed.

(*i*) Ramsey (1951) has calculated the density distribution for Jupiter and Saturn on the assumption that they are composed of a homogeneous mixture of hydrogen and helium. From these calculations it was found that $a = 0 \cdot 28$ for Jupiter and has a slightly higher value for Saturn. These values are within the limits, for a homogeneous sphere, imposed by this paper but are so close to the lower limit as to suggest very forcibly that Jupiter is nearly the largest planet possible. In this connexion it should be borne in mind that, since we are considering $\mathrm{d}\varkappa/\mathrm{d}p \rightarrow 5/3$ as $p \rightarrow \infty$, the lower limit of a does not correspond to the maximum radius, but to the zero radius which is attained on shrinking of the sphere after the maximum radius has been reached. Hence the maximum radius corresponds to a value of a greater than $0 \cdot 23$.

It is difficult to ascertain ρ_0 and $\varkappa_0$ for hydrogen since for a sphere of any size the hydrogen atoms are likely to form a metallic lattice. However, the value of 85,000 km shown by Ramsey (1950), using quantum mechanical calculations, to be the maximum radius of a hydrogen planet is not in conflict with the upper bound found in this paper.

VIII. References

Birch, F. (1939).—The variation of seismic velocities within a simplified earth model, in accordance with the theory of finite strain. *Bull. Seism. Soc. Amer.* **29** : 463-79.

Birch, F. (1952).—Elasticity and constitution of the earth's interior. *J. Geophys. Res.* **57** : 227-86.

Chandrasekhar, S. (1939).—" An Introduction to the Study of Stellar Structure." (Univ. Chicago Press.)

Keane, A. (1954).—An investigation of finite strain in an isotropic material subjected to hydrostatic pressure and its seismological applications. *Aust. J. Phys.* **7** : 322-33.

Ramsey, W. H. (1950).—The planets and the white dwarfs. *Mon. Not. R. Astr. Soc.* **110** : 444-54.

Ramsey, W. H. (1951).—On the constitution of the major planets. *Mon. Not. R. Astr. Soc.* **111** : 427-47.