EFFICIENCIES IN THE METHOD OF GROUPING

By P. G. GUEST*

[*Manuscript received July 27, 1953*]

Summary

The efficiencies obtained in curve fitting by the method of grouping are discussed in terms of two parameters $\varkappa_2$, $\varkappa_3$ which specify the departure from uniform spacing. For polynomials of the first and second degree the efficiencies practically always exceed 0·7, but the efficiencies for the third degree polynomial may be less than this value if the spacing is markedly non-uniform.

I. INTRODUCTION

In an earlier paper (Guest 1952) a method of fitting polynomials to unequally spaced observations was described, the method being named the method of grouping. Although intended for use in cases where the spacing was non-uniform, it was only possible at that time to discuss the efficiencies for cases in which the variation from uniformity was random and the standard errors did not differ markedly from the errors in the equally spaced case.

Since the publication of this paper a method of treating cases in which the spacing is non-uniform has been devised, the departure from uniform spacing being characterized by two parameters $\varkappa_2$, $\varkappa_3$. The behaviour of the standard errors in the least squares problem has been described in terms of the two parameters (Guest 1953). In the present paper the calculation of the efficiencies of the values obtained by the method of grouping will be carried out in terms of these same parameters.

II. CALCULATION OF THE STANDARD ERRORS

The coefficients b_{pj} in the fitted polynomial

$$u_p(x)=\sum_{j=0}^{p} b_{pj}x^j$$

are determined in the method of grouping by the solution of the "normal" equations

$$\sum_r W_k(x)\left\{y(x)-\sum_{j=0}^{p} b_{pj}x^j\right\}=0, \qquad k=0 \text{ to } p, \quad \dots\dots\dots\dots \quad (1)$$

where the $y(x)$ represent the observations and the $W_k(x)$ are step functions. The methods of solving these equations consist in eliminating in turn the coefficients b_{p0}, b_{p1}, etc. The most convenient method is some variant of what Dwyer

* Physics Department, University of Sydney.

(1951) calls the method of single division. Equations (1) are in effect converted to the set

$$\sum_x W_{k.k}(x)\left\{y(x)-\sum_j b_{pj}x^j\right\}=0, \quad k=0 \text{ to } p, \quad \ldots\ldots \quad (2)$$

where the functions $W_{k.k}(x)$ are linear combinations of the functions $W_k(x)$ such that

$$\sum_x W_{k.k}(x)x^m=0, \quad m<k. \quad \ldots\ldots \quad (3)$$

$W_{k.k}(x)$ may be expanded in the form

$$W_{k.k}(x)=W_k(x)+\sum_{m=0}^{k-1}\alpha_{km}W_{m.m}(x), \quad \ldots\ldots \quad (4)$$

and it follows from (3) that

$$\alpha_{km}=-\sum W_k(x)x^m/\sum W_{m.m}(x)x^m. \quad \ldots\ldots \quad (5)$$

The value α_{kk} is defined to be -1. From the coefficients α_{km} the coefficients β_{km} defined by the equations

$$W_{k.k}(x)=\sum_{m=0}^{k}\beta_{km}W_m(x) \quad \ldots\ldots \quad (6)$$

may be derived. In fact

$$\beta_{km}=\sum_{r=m}^{k-1}\alpha_{kr}\beta_{rm}. \quad \ldots\ldots \quad (7)$$

For the coefficient of degree p, equation (2) with $k=p$ gives

$$b_{pp}=\sum_x W_{p.p}(x)y(x)\Big/\sum_x W_{p.p}(x)x^p,$$

and so

$$\sigma^2(b_{pp})/\sigma^2(y)=\sum_x\left[W_{p.p}(x)\right]^2\Big/\left[\sum_x W_{p.p}(x)x^p\right]^2. \quad \ldots\ldots \quad (8)$$

The expression $\sum_x\left[W_{p.p}(x)\right]^2$ can be evaluated from (6) when the values of $\sum_x W_k^2(x)$, $\sum_x W_k(x)W_m(x)$ are known.

If the efficiencies of the other coefficients b_{pj} or the efficiencies of the fitted values are required, it is necessary to complete the inversion of the matrix $\sum W_k(x)x^m$. Functions $W_{k.p}(x)$ may be defined for which

$$\sum_x W_{k.p}(x)x^m=0, \quad m\leqslant p, \quad m\neq k, \quad \ldots\ldots \quad (9)$$

and then

$$b_{pk}=\frac{\sum W_{k.p}(x)y(x)}{\sum W_{k.p}(x)x^k}=\sum_x\sum_{m=0}^{p}\lambda_{km}W_m(x)y(x), \quad \ldots\ldots \quad (10)$$

where λ_{km} are the elements of the inverse matrix. These elements can be built up from the quantities β_{jk}, $\Sigma W_{j.j}(x)x^k$ in the following way. $W_{k.p}(x)$ is expanded in the form

$$W_{k.p}(x)=W_{k.k}(x)+\sum_{k+1}^{p}\alpha_{km}W_{m.p}(x),$$

where, from (9)

$$\alpha_{km}=-\frac{\Sigma W_{k.k}(x)x^m}{\Sigma W_{m.p}(x)x^m}=-\frac{\Sigma W_{k.k}(x)x^m}{\Sigma W_{m.m}(x)x^m}.$$

Therefore

$$W_{k.p}(x)\Big/\sum_x W_{k.p}(x)x^k=\left[W_{k.k}(x)-\sum_m\left\{\sum_x W_{k.k}(x)x^m\right\}\frac{W_{m.p}(x)}{\sum_x W_{m.m}(x)x^m}\right]\Big/\sum_x W_{k.k}(x)x^k$$

$$=\left[\sum_r\beta_{kr}W_r(x)-\sum_m\left\{\sum_x W_{k.k}(x)x^m\right\}\sum_r\lambda_{mr}W_r(x)\right]\Big/\sum_x W_{k.k}(x)x^k$$

$$=\sum_r\lambda_{kr}W_r(x).$$

Hence

$$\lambda_{kr}=\left[\beta_{kr}-\sum_{k+1}^{p}\lambda_{mr}\left\{\sum_x W_{k.k}(x)x^m\right\}\right]\Big/\sum_x W_{k.k}(x)x^k. \quad\ldots\ldots\ldots \quad (11)$$

The standard error of the coefficient b_{pk} is given by

$$\sigma^2(b_{pk})/\sigma^2(y)=\sum_x\left\{\sum_{m=0}^{p}\lambda_{km}W_m(x)\right\}^2, \quad\ldots\ldots\ldots\ldots \quad (12)$$

and the standard error of the fitted value by

$$\sigma^2[u_p(x)]=E\left[\sum_j b_{pj}x^j\right]^2=\sum_{j,k}x^jx^kE[b_{pj}b_{pk}],$$

or

$$\sigma^2[u_p(x)]/\sigma^2(y)=\sum_{j,k}x^jx^k\sum_x\left\{\sum_r\lambda_{jr}W_r(x)\right\}\left\{\sum_s\lambda_{ks}W_s(x)\right\}. \quad\ldots\ldots \quad (13)$$

The method of matrix inversion outlined above is the same as that described by Fox and Hayes (1951), but they deal with a general matrix for which the functions $W_j(x)$ are not defined. The present discussion brings out the significance of the intermediate terms occurring in their method. In the usual method of calculation the quantities are arranged in a square array, as shown below :

$\Sigma W_{0.0}$	$\Sigma W_{0.0}x$	$\Sigma W_{0.0}x^2$	$\Sigma W_{0.0}x^3$
$(-)\alpha_{10}$	$\Sigma W_{1.1}x$	$\Sigma W_{1.1}x^2$	$\Sigma W_{1.1}x^3$
$(-)\alpha_{20}$	$(-)\alpha_{21}$	$\Sigma W_{2.2}x^2$	$\Sigma W_{2.2}x^3$
$(-)\alpha_{30}$	$(-)\alpha_{31}$	$(-)\alpha_{32}$	$\Sigma W_{3.3}x^3$.

The lower triangular matrix $(-)\underset{\sim}{\alpha}$ is then inverted; the elements of $-\underset{\sim}{\alpha}^{-1}$ are the quantities β_{km}, as is clear from (7). Finally the rows λ_{jk} are built up in turn, beginning with λ_{pk}. However, when a large number of inversions have to be made it is more convenient to tabulate the α_{km}, β_{km}, etc., in columns and perform the same calculations for all the matrices at the same time.

It is illuminating to put some of the above equations into matrix notation. The quantities $\sum_x W_j(x)x^k$ form a matrix $\mathbf{W}$, the quantities α_{jk} a lower triangular matrix $\underset{\sim}{\alpha}$, and the quantities $\sum_x W_{j.j}(x)x^k$ an upper triangular matrix $\underset{\sim}{\omega}$. Then, from equation (4)

$$\sum_x W_{k.k}(x)x^r = \sum_x W_k(x)x^r + \sum_{m=0}^{k-1} \alpha_{km} \sum_x W_{m.m}(x)x^r,$$

or

$$\mathbf{W} = -\underset{\sim}{\alpha}\underset{\sim}{\omega}.$$

Then from equation (7)

$$\underset{\sim}{\alpha}\underset{\sim}{\beta} = -\mathbf{I},$$

and so

$$\underset{\sim}{\beta} = -\underset{\sim}{\alpha}^{-1}.$$

Finally equation (11) may be written

$$\omega_{kk}\lambda_{kr} = \beta_{kr} - \sum_m \omega_{km}\lambda_{mr},$$

and so

$$\underset{\sim}{\beta} = \underset{\sim}{\omega}\underset{\sim}{\lambda},$$

and

$$\begin{aligned}\underset{\sim}{\lambda} &= \underset{\sim}{\omega}^{-1}\underset{\sim}{\beta} = -\underset{\sim}{\omega}^{-1}\underset{\sim}{\alpha}^{-1}\\ &= \mathbf{W}^{-1}.\end{aligned}$$

III. Efficiencies in Terms of the Parameters $\varkappa_2$, $\varkappa_3$

The symbol ε will be used to denote the variable which takes the integral or half-integral values from $+\frac{1}{2}(n-1)$ to $-\frac{1}{2}(n-1)$ at the points of observation, where n is the number of observations. In the method of grouping (Guest 1952, Section III)

$$\left.\begin{aligned}\sum_\varepsilon W_j\varepsilon^m &= \frac{n^{m+1}}{(m+1)2^m}[1-\alpha_j^{m+1}-\beta_j^{m+1}+\ldots\ldots], && m+j \text{ even,}\\ &=0, && m+j \text{ odd,}\end{aligned}\right\}\ldots(14)$$

where α_j, β_j, etc., are the parameters which determine the location of the groups. Equation (14) can be written in the form

$$\sum_\varepsilon W_j\varepsilon^m = \frac{n^{m+1}}{(m+1)2^m}\psi_{jm}.$$

The quantities ψ_{jm} may be readily calculated from the values of the parameters α_j, β_j, etc., given in the previous paper (Guest 1952). The numerical values of ψ_{jm} are listed in Table 1. If the variable $e=2\varepsilon/n$ is introduced, then

$$n^{-1}\Sigma W_j e^m=\psi_{jm}/(m+1). \qquad (15)$$

In accordance with the treatment given in an earlier paper (Guest 1953, p. 132), the independent variable x is replaced by the variable ξ, where

$$\xi=\varepsilon+\varkappa_2 n^{-1}\left(\varepsilon^2-\frac{1}{12}n^2\right)+2\varkappa_3 n^{-2}\left(\varepsilon^3-\frac{1}{4}n^2\varepsilon\right), \qquad (16)$$

and $\varkappa_2$, $\varkappa_3$ are the parameters which specify the departure from uniform spacing.

Now

$$12n^{-1}\xi=6e+\varkappa_2(3e^2-1)+3\varkappa_3(e^3-e),$$

and so

$$n^{-1}\Sigma W_j(12n^{-1}\xi)=3\psi_{j1}+\varkappa_3\left(\frac{3}{4}\psi_{j3}-\frac{3}{2}\psi_{j1}\right)+\varkappa_2(\psi_{j2}-\psi_{j0})$$

$$\begin{aligned}n^{-1}\Sigma W_j(12n^{-1}\xi)^2=&\,12\psi_{j2}+\varkappa_3\left(\frac{36}{5}\psi_{j4}-12\psi_{j2}\right)+\varkappa_3^2\left(\frac{9}{7}\psi_{j6}-\frac{18}{5}\psi_{j4}+3\psi_{j2}\right)\\&+\varkappa_2^2\left(\frac{9}{5}\psi_{j4}-2\psi_{j2}+\psi_{j0}\right)\\&+\varkappa_2[(9\psi_{j3}-6\psi_{j1})+\varkappa_3(3\psi_{j5}-6\psi_{j3}+3\psi_{j1})],\end{aligned}$$

$$\begin{aligned}n^{-1}\Sigma W_j(12n^{-1}\xi)^3=&\,54\psi_{j3}+\varkappa_3(54\psi_{j5}-81\psi_{j3})+\varkappa_3^2\left(\frac{81}{4}\psi_{j7}-54\psi_{j5}+\frac{81}{2}\psi_{j3}\right)\\&+\varkappa_3^3\left(\frac{27}{10}\psi_{j9}-\frac{81}{8}\psi_{j7}+\frac{27}{2}\psi_{j5}-\frac{27}{4}\psi_{j3}\right)\\&+\varkappa_2^2\left[(27\psi_{j5}-27\psi_{j3}+9\psi_{j1})+\varkappa_3\left(\frac{81}{8}\psi_{j7}-\frac{45}{2}\psi_{j5}+\frac{63}{4}\psi_{j3}-\frac{9}{2}\psi_{j1}\right)\right]\\&+\varkappa_2\left[\left(\frac{324}{5}\psi_{j4}-36\psi_{j2}\right)+\varkappa_3\left(\frac{324}{7}\psi_{j6}-\frac{432}{5}\psi_{j4}+36\psi_{j2}\right)\right.\\&\qquad+\varkappa_3^2(9\psi_{j8}-27\psi_{j6}+27\psi_{j4}-9\psi_{j2})\\&\qquad\left.+\varkappa_2^2\left(\frac{27}{7}\psi_{j6}-\frac{27}{5}\psi_{j4}+3\psi_{j2}-\psi_{j0}\right)\right].\end{aligned}$$

Using Table 1, it is a simple matter to calculate $\Sigma W_j\xi^m$ as a function of $\varkappa_2$, $\varkappa_3$. These expressions are tabulated in Table 2, together with the quantities ΣW_j^2, $\Sigma W_j W_k$, which are needed for the evaluation of the standard errors.

TABLE 1

THE QUANTITIES ψ_{jm}

			$p=1, 2$	$p=3$		
ψ_{20}	$-0\cdot 2943$	ψ_{11}	$+0\cdot 888889$	$+0\cdot 502270$	ψ_{31}	$-0\cdot 514459$
ψ_{22}	$+0\cdot 385331$	ψ_{13}	$+0\cdot 987654$	$+0\cdot 752265$	ψ_{33}	$-0\cdot 181496$
ψ_{24}	$+0\cdot 669354$	ψ_{15}	$+0\cdot 998628$	$+0\cdot 876695$	ψ_{35}	$+0\cdot 078761$
ψ_{26}	$+0\cdot 809080$	ψ_{17}	$+0\cdot 999848$	$+0\cdot 938627$	ψ_{37}	$+0\cdot 275442$
ψ_{28}	$+0\cdot 885873$	ψ_{19}	$+0\cdot 999983$	$+0\cdot 969453$	ψ_{39}	$+0\cdot 425384$

TABLE 2

$\Sigma W_j\xi^m$

$n^{-1}\Sigma W_0$	1
$n^{-1}\Sigma W_1$	0
$n^{-1}\Sigma W_2$	$-0\cdot 2943$
$n^{-1}\Sigma W_3$	0
$4n^{-2}\Sigma W_0\xi$	0
$4n^{-2}\Sigma W_1\xi$	$0\cdot 888889-0\cdot 197531\varkappa_3 \quad (p=1, 2)$
	$0\cdot 502270-0\cdot 063069\varkappa_3 \quad (p=3)$
$4n^{-2}\Sigma W_2\xi$	$0\cdot 226544\varkappa_2$
$4n^{-2}\Sigma W_3\xi$	$-0\cdot 514459+0\cdot 211856\varkappa_3$
$12n^{-3}\Sigma W_0\xi^2$	$1-0\cdot 400000\varkappa_3+0\cdot 057143\varkappa_3^2+0\cdot 066667\varkappa_2^2$
$12n^{-3}\Sigma W_1\xi^2$	$\varkappa_2[0\cdot 296296-0\cdot 021948\varkappa_3] \quad (p=1, 2)$
	$\varkappa_2[0\cdot 313064-0\cdot 031391\varkappa_3] \quad (p=3)$
$12n^{-3}\Sigma W_2\xi^2$	$0\cdot 385331+0\cdot 016281\varkappa_3-0\cdot 017786\varkappa_3^2+0\cdot 011656\varkappa_2^2$
$12n^{-3}\Sigma W_3\xi^2$	$\varkappa_2[0\cdot 121108-0\cdot 018177\varkappa_3]$
$32n^{-4}\Sigma W_0\xi^3$	$\varkappa_2[0\cdot 533333-0\cdot 076190\varkappa_3+0\cdot 008466\varkappa_2^2]$
$32n^{-4}\Sigma W_1\xi^3$	$0\cdot 752265-0\cdot 251702\varkappa_3+0\cdot 039489\varkappa_3^2-0\cdot 002379\varkappa_3^3$
	$+\varkappa_2^2[0\cdot 145927-0\cdot 011742\varkappa_3]$
$32n^{-4}\Sigma W_2\xi^3$	$\varkappa_2[0\cdot 546337-0\cdot 120582\varkappa_3+0\cdot 013561\varkappa_3^2+0\cdot 017713\varkappa_2^2]$
$32n^{-4}\Sigma W_3\xi^3$	$-0\cdot 181496+0\cdot 351005\varkappa_3-0\cdot 111592\varkappa_3^2+0\cdot 012001\varkappa_3^3$
	$+\varkappa_2^2[0\cdot 044385+0\cdot 008764\varkappa_3]$
$n^{-1}\Sigma W_0^2$	1
$n^{-1}\Sigma W_1^2$	$0\cdot 666667 \quad (p=1, 2)$
	$0\cdot 2945 \quad (p=3)$
$n^{-1}\Sigma W_2^2$	$0\cdot 7265$
$n^{-1}\Sigma W_3^2$	$0\cdot 7894$
$n^{-1}\Sigma W_0W_2$	$-0\cdot 2943$
$n^{-1}\Sigma W_1W_3$	$-0\cdot 0417$

The standard errors of the coefficients b_{pp} can be calculated for selected values of $\varkappa_2$, $\varkappa_3$ from the values $\Sigma W_j\xi^k$, using the scheme developed in Section II. The efficiencies can then be calculated by comparison with the corresponding errors obtained for the least squares curve in the earlier paper. The efficiencies obtained in this way for the coefficients b_{11}, b_{22}, b_{33}, are listed in Table 3.

In Table 4 the efficiencies of the fitted values have been tabulated for various values of $\varkappa_2$, $\varkappa_3$. For the second and third degree polynomials the variable used is

$$k = e - \varkappa_2/5,$$

which was introduced in the treatment of the least squares standard errors (Guest 1953, equation (**41**)). The range of interpolation, that is, the range of values of k within which the observations occur, is roughly from $k=+1$ to $k=-1$.

TABLE 3

EFFICIENCIES OF THE COEFFICIENTS b_{pp}

$\varkappa_3$ \ $\varkappa_2^2$	b_{11}					b_{22}					b_{33}		
	0	0·5	1·0	1·5	2·0	0	0·25	0·5	0·75	1·0	0	0·25	0·5
−1·0	0·911	0·891	0·871	0·853	0·835	0·853	0·801	0·751	0·705	0·660	0·813	0·641	0·498
−0·8	0·909	0·887	0·866	0·847	0·828	0·872	0·819	0·768	0·720	0·675	0·861	0·692	0·548
−0·6	0·906	0·882	0·860	0·839	0·819	0·887	0·834	0·783	0·734	0·687	0·897	0·735	0·593
−0·4	0·901	0·876	0·853	0·830	0·809	0·897	0·844	0·793	0·744	0·696	0·921	0·769	0·632
−0·2	0·896	0·869	0·844	0·820	0·798	0·902	0·851	0·801	0·751	0·703	0·934	0·793	0·663
0	0·889	0·860	0·833	0·808	0·784	0·902	0·853	0·804	0·755	0·707	0·937	0·807	0·685
0·2	0·880	0·849	0·821	0·794	0·769	0·897	0·851	0·803	0·755	0·708	0·929	0·810	0·699
0·4	0·869	0·836	0·806	0·777	0·751	0·888	0·844	0·799	0·752	0·705	0·913	0·804	0·702
0·6	0·855	0·820	0·788	0·758	0·731	0·874	0·833	0·790	0·745	0·698	0·888	0·789	0·696
0·8	0·839	0·801	0·767	0·736	0·707	0·855	0·818	0·777	0·733	0·688	0·856	0·765	0·681
1·0	0·818	0·779	0·743	0·710	0·680	0·833	0·799	0·760	0·718	0·673	0·817	0·733	0·657
1·2	0·794	0·752	0·715	0·681		0·808	0·776	0·739	0·698	0·653	0·770	0·693	0·626
1·4	0·764	0·721	0·682			0·780	0·749	0·713	0·672	0·628	0·715	0·645	0·587
1·6	0·729	0·684	0·644			0·751	0·719	0·683	0·641	0·596			
1·8	0·688	0·642	0·602			0·719	0·686	0·647	0·604	0·556			
2·0	0·640	0·594	0·554			0·687	0·649	0·606	0·558	0·507			

In a practical example the efficiencies may be expected to differ somewhat from the values given in Tables 3 and 4, because of the neglect in the present discussion of higher parameters $\varkappa_4$, $\varkappa_5$, etc. In Table 5 are shown the efficiencies of the coefficients b_{pp} for the three examples discussed in an earlier paper (Guest 1953), with the values calculated from the parameters $\varkappa_2$, $\varkappa_3$ in brackets. It is seen that there is in each case a reasonable agreement between the two values for the efficiency.

IV. CONCLUSION

From Table 3 it will be seen that the efficiencies of the coefficients b_{pp} are always less than the corresponding efficiencies in the equally spaced case ($\varkappa_2=0$, $\varkappa_3=0$), except for the coefficient b_{11} when $\varkappa_3$ is negative, where the efficiency may be slightly higher. As a consequence, the value of 90 per cent. suggested in the earlier paper for the efficiencies must be regarded as an upper limit to the efficiencies which would be found in any practical example.

TABLE 4

EFFICIENCIES OF THE FITTED VALUES

First Degree Polynomial

$\lvert\varkappa_2\rvert$	0					0·5					1·0				
$\varkappa_3$ / $\lvert e\rvert$	−1·0	−0·5	0	+0·5	+1·0	−1·0	−0·5	0	+0·5	+1·0	−1·0	−0·5	0	+0·5	+1·0
1·4	0·923	0·917	0·903	0·880	0·840	0·914	0·906	0·891	0·865	0·822	0·888	0·875	0·854	0·821	0·772
1·2	0·927	0·920	0·908	0·885	0·847	0·918	0·910	0·895	0·871	0·830	0·893	0·881	0·860	0·829	0·781
1·0	0·932	0·926	0·914	0·893	0·857	0·924	0·916	0·903	0·879	0·840	0·900	0·889	0·870	0·840	0·794
0·8	0·940	0·935	0·924	0·905	0·873	0·933	0·926	0·914	0·893	0·857	0·912	0·901	0·884	0·857	0·815
0·6	0·952	0·948	0·939	0·924	0·897	0·946	0·941	0·931	0·913	0·884	0·929	0·920	0·906	0·883	0·848
0·4	0·969	0·967	0·961	0·951	0·933	0·966	0·962	0·955	0·944	0·924	0·954	0·949	0·939	0·924	0·899
0·2	0·990	0·989	0·987	0·983	0·977	0·988	0·987	0·985	0·981	0·974	0·984	0·982	0·979	0·973	0·964
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Second Degree Polynomial

$\lvert\varkappa_2\rvert$	0					0·5					1·0				
$\varkappa_3$	−1·0	−0·5	0	+0·5	+1·0	−1·0	−0·5	0	+0·5	+1·0	−1·0	−0·5	0	+0·5	+1·0
$\lvert k\rvert$								($k\varkappa_2$ negative)							
1·4	0·868	0·898	0·903	0·880	0·833	0·835	0·871	0·887	0·877	0·838	0·726	0·757	0·775	0·775	0·748
1·2	0·879	0·902	0·904	0·881	0·835	0·854	0·883	0·896	0·885	0·846	0·766	0·786	0·799	0·796	0·765
1·0	0·901	0·911	0·908	0·884	0·838	0·893	0·905	0·911	0·898	0·858	0·855	0·847	0·844	0·831	0·795
0·8	0·940	0·930	0·918	0·893	0·848	0·950	0·942	0·935	0·919	0·879	0·962	0·948	0·927	0·899	0·852
0·6	0·952	0·952	0·939	0·915	0·874	0·938	0·951	0·950	0·940	0·911	0·872	0·914	0·943	0·957	0·942
0·4	0·924	0·946	0·950	0·938	0·912	0·886	0·914	0·926	0·927	0·921	0·779	0·815	0·840	0·859	0·877
0·2	0·904	0·935	0·946	0·941	0·925	0·863	0·896	0·911	0·911	0·904	0·750	0·785	0·807	0·816	0·810
								($k\varkappa_2$ positive)							
0	0·898	0·931	0·943	0·939	0·924	0·861	0·898	0·916	0·918	0·906	0·754	0·793	0·821	0·834	0·826
0·2	0·904	0·935	0·946	0·941	0·925	0·876	0·913	0·934	0·936	0·920	0·786	0·831	0·865	0·883	0·876
0·4	0·924	0·946	0·950	0·938	0·912	0·911	0·941	0·951	0·942	0·910	0·858	0·902	0·929	0·937	0·918
0·6	0·952	0·952	0·939	0·915	0·874	0·950	0·948	0·928	0·895	0·848	0·942	0·946	0·927	0·894	0·848
0·8	0·940	0·930	0·918	0·893	0·848	0·922	0·902	0·877	0·842	0·794	0·894	0·860	0·819	0·775	0·725
1·0	0·901	0·911	0·908	0·884	0·838	0·865	0·862	0·848	0·819	0·774	0·786	0·769	0·743	0·708	0·663
1·2	0·879	0·902	0·904	0·881	0·835	0·833	0·845	0·839	0·813	0·768	0·726	0·725	0·710	0·683	0·641
1·4	0·868	0·898	0·903	0·880	0·833	0·818	0·838	0·837	0·813	0·768	0·697	0·705	0·697	0·675	0·635

Third Degree Polynomial

$\lvert\varkappa_2\rvert$	0					0·5				
$\varkappa_3$	−1·0	−0·5	0	+0·5	+1·0	−1·0	−0·5	0	+0·5	+1·0
$\lvert k\rvert$					($k\varkappa_2$ negative)					
1·4	0·830	0·910	0·926	0·887	0·805	0·625	0·747	0·818	0·826	0·779
1·2	0·842	0·909	0·919	0·881	0·802	0·643	0·756	0·823	0·832	0·787
1·0	0·870	0·902	0·905	0·871	0·802	0·736	0·790	0·836	0·842	0·802
0·8	0·874	0·889	0·887	0·870	0·834	0·872	0·897	0·881	0·866	0·834
0·6	0·844	0·902	0·917	0·914	0·901	0·711	0·835	0·895	0·903	0·884
0·4	0·852	0·914	0·936	0·932	0·908	0·695	0·816	0·891	0·918	0·911
0·2	0·879	0·925	0·942	0·936	0·913	0·759	0·858	0·919	0·942	0·939
					($k\varkappa_2$ positive)					
0	0·898	0·931	0·943	0·939	0·924	0·889	0·927	0·938	0·923	0·886
0·2	0·879	0·925	0·942	0·936	0·913	0·907	0·915	0·903	0·873	0·824
0·4	0·852	0·914	0·936	0·932	0·908	0·821	0·860	0·871	0·864	0·836
0·6	0·844	0·902	0·917	0·914	0·901	0·767	0·834	0·872	0·891	0·894
0·8	0·874	0·889	0·887	0·870	0·834	0·820	0·874	0·888	0·875	0·843
1·0	0·870	0·902	0·905	0·871	0·802	0·887	0·879	0·850	0·795	0·714
1·2	0·842	0·909	0·919	0·881	0·802	0·800	0·836	0·826	0·774	0·688
1·4	0·830	0·910	0·926	0·887	0·805	0·752	0·813	0·817	0·772	0·687

TABLE 5

EFFICIENCIES IN PRACTICAL EXAMPLES

Values calculated from $\varkappa_2$, $\varkappa_3$ in brackets

Example	n	$\varkappa_2^2$	$\varkappa_3$	b_{11}	b_{22}	b_{33}
1	16	0·015	—0·642	0·878 (0·906)	0·878 (0·881)	0·902 (0·879)
2	67	0·345	+0·740	0·825 (0·818)	0·824 (0·806)	0·751 (0·739)
3	16	0·221	—0·392	0·884 (0·890)	0·839 (0·850)	0·823 (0·788)

The effect of departures from uniform spacing may be roughly summarized in the following way:

Departure from Uniformity		Efficiency b_{11}	b_{22}	b_{33}
Slight	$\mid\varkappa_2\mid, \mid\varkappa_3\mid < 0{\cdot}25$	$>0{\cdot}875$	$>0{\cdot}875$	$>0{\cdot}900$
Moderate	$\mid\varkappa_2\mid, \mid\varkappa_3\mid < 0{\cdot}5$	$>0{\cdot}850$	$>0{\cdot}840$	$>0{\cdot}750$
Pronounced	$\mid\varkappa_2\mid, \mid\varkappa_3\mid < 0{\cdot}75$	$>0{\cdot}800$	$>0{\cdot}750$	$>0{\cdot}550$

Since the efficiency of the fitted value $u_b(x)$ is at worst only slightly less than the efficiency of the coefficient b_{pp}, the limiting efficiencies of the fitted values will also be given roughly by the above table. However, from Table 4 it will be seen that the efficiency of the fitted value varies quite rapidly with the location of the point (i.e. the coordinate k) in the second and third degree polynomials when the departure from uniformity becomes pronounced.

The value that would be considered acceptable for the efficiency depends very much on the purpose for which the curve is required. If the curve is to summarize the results of 6 months' research, then clearly the least squares curve should be calculated. If a large number of curves are to be plotted, then the method of grouping may well be more appropriate because of the saving in time. Jeffreys' (1948) statement on this point is worth quoting in full.

"If [the estimate] a' has an efficiency of 50 per cent., a' will habitually differ from [the least squares estimate] a by more than the standard error of the latter. This is very liable to be serious. No general rule can be given; we have in particular cases to balance accuracy against the time that would be needed for an accurate calculation, but as a rough guide it may be said that efficiencies over 90 per cent. are practically always acceptable, those between 70 and 90 per cent. usually acceptable, but those under 50 per cent. should be avoided."

Cases in which $\mid\varkappa_2\mid$ or $\mid\varkappa_3\mid$ exceed unity will be very rare. It can be said then that the efficiencies for polynomials of the first and second degree fall into the "usually acceptable" category, while for the third degree polynomial the efficiencies will only fall into this category when the departure from uniformity is not very pronounced.

For the fourth degree polynomial the representation in terms of the two parameters $\varkappa_2$, $\varkappa_3$ is not very satisfactory, but a calculation of the standard errors for the case $\varkappa_3=0$ has shown that the drop in efficiency as $|\varkappa_2|$ increases is even more pronounced than is the case with the third degree polynomial. Consequently the method of grouping should not be used with a polynomial of the fourth degree unless the spacing is roughly uniform.

V. References

Dwyer, P. S. (1951).—" Linear Computations." (John Wiley & Sons: New York.)

Fox, L., and Hayes, J. G. (1951).—*J. R. Statist. Soc.* B **13**: 83.

Guest, P. G. (1952).—*Aust. J. Sci. Res.* A **5**: 238.

Guest, P. G. (1953).—*Aust. J. Phys.* **6**: 131.

Jeffreys, H. (1948).—" Theory of Probability." 2nd Ed. p. 179. (Oxford Univ. Press.)