THE DISTRIBUTION OF DISLOCATIONS IN LINEAR ARRAYS

By A. K. HEAD* and N. LOUAT*

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Summary

An approximate method is given for finding the equilibrium distribution of arrays of dislocations. The analysis is based on the assumption that an array of discrete dislocations may be replaced by a continuous distribution of smeared dislocation. The solutions of a number of problems of physical interest are investigated, including some in which dislocations of opposite sign are involved.

I. INTRODUCTION

Eshelby, Frank, and Nabarro (1951) have considered the problem of the position taken up by a set of identical straight dislocations, which are constrained to lie in some part of the same slip plane, under the combined action of their repulsions and the force exerted on them by a given applied shear stress.

Their method of solution, although exact, is subject to a number of disabilities in practice. In any particular problem it is apparently necessary to guess a function $q(n, x)$ which will give a polynomial solution to the differential equation representing the problem. If an appropriate $q(n, x)$ can be chosen then the dislocations will lie at the zeros of the polynomial solution. However, this polynomial may be one whose zeros are not tabulated and, if the number of dislocations, n, is large, much computation is then necessary. Furthermore, one does not usually obtain a general picture of dislocation distribution from such a computation.

We shall consider here an approximate method for the solution of the problem of Eshelby, Frank, and Nabarro. The approximation made is to replace the discrete distribution of finite dislocations by a continuous distribution of infinitesimal dislocations with the same total Burgers vector. The problem is then to find the density of dislocations at any point such that the distribution is in equilibrium under its mutual forces and that applied externally. This leads to the problem of inverting a singular integral equation, which is a routine procedure.

This approximation can be expected to involve little error when the distance between dislocations is of the same order as their width. Since these conditions are likely to occur near the head of any array of practical significance, it is also likely that the dislocation stresses near this region can be evaluated with reasonable accuracy if the equilibrium distribution of smeared-out dislocation can first be determined. One advantage of this method is that it will deal as easily with

* Aeronautical Research Laboratories, Department of Supply, Melbourne.

distributions of dislocation of opposite sign as with those of the same sign. The method of Eshelby, Frank, and Nabarro will not treat the case of the interaction of dislocations of opposite sign.

II. Formulation of Problem

Suppose that the plane $y=0$ is the slip plane, that the dislocation lines are parallel to the z-axis, and that they can move along the x-axis. Let $f(x)$ be the dislocation density at any point on the x-axis, with the convention that $f(x)$ be positive in regions of positive dislocations and vice versa. Let $P(x)$ be the appropriate component of the applied stress tending to move the dislocations along the x-axis, and take $P(x)$ to be positive if it tends to move a positive dislocation in the positive direction of the x-axis.

A small element of dislocation of strength ε/λ at x produces a stress at x_0 given by

$$\sigma=\frac{A\varepsilon}{x_0-x}, \qquad \ldots\ldots\ldots\ldots \quad (1)$$

where $A=\mu\lambda/2\pi$ for screw dislocations and $A=\mu\lambda/2\pi(1-\nu)$ for edge dislocations, where μ is the shear modulus of the material assumed isotropic, ν is Poisson's ratio, and λ is the Burgers vector of a unit dislocation. Hence the stress at x_0 due to the applied stress and the distributed dislocation is given by

$$S(x_0)=P(x_0)+A\int_D \frac{f(x)}{x_0-x}\mathrm{d}x+T(x_0), \qquad \ldots\ldots \quad (2)$$

where the integral is taken over all regions D of the x-axis where there is dislocation pres
form of a Dirac δ function, which may be necessary for equilibrium at the ends of an array.

Since a dislocation will move if there is any net stress at its centre (excluding that produced by itself), the dislocation distribution can only be in equilibrium if $S(x_0)=0$ in the regions D of dislocation. Therefore we must have

$$\int_D \frac{f(x)}{x-x_0}\mathrm{d}x-\frac{T(x_0)}{A}=\frac{P(x_0)}{A} \qquad \ldots\ldots\ldots\ldots \quad (3)$$

for all points x_0 in D and in particular for those at which $T(x)=0$. Since we exclude the self stress of a dislocation from the condition of equilibrium, the Cauchy principal value of the singular integral in (3) is to be taken. The inversion of this singular integral equation to find $f(x)$ for a given $P(x)$, with $T(x)=0$, is given by the following theorem.

III. Inversion Theorem

Singular integral equations such as (3) have been investigated by Muskhelishvili (1953*a*, 1953*b*) and the inversion theorem for (3) is the following (Muskhelishvili 1953*a*, p. 251).

Suppose $P(x)$ is a known and $f(x)$ an unknown function and that D consists of p finite segments of the x-axis (a_1, b_1), (a_2, b_2), . . ., (a_p, b_p). Suppose that at q of the end-points of the segments, denoted by $c_1, c_2, \ldots, c_q$, $f(x)$ is to remain bounded, and that at the remaining $2p-q$ end-points, denoted by c_{q+1}, $c_{q+2}, \ldots, c_{2p}$, $f(x)$ may be unbounded. Let

$$R_1(x)=\prod_{k=1}^{q}(x-c_k),\quad R_2(x)=\prod_{k=q+1}^{2p}(x-c_k).$$

Then, if $p-q\geqslant 0$, solutions of (**3**), bounded at $c_1, \ldots, c_q$, always exist and are given by

$$f(x_0)=\frac{-1}{\pi^2 A}\sqrt{\left\{\frac{R_1(x_0)}{R_2(x_0)}\right\}}\int_D\sqrt{\left\{\frac{R_2(x)}{R_1(x)}\right\}}\frac{P(x)\mathrm{d}x}{x-x_0}+\sqrt{\left\{\frac{R_1(x_0)}{R_2(x_0)}\right\}}Q_{p-q-1}(x_0), \quad \ldots (\mathbf{4})$$

where $Q_{p-q-1}(x_0)$ is an arbitrary polynomial of degree not greater than $p-q-1$ (it is identically zero for $p=q$).

If $p-q<0$, a unique solution, bounded at $c_1, \ldots, c_q$, exists if and only if $P(x)$ satisfies the conditions

$$\int_D\sqrt{\left\{\frac{R_2(x)}{R_1(x)}\right\}}x^n P(x)\mathrm{d}x=0, \quad \text{for } n=0,\ 1, \ldots, (q-p-1), \quad \ldots\ldots (\mathbf{5})$$

and if this is so the solution is given by (**4**) with $Q_{p-q-1}(x)\equiv 0$. Moreover at a bounded end-point, $f(x)$ vanishes.

IV. End Conditions

Most of the problems considered by Eshelby, Frank, and Nabarro are of the type where a group of dislocations would move off to infinity owing either to their mutual repulsion or to an applied stress. They are prevented from doing so by a barrier in the form of a dislocation which is locked in position by a localized stress.

When setting up the same problem in our approximation we can proceed in one of two ways. Either we can leave the locked dislocation intact and only smear the free dislocations, or we can smear the locked dislocation too, in which case the barrier to the dislocation distribution becomes the localized stress field which locked the dislocation. This latter type of barrier, which we shall term a block, we take as a repulsive stress field which rises suddenly from zero to infinity. The effect of such a barrier has been taken into account in (**2**) and (**3**) by the term $T(x)$. For a locked dislocation it is found that the appropriate boundary condition on $f(x)$ is that it becomes zero at a small distance from the locked dislocation, this distance being a function of the stress forcing the distribution against the barrier.

V. Examples

The general method will be illustrated by some problems of physical interest. Some of these have already been considered by Eshelby, Frank, and Nabarro, and a comparison of the results obtained by the two methods indicates that the approximate method is little in error.

(i) n *positive dislocations in the potential trough given by* P(x) = —Cx. $f(x)$ will be symmetrical about $x=0$, and we assume it becomes zero at $x=\pm a$, where a will depend on n.

With the notation of Section III we have

$$p=1, \qquad q=2,$$
$$R_1(x)=(x^2-a^2), \quad R_2(x)=1,$$

and since $p-q<0$ a solution will exist only if $P(x)$ satisfies (5). As this is so it follows that $f(x)$ is given by (4) to be

$$f(x)=\frac{C}{A\pi}\sqrt{(a^2-x^2)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

The constant a is determined by

$$\int_{-a}^{a} f(x)\mathrm{d}x=n,$$

which gives

$$a=\sqrt{(2nA/C)}. \quad \ldots\ldots\ldots\ldots\ldots \quad (7)$$

It is not possible to compare (6) with the corresponding result of Eshelby, Frank, and Nabarro, but they find that all dislocations lie in a region

$$|\,x\,|<\sqrt{\{(2n+1)A/C\}},$$

which is nearly equal to the value given by (7).

(ii) n *positive dislocations between blocks at* x $=\pm$a *with no applied shear stress*. Since $P(x)=0$ we get a non-zero solution for $f(x)$ only if we allow it to be unbounded at $x=\pm a$.

Then

$$p=1,\ q=0,\ p-q>0,$$
$$R_1(x)=1,\ R_2(x)=x^2-a^2,$$
$$f(x)=Q_0/\sqrt{(a^2-x^2)},$$

where Q_0 is the arbitrary constant which the arbitrary polynomial Q_{p-q-1} in (4) becomes. Q_0 is determined by

$$n=\int_{-a}^{a} f(x)\mathrm{d}x,$$

which gives

$$Q_0=n/\pi.$$

(iii) n *positive dislocations between unit positive dislocations locked at* x $=\pm$a. Let $f(x)$ become zero at $x=\pm b$ $(b<a)$. Then

$$p=1,\ q=2,\ p-q<0,$$
$$R_1(x)=x^2-b^2,\ R_2(x)=1,$$

and a solution will exist if

$$P(x)=A\left\{\frac{1}{x+a}+\frac{1}{x-a}\right\}$$

satisfies (5). This it does and (4) gives

$$f(x)=\frac{2a}{\pi\sqrt{(a^2-b^2)}}\ \frac{\sqrt{(b^2-x^2)}}{a^2-x^2},$$

and b is determined by

$$n=\int_{-b}^{b} f(x)\mathrm{d}x$$

to be

$$b=a\sqrt{\{1-(\tfrac{1}{2}n+1)^{-2}\}}.$$

(iv) n *positive dislocations on the positive half of the* x-*axis, forced against a block at* x=0 *by a uniform stress* P(x) = $-\sigma$. Let $f(x)$ be bounded at $x=a$ and unbounded at $x=0$. Then

$$p=1,\ q=1,\ p-q=0,$$
$$R_1(x)=x-a,\ R_2(x)=x,$$

and (**4**) gives

$$f(x)=\frac{\sigma}{\pi A}\sqrt{\left(\frac{a-x}{x}\right)}. \qquad \text{(8)}$$

a is determined by

$$n=\int_{0}^{a} f(x)\mathrm{d}x$$

as

$$a=2nA/\sigma. \qquad \text{(9)}$$

Eshelby, Frank, and Nabarro have considered this case and (**8**) and (**9**) are in agreement with the expressions they have found by the exact treatment.

The total stress at any point on the x-axis due to the dislocation distribution and the applied stress is given by (**2**) as

$$S(x)=-\sigma\sqrt{\left(\frac{x-a}{x}\right)}, \qquad x>a,\ x<0,$$
$$=0, \qquad a>x>0.$$

(v) *Blocks at* x $=\pm$a *and a dislocation source at* x=0. A uniform stress $P(x)=\sigma$ causes the source to generate equal numbers of positive and negative dislocations which move off in opposite directions until held up by the blocks. The source continues to generate dislocations until the net stress at the source is reduced to zero. $f(x)$ will be unbounded at $x=\pm a$, so

$$p=1,\ q=0,\ p-q>0,$$
$$R_1(x)=1,\ R_2(x)=x^2-a^2,$$
$$f(x)=\frac{\sigma}{\pi A}\ \frac{x}{\sqrt{(a^2-x^2)}}+\frac{Q_0}{\sqrt{(a^2-x^2)}},$$

where the arbitrary constant Q_0 is determined by the position of the source. Since $f(x)$ changes sign on either side of the source it will be zero at the position of the source. Hence Q_0 is zero for the simple symmetrical case.

The number of positive dislocations generated is given by

$$n=\int_{0}^{a} f(x)\mathrm{d}x=\sigma a/\pi A, \qquad \text{(10)}$$

and is equal to the number of negative dislocations.

The stress on the x-axis beyond the blocks is given by (**2**) as

$$S(x)=\sigma\mid x\mid/\sqrt{(x^2-a^2)}.$$

This is identical with the expression given by Starr (1928) for the stress in front of an infinitely narrow two dimensional crack extending from $x=-a$ to $x=a$ under a uniform applied stress. This is not surprising since it may be seen that the two problems are physically the same.

(vi) *Blocks at* x $=\pm a$ *with* n *positive dislocations between* x $=$ b *and* x $=$ a, *and* n *negative dislocations, between* x $=-$b *and* x $=-$a, *held apart by applied stress* P(x) $=\sigma$. Let $f(x)$ be unbounded at $x=\pm a$ and bounded at $x=\pm b$. Then

$$p=2,\ q=2,\ p-q=0,$$
$$R_1(x)=(x^2-b^2),\ R_2(x)=(x^2-a^2),$$
$$f(x)=\pm\frac{\sigma}{\pi A}\sqrt{\left(\frac{x^2-b^2}{a^2-x^2}\right)},$$

where the sign is to be taken as positive for $b<x<a$ and as negative for $-a<x<-b$. b is related to the number of dislocation pairs by the relation

$$n=\frac{\sigma}{\pi A}\int_b^a\sqrt{\left(\frac{x^2-b^2}{a^2-x^2}\right)}\mathrm{d}x$$
$$=\frac{\sigma}{\pi A}\left\{aE[\surd(1-b^2/a^2)]-\frac{b^2}{a}K[\surd(1-b^2/a^2)]\right\},$$

where K and E are complete elliptic integrals of the first and second kinds respectively. If n is small, or to be more exact if b is nearly equal to a, this becomes

$$n\simeq\frac{\sigma}{2A}(a-b).$$

For b equal to zero this gives

$$n\simeq\sigma a/2A,$$

which is approximately the value given by (**10**).

(vii) n *positive dislocations beyond* x $=$ a *and* n *negative dislocations beyond* x $=-$a, *driven together by a uniform stress* P(x) $=-\sigma$, *but prevented from coalescing by blocks at* x $=\pm$a. Let the distribution be bounded at $x=\pm b$ and unbounded at $x=\pm a$. Then

$$p=2,\ q=2,\ p-q=0,$$
$$R_1(x)=(x^2-b^2),\ R_2(x)=(x^2-a^2),$$
$$f(x)=\pm\frac{\sigma}{\pi A}\sqrt{\left(\frac{b^2-x^2}{x^2-a^2}\right)},$$

and b is given by

$$n=\frac{\sigma}{\pi A}\int_a^b\sqrt{\left(\frac{b^2-x^2}{x^2-a^2}\right)}\mathrm{d}x$$
$$=\frac{\sigma b}{\pi A}\{K[\surd(1-a^2/b^2)]-E[\surd(1-a^2/b^2)]\}$$
$$\simeq\frac{\sigma b}{\pi A}\left\{\ln\left(\frac{4b}{a}\right)-1\right\},\quad\text{for } b\gg a.$$

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VII. References

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