THE RELATIVISTIC ELECTROMAGNETIC EQUATIONS IN A MATERIAL MEDIUM

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Summary

It is shown how the general relativistic electromagnetic equations for a material medium can be expressed in the form of a single four-vector density equation. The field tensor has six different complex components instead of three, as in the case of a free medium. The classical equations are obtained by separating the real and imaginary parts.

I. INTRODUCTION

It has already been shown how, for empty space, Maxwell's equations in General Relativity may be expressed as a single complex vector density equation from which the complete set of classical equations can be deduced by equating the real and imaginary parts (Taylor 1952). It was pointed out (loc. cit., Section VI) that this seems to be the natural generalization of the quaternionic form used by Silberstein (1924, pp. 46, 206) for classical and special relativistic theory. The purpose of the present investigation is to extend this method of the complex field components to the case of a material medium, for which

$$\mathbf{D}=\varepsilon\mathbf{E}, \quad \mathbf{B}=\mu\mathbf{H}.$$

Some work related to this topic has been attempted before. Following the success of the quaternionic notation in dealing with the electromagnetic equations for a vacuum in classical theory and in Special Relativity, Silberstein (1907) tried a similar treatment for material media. He considered the classical equation (our notation)

$$\frac{\partial\eta}{\partial t}=-\mathrm{i}v\ \mathrm{curl}\ \eta,$$

where

$$\eta=\sqrt{\varepsilon}\mathbf{E}+\mathrm{i}\sqrt{\mu}\mathbf{H}$$

and

$$v=c/\sqrt{\varepsilon\mu}.$$

This is obviously unsuitable for inclusion in a relativistic theory, and besides, the equations div $\mathbf{B}=0$, div $\mathbf{D}=\rho$ cannot be brought into this scheme. Using a different line of attack (1924, p. 260) he obtained the complete set of Maxwell's equations for a ponderable medium in Special Relativity in terms of *two* quaternionic equations involving the two different bivectors (our notation)

$$\mathbf{B}\pm\mathrm{i}\mathbf{E}, \quad \mathbf{H}\pm\mathrm{i}\mathbf{D}. \qquad \text{(a)}$$

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In the usual method there are two tensor equations depending on the two antisymmetrical tensors, each with six independent components, corresponding to the bivectors (**a**). In our notation, the components of these tensors are, respectively,

$$\left.\begin{matrix} 0 & -B_3 & B_2 & iE_1 \\ B_3 & 0 & -B_1 & iE_2 \\ -B_2 & B_1 & 0 & iE_3 \\ -iE_1 & -iE_2 & -iE_3 & 0 \end{matrix} \quad \text{and} \quad \begin{matrix} 0 & -H_3 & H_2 & iD_1 \\ H_3 & 0 & -H_1 & iD_2 \\ -H_2 & H_1 & 0 & iD_3 \\ -iD_1 & -iD_2 & -iD_3 & 0 \end{matrix}\right\} . \qquad \ldots\ldots\ldots\ldots \quad (\mathbf{b})$$

In this form the equations may be readily extended to the continuum of General Relativity. Examples of this tensor method are given in works by Silberstein (1924, pp. 288, 462), McConnell (1936), Costa de Beauregard (1949), and Schouten (1951). The method is based on the investigations of Minkowski (1910) in Special Relativity. He uses a matrix notation, obtaining two equations (loc. cit., p. 38, equations {A} and {B}) in terms of the two matrices corresponding to (**b**). By simply adding Minkowski's two equations we have the equivalent of the single field equation of the present theory (equation (**1**)), for the special case of Cartesian coordinate systems.

Sommerfeld (1948) mentions two bivectors similar to (**a**), but gives the electromagnetic equations in their usual tensor form.

II. Maxwell's Equations and the Field Components

Consider the equation*

$$\frac{\partial}{\partial x^{\nu}}(\mathrm{D}^{\mu\nu})=\mathrm{J}^{\mu}, \qquad \ldots\ldots\ldots\ldots \quad (\mathbf{1})$$

where J^{μ} is the current four-vector density

$$\mathrm{J}^{\mu}=-\frac{1}{c}(j_1,\ j_2,\ j_3,\ ic\rho), \qquad \ldots\ldots\ldots\ldots \quad (\mathbf{2})$$

and $\mathrm{D}^{\mu\nu}$ is the field tensor density derived from the field tensor $D^{\mu\nu}$, whose components in the geodesic Cartesian system (x, y, z, ict) are

$$D^{\mu\nu}=\left.\begin{matrix} 0 & F_3 & -F_2 & N_1 \\ -F_3 & 0 & F_1 & N_2 \\ F_2 & -F_1 & 0 & N_3 \\ -N_1 & -N_2 & -N_3 & 0 \end{matrix}\right\}, \qquad \ldots\ldots\ldots\ldots \quad (\mathbf{3})$$

where

$$\left.\begin{matrix} F_1=iE_1-H_1, \text{ etc.}, \\ N_1=iD_1-B_1, \text{ etc.} \end{matrix}\right\} \qquad \ldots\ldots\ldots\ldots \quad (\mathbf{4})$$

It will be noticed that (**3**) may be obtained from (**b**) by combining one with the dual of the other. However, (**4**) are not the components of Silberstein's bivectors (**a**) except in a free medium, the case previously considered.

If (**1**) is written out in the geodesic Cartesian system, we have, from the real part,

$$\frac{\partial \mathbf{D}}{\partial t}+\mathbf{j}=c \text{ curl } \mathbf{H},$$

$$\text{div } \mathbf{B}=0,$$

* Roman type shall denote tensor densities, while italics shall refer to tensors.

and from the imaginary part,

$$\frac{\partial \mathbf{B}}{\partial t} = -c \text{ curl } \mathbf{E}$$

$$\text{div } \mathbf{D} = \rho.$$

Equations (**1**), (**2**), (**3**), and (**4**) therefore define the complete set of Maxwell's equations in macroscopic terms, for a material medium at rest relative to the observer. Instead of two sets of equations depending on two field tensors, each with six terms, we now have one equation depending on one tensor with six distinct components.

If it is assumed that (**3**) and (**4**) determine the classical components of the field in *any* system of coordinates, an inconsistent set of equations is obtained for the transformation of the field components. We must therefore postulate some other tensor, having fewer distinct terms, whose transformation is to define the field in all systems of coordinates. This tensor shall be, in the geodesic Cartesian system,

$$E^{\mu\nu} = \left.\begin{matrix} 0 & M_3 & -M_2 & M_1 \\ -M_3 & 0 & M_1 & M_2 \\ M_2 & -M_1 & 0 & M_3 \\ -M_1 & -M_2 & -M_3 & 0 \end{matrix}\right\}, \quad \ldots\ldots\ldots\ldots \quad (\mathbf{5})$$

where

$$M_1 = \mathrm{i}E_1 - B_1, \text{ etc.} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{6})$$

This is constructed from the first of (**b**), the tensor which defines the transformation of the field components in the usual relativistic theory of the material medium, and it has the same form as the $F^{\mu\nu}$ which was used in the case of fields in free space (Taylor 1952).

The relation of the field components in one Cartesian system of coordinates (x^μ) to the components in another system (x'^μ) is then given by the transformation equations

$$E'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} E^{\alpha\beta},$$

where, according to our postulate we have

$$E'^{12} = M'_1 = \mathrm{i}E'_1 - B'_1, \text{ etc.}$$

as well as (**5**) and (**6**). Writing this out, using the coefficients $\partial x'^\mu / \partial x^\alpha$ calculated from the Lorentz transformation, we obtain the required classical equations

$$E'_1 = E_1, \qquad B'_1 = B_1,$$

$$E'_{2,3} = \frac{E_{2,3} \mp (v/c) B_{3,2}}{\sqrt{1 - v^2/c^2}}, \quad B'_{2,3} = \frac{B_{2,3} \pm (v/c) E_{3,2}}{\sqrt{1 - v^2/c^2}},$$

for the case where the space of (x'^μ) is moving relative to the space of (x^μ) with velocity v in the x^1 direction.

The fact that the components of $E^{\mu\nu}$ are arranged as shown in (**5**) can be stated in the covariant form

$$\mathrm{E}^{\mu\nu} = \tfrac{1}{2} \varepsilon^{\mu\nu\sigma\tau} E_{\sigma\tau}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{7})$$

A tensor which satisfies an equation of this type is described by the term " self-dual " since the operation indicated on the right-hand side (that is, the formation of the "dual") reproduces the original tensor. Guided by (7) we shall try expressing $E^{\mu\nu}$ in terms of a potential function as follows:

$$\mathrm{E}^{\mu\nu}=(\sqrt{g}g^{\mu\sigma}g^{\nu\tau}+\tfrac{1}{2}\varepsilon^{\mu\nu\sigma\tau})\left(\frac{\partial\varkappa_\sigma}{\partial x^\tau}-\frac{\partial\varkappa_\tau}{\partial x^\sigma}\right). \qquad (8)$$

This identically satisfies (7). By comparing with the similar treatment for $F^{\mu\nu}$ in the case of empty space, we find that (8) includes the classical equations

$$\mathbf{B}=\text{curl }\mathbf{A},$$
$$\mathbf{E}=-\text{grad } V-\dot{\mathbf{A}}/c,$$

where

$$\varkappa_\mu=(A_1,\ A_2,\ A_3,\ \mathrm{i}V).$$

We now show that the six-component system $D^{\mu\nu}$ can be expressed in terms of $E^{\mu\nu}$ and another three-component system, $B^{\mu\nu}$. Take, for the case of the geodesic Cartesian system at rest in the medium,

$$B^{\mu\nu}=\begin{matrix} 0 & 0 & 0 & L_1 \\ 0 & 0 & 0 & L_2 \\ 0 & 0 & 0 & L_3 \\ -L_1 & -L_2 & -L_3 & 0, \end{matrix}$$

where

$$L_1=N_1-F_1=\mathrm{i}(D_1-E_1)-(B_1-H_1),\ \text{etc.}$$

This tensor vanishes in an empty region, and is of an appropriate form for matter at rest, since it attributes a special distinction to components involving a time index. It therefore seems a suitable tensor for describing the departure of the field, in the presence of matter, from the completely isotropic (i.e. invariant in form under coordinate transformations) part represented by $E^{\mu\nu}$. We then have

$$\mathrm{D}^{\mu\nu}=\mathrm{E}^{\mu\nu}+\mathbf{I}\mathrm{B}^{\mu\nu}-\tfrac{1}{2}\varepsilon^{\mu\nu\sigma\tau}\mathbf{R}B_{\sigma\tau}, \qquad (9)$$

where the $\mathbf{I}$, $\mathbf{R}$ operators denote that the imaginary and real parts, respectively, are taken, the factor i being preserved in the imaginary part so that $B^{\mu\nu}=(\mathbf{R}+\mathbf{I})B^{\mu\nu}$. For the Cartesian system,

$$\mathbf{I}B^{\mu\nu}=\begin{matrix} 0 & 0 & 0 & \mathrm{i}(D_1-E_1) \\ 0 & 0 & 0 & \mathrm{i}(D_2-E_2) \\ 0 & 0 & 0 & \mathrm{i}(D_3-E_3) \\ -\mathrm{i}(D_1-E_1) & -\mathrm{i}(D_2-E_2) & -\mathrm{i}(D_3-E_3) & 0\ , \end{matrix}$$

and

$$-\tfrac{1}{2}\varepsilon^{\mu\nu\sigma\tau}\mathbf{R}B_{\sigma\tau}=\begin{matrix} 0 & (B_3-H_3) & -(B_2-H_2) & 0 \\ -(B_3-H_3) & 0 & (B_1-H_1) & 0 \\ (B_2-H_2) & -(B_1-H_1) & 0 & 0 \\ 0 & 0 & 0 & 0\ . \end{matrix}$$

Hence (9) implies

$$D^{12}=\mathrm{i}E_3-B_3+0+(B_3-H_3)=\mathrm{i}E_3-H_3,\ \text{etc.}$$
$$D^{14}=\mathrm{i}E_1-B_1+\mathrm{i}(D_1-E_1)+0=\mathrm{i}D_1-B_1,\ \text{etc.}$$

as required.

Using (9) the field equations can be expressed by one equation involving two tensors with complex terms, each having three distinct components.

It will be observed that the space-space components of the tensor $D^{\mu\nu}$, given by equation (**3**), are the components of the self-dual tensor $F^{\mu\nu}$, which has been used in the treatment of fields in free space. Since $F^{\mu\nu}$ is independent of the properties of the medium, and since it is a characteristic of the self-dual part that it does not indicate motion relative to the medium (being preserved in the same form for transformations of coordinates), it might at first seem more reasonable to choose $F^{\mu\nu}$ instead of $E^{\mu\nu}$ as the self-dual portion of $D^{\mu\nu}$. However, if this were done, results would not agree with those of classical theory except when $\mu=1$. Hence $E^{\mu\nu}$, but not $F^{\mu\nu}$, is an isotropic tensor in a material medium in which $\mu\neq 1$.

III. Relations between the Field Components

The next step is to express the classical equations

$$\mathbf{D}=\varepsilon\mathbf{E}, \qquad \mathbf{B}=\mu\mathbf{H}$$

for a homogeneous medium, as a single tensor equation. Since there are six different equations involved here, we shall attempt to solve the problem by assuming a completely self-dual set such as

$$\mathrm{D}^{\mu\nu}+\tfrac{1}{2}\varepsilon^{\mu\nu\sigma\tau}D_{\sigma\tau}=a\mathbf{R}\mathrm{E}^{\mu\nu}+b\mathbf{I}\mathrm{E}^{\mu\nu}, \qquad \ldots\ldots\ldots\ldots (\mathbf{10})$$

the $\mathrm{E}^{\mu\nu}$ being split up into its (self-dual) real and imaginary parts to admit the appearance of two constants. The comparison of (**10**) with the classical equations provides a, b in terms of ε, μ.

When the indices $\mu,\nu=1,2$ or $3,4$ in a Cartesian system at rest in the medium,

$$F_3+N_3=a\mathbf{R}M_3+b\mathbf{I}M_3.$$

Therefore

$$(\mathrm{i}E_3-H_3)+(\mathrm{i}D_3-B_3)=-aB_3+b\mathrm{i}E_3,$$

with similar expressions for the other components. The real and imaginary parts of this will respectively reduce to the third components of the classical equations quoted above, provided

$$\left.\begin{aligned} a&=1+1/\mu=1+\varphi, \text{ say,}\\ b&=1+\varepsilon. \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots (\mathbf{11})$$

From (**10**) and (**11**) therefore,

$$\mathrm{D}^{\mu\nu}+\tfrac{1}{2}\varepsilon^{\mu\nu\sigma\tau}D_{\sigma\tau}=\mathrm{E}^{\mu\nu}+\varphi\mathbf{R}\mathrm{E}^{\mu\nu}+\varepsilon\mathbf{I}\mathrm{E}^{\mu\nu}. \qquad \ldots\ldots\ldots\ldots (\mathbf{12})$$

Other forms may be obtained by substituting from (**9**) in this equation, or they may be derived directly by the method just described. For example, commencing with the completely self-dual equation

$$\mathrm{D}^{\mu\nu}+\tfrac{1}{2}\varepsilon^{\mu\nu\sigma\tau}D_{\sigma\tau}=\alpha\mathrm{E}^{\mu\nu}+\beta(\mathrm{B}^{\mu\nu}+\tfrac{1}{2}\varepsilon^{\mu\nu\sigma\tau}B_{\sigma\tau}),$$

we find

$$\alpha=\frac{2(\varepsilon-\mu)}{\mu\varepsilon-2\mu+1}, \qquad \beta=\frac{\mu\varepsilon-1}{\mu\varepsilon-2\mu+1}.$$

In the case of a crystalline medium we require an equation reducing in the rest system to

$$D_i=\varepsilon_{ij}E_j, \qquad B_i=\mu_{ij}H_j,$$

where $i,j=1,2,3$. The second equation may be solved to give

$$H_i=\varphi_{ij}B_j.$$

There is also the additional requirement that the nine-component systems ε_{ij}, φ_{ij} should be symmetrical.

Consider the equation*

$$\mathrm{D}^{\mu\nu}+\tfrac{1}{2}\varepsilon^{\mu\nu\sigma\tau}D_{\sigma\tau}-\mathrm{E}^{\mu\nu}=\tfrac{1}{4}(\varphi^{\mu\nu}_{\alpha\beta}\mathbf{R}+\varepsilon^{\mu\nu}_{\alpha\beta}\mathbf{I})\mathrm{E}^{\alpha\beta}. \quad \ldots\ldots\ldots \quad (\mathbf{13})$$

Since there are only three independent terms in the tensor which constitutes the whole of the left-hand side, and only three in the $\mathrm{E}^{\alpha\beta}$, the $\varphi^{\mu\nu}_{\alpha\beta}$ and $\varepsilon^{\mu\nu}_{\alpha\beta}$ must each have only 3×3 independent components, as required.

Using geodesic coordinates at rest in the medium, (**13**) gives for $\mu,\nu=1,2$; $2,1$; $3,4$; or $4,3$

$$D^{12}+D^{34}-E^{12}=\tfrac{1}{4}(\varphi^{12}_{\alpha\beta}\mathbf{R}+\varepsilon^{12}_{\alpha\beta}\mathbf{I})E^{\alpha\beta}.$$

In the sums over α,β there are four terms equal to that in which $\alpha,\beta=1,2$. Similarly for $\alpha,\beta=2,3$ and $3,1$. Hence

$$D^{12}+D^{34}-E^{12}=(\varphi^{12}_{12}\mathbf{R}+\varepsilon^{12}_{12}\mathbf{I})E^{12}+(\varphi^{12}_{23}\mathbf{R}+\varepsilon^{12}_{23}\mathbf{I})E^{23}+(\varphi^{12}_{31}\mathbf{R}+\varepsilon^{12}_{31}\mathbf{I})E^{31}.$$

Equating real and imaginary parts, this gives

$$H_3=\varphi^{12}_{12}B_3+\varphi^{12}_{31}B_2+\varphi^{12}_{23}B_1,$$
$$D_3=\varepsilon^{12}_{12}E_3+\varepsilon^{12}_{31}E_2+\varepsilon^{12}_{23}E_1.$$

The formulae for H_2, D_2 and H_1, D_1 are found similarly. The results can be summarized by

$$\left.\begin{matrix}\varepsilon_{ij}=\varepsilon^{23}_{23} & \varepsilon^{23}_{31} & \varepsilon^{23}_{12} & = & \varepsilon^{14}_{14} & \varepsilon^{14}_{24} & \varepsilon^{14}_{34}\\ \varepsilon^{31}_{23} & \varepsilon^{31}_{31} & \varepsilon^{31}_{12} & & \varepsilon^{24}_{14} & \varepsilon^{24}_{24} & \varepsilon^{24}_{34}\\ \varepsilon^{12}_{23} & \varepsilon^{12}_{31} & \varepsilon^{12}_{12} & & \varepsilon^{34}_{14} & \varepsilon^{34}_{24} & \varepsilon^{34}_{34}\end{matrix}\right\}, \text{ etc.,}$$

each independent $\varepsilon^{\mu\nu}_{\alpha\beta}$ corresponding to one of the ε_{ij}. Similarly for the φ_{ij}.

The symmetry of ε_{ij} shows that

$$\varepsilon^{\mu\nu}_{\alpha\beta}=\varepsilon^{\alpha\beta}_{\mu\nu},$$

and similarly

$$\varphi^{\mu\nu}_{\alpha\beta}=\varphi^{\alpha\beta}_{\mu\nu}.$$

These two latter equations are not transformable in general coordinate systems.

In the usual theory, different methods of deriving covariant forms of the classical relations $\mathbf{D}=\varepsilon\mathbf{E}$, $\mathbf{B}=\mu\mathbf{H}$ have been used. Schouten (1951) obtains a single tensor equation with six components, while McConnell (1936) takes, for the more general case of the crystalline medium, two tensor equations.

IV. The Ponderomotive Force

Consider

$$W^{\mu}=\frac{1}{g}\mathrm{D}^{\mu\nu}\mathrm{J}_{\nu}=D^{\mu\nu}J_{\nu}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{14})$$

Take the case of geodesic Cartesian coordinates. In such a system a force/power four-vector P^{μ} would be given by

$$P^{\mu}=\mathbf{R}(W^1,\ W^2,\ W^3),\ \mathbf{I}(W^4).$$

* The dielectric constants must not be confused with the ε-systems, in which the suffixes will always appear as superscripts.

Now

$$\begin{aligned} P^1 &= \mathbf{R}(D^{12}J^2+D^{13}J^3+D^{14}J^4) \\ &= \mathbf{R}\left\{-\frac{1}{c}(F_3j_2-F_2j_3)-N_1\mathrm{i}\rho\right\} \\ &= \frac{1}{c}(H_3j_2-H_2j_3)+D_1\rho. \end{aligned}$$

Also

$$\begin{aligned} P^4 &= \mathbf{I}(D^{41}J^1+D^{42}J^2+D^{43}J^3) \\ &= \frac{\mathrm{i}}{c}\mathbf{j}\,.\,\mathbf{D}. \end{aligned}$$

Therefore

$$P^{\mu}=\left(\rho\mathbf{D}+\frac{1}{c}\mathbf{j}\times\mathbf{H},\ \frac{\mathrm{i}}{c}\mathbf{j}\,.\,\mathbf{D}\right).$$

Hence (**14**) corresponds to the classical force/power per unit volume.

The conservation of charge is deduced exactly as before.

The law of propagation of electromagnetic waves has lost its fundamental significance—the velocity is no longer the same for all observers—and so the wave equations cannot be derived as neatly as before from the general four-dimensional equations. They can best be obtained by way of the classical equations.

V. The Propagation of Potential

We now attempt the deduction of the wave equation for the potential function from the four-dimensional expressions. Consider the case of a homogeneous isotropic medium at rest with respect to a geodesic system of coordinates. Equation (**12**) becomes

$$D^{\mu\nu}+\tfrac{1}{2}\varepsilon^{\mu\nu\sigma\tau}D^{\sigma\tau}-E^{\mu\nu}=(\varphi\mathbf{R}+\varepsilon\mathbf{I})E^{\mu\nu}. \quad \ldots\ldots\ldots \quad (\mathbf{15})$$

Also, from (**9**),

$$D^{\mu\nu}=E^{\mu\nu}+\mathbf{I}B^{\mu\nu}-\tfrac{1}{2}\varepsilon^{\mu\nu\sigma\tau}\mathbf{R}B^{\sigma\tau}. \quad \ldots\ldots\ldots\ldots \quad (\mathbf{16})$$

From the definition of the potential function (**8**) and equation (**7**) which it includes,

$$E^{\mu\nu}=\frac{\partial\varkappa_{\mu}}{\partial x^{\nu}}-\frac{\partial\varkappa_{\nu}}{\partial x^{\mu}}+\tfrac{1}{2}\varepsilon^{\mu\nu\sigma\tau}\left(\frac{\partial\varkappa_{\sigma}}{\partial x^{\tau}}-\frac{\partial\varkappa_{\tau}}{\partial x^{\sigma}}\right), \quad \ldots\ldots\ldots\ldots \quad (\mathbf{17})$$

$$E^{\mu\nu}=\tfrac{1}{2}\varepsilon^{\mu\nu\sigma\tau}E^{\sigma\tau}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{17'})$$

From (**16**) and (**17′**)

$$\tfrac{1}{2}\varepsilon^{\mu\nu\sigma\tau}D^{\sigma\tau}=E^{\mu\nu}+\tfrac{1}{2}\varepsilon^{\mu\nu\sigma\tau}\mathbf{I}B^{\sigma\tau}-\mathbf{R}B^{\mu\nu}. \quad \ldots\ldots\ldots \quad (\mathbf{18})$$

Substituting from (**18**) in (**15**),

$$D^{\mu\nu}+\tfrac{1}{2}\varepsilon^{\mu\nu\sigma\tau}\mathbf{I}B^{\sigma\tau}-\mathbf{R}B^{\mu\nu}-(\varphi\mathbf{R}+\varepsilon\mathbf{I})E^{\mu\nu}=0. \quad \ldots\ldots \quad (\mathbf{19})$$

We shall let Greek suffixes range from 1 to 4, and Latin suffixes from 1 to 3. In (**19**) we take $\mu,\nu=4,i$. Then in the second term σ,τ must be of the type j,k. But $B^{jk}=0$. Hence

$$D^{4i}-\mathbf{R}B^{4i}-\varphi\mathbf{R}E^{4i}-\varepsilon\mathbf{I}E^{4i}=0. \quad \ldots\ldots\ldots\ldots \quad (\mathbf{20})$$

Assuming that there are no currents or free charges, we have, from (**1**)

$$\partial D^{4i}/\partial x^{i}=0,$$

and so, operating on (**20**) with $\partial/\partial x^i$, and choosing the imaginary part,

$$-\varepsilon\mathbf{I}\frac{\partial E^{4i}}{\partial x^i}=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

But, from (**17**)

$$E^{4i}=\frac{\partial\varkappa_4}{\partial x^i}-\frac{\partial\varkappa_i}{\partial x^4}+\tfrac{1}{2}\varepsilon^{4ijk}\left(\frac{\partial\varkappa_j}{\partial x^k}-\frac{\partial\varkappa_k}{\partial x^j}\right).$$

Hence

$$\frac{\partial E^{4i}}{\partial x^i}=\frac{\partial}{\partial x^i}\left(\frac{\partial\varkappa_4}{\partial x^i}-\frac{\partial\varkappa_i}{\partial x^4}\right).$$

For a reason which will appear later we cannot here use $\partial\varkappa_\mu/\partial x^\mu=0$, as in the case of empty space. Since $\varkappa_4$ and x^4 are both imaginary, (**21**) becomes

$$\frac{\partial}{\partial x^i}\left(\frac{\partial\varkappa_4}{\partial x^i}-\frac{\partial\varkappa_i}{\partial x^4}\right)=0.$$

Writing $\varkappa_i=\mathbf{A}$, $\varkappa_4=\mathrm{i}V$, this gives

$$\nabla^2V+\frac{1}{c}\operatorname{div}\dot{\mathbf{A}}=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (22)$$

Now let us take $\mu,\nu=i,j$ in (**15**) and (**16**), noting that $B^{ij}=0$, and also that $\varepsilon^{ij\sigma\tau}$ has the two sets of components $\varepsilon^{ijk4}=-\varepsilon^{ij4k}$. Subtracting the two equations thus obtained in order to eliminate D^{ij},

$$\varepsilon^{ijk4}D^{k4}-(\varphi\mathbf{R}+\varepsilon\mathbf{I})E^{ij}-\varepsilon^{ijk4}\mathbf{R}B^{k4}=0.$$

We next differentiate this with respect to x^4. Now

$$\partial D^{k\nu}/\partial x^\nu=0,$$

and so

$$\partial D^{k4}/\partial x^4=-\partial D^{kn}/\partial x^n.$$

Hence

$$-\varepsilon^{ijk4}\frac{\partial D^{kn}}{\partial x^n}-\frac{\partial}{\partial x^4}(\varphi\mathbf{R}+\varepsilon\mathbf{I})E^{ij}-\varepsilon^{ijk4}\frac{\partial}{\partial x^4}(\mathbf{R}B^{k4})=0.$$

From (**19**), using $B^{kn}=0$ again,

$$D^{kn}=-\varepsilon^{knp4}\mathbf{I}B^{p4}+(\varphi\mathbf{R}+\varepsilon\mathbf{I})E^{kn}.$$

Hence

$$-\varepsilon^{ijk4}\frac{\partial}{\partial x^n}\{-\varepsilon^{knp4}\mathbf{I}B^{p4}+(\varphi\mathbf{R}+\varepsilon\mathbf{I})E^{kn}\}-\frac{\partial}{\partial x^4}(\varphi\mathbf{R}+\varepsilon\mathbf{I})E^{ij}-\varepsilon^{ijk4}\frac{\partial}{\partial x^4}(\mathbf{R}B^{k4})=0.$$

The real part of this is obviously

$$-\varepsilon^{ijk4}\frac{\partial}{\partial x^n}(\varphi\mathbf{R}E^{kn})-\frac{\partial}{\partial x^4}(\varepsilon\mathbf{I}E^{ij})=0. \quad \ldots\ldots\ldots\ldots \quad (23)$$

Now, from (**17**)

$$E^{ij}=\frac{\partial\varkappa_i}{\partial x^j}-\frac{\partial\varkappa_j}{\partial x^i}+\left(\frac{\partial\varkappa_k}{\partial x^4}-\frac{\partial\varkappa_4}{\partial x^k}\right),$$

where i,j,k is an *even* permutation of 1,2,3. Hence,

$$\mathbf{I}E^{ij}=\left(\frac{\partial\varkappa_k}{\partial x^4}-\frac{\partial\varkappa_4}{\partial x^k}\right),$$

$$\frac{\partial}{\partial x^n}(\mathbf{R}E^{1n})=\frac{\partial}{\partial x^2}\left(\frac{\partial\varkappa_1}{\partial x^2}-\frac{\partial\varkappa_2}{\partial x^1}\right)+\frac{\partial}{\partial x^3}\left(\frac{\partial\varkappa_1}{\partial x^3}-\frac{\partial\varkappa_3}{\partial x^1}\right).$$

Putting $i,j=2,3$ in (**23**) and substituting from these,

$$-\varphi\left\{\frac{\partial}{\partial x^2}\left(\frac{\partial\varkappa_1}{\partial x^2}-\frac{\partial\varkappa_2}{\partial x^1}\right)+\frac{\partial}{\partial x^3}\left(\frac{\partial\varkappa_1}{\partial x^3}-\frac{\partial\varkappa_3}{\partial x^1}\right)\right\}-\varepsilon\frac{\partial}{\partial x^4}\left(\frac{\partial\varkappa_1}{\partial x^4}-\frac{\partial\varkappa_4}{\partial x^1}\right)=0.$$

Substituting for x^μ and $\varkappa_4$ this is seen to be the x-component of

$$\frac{\varepsilon}{c^2}\ddot{\mathbf{A}}+\frac{\varepsilon}{c}\text{ grad }\dot{V}+\varphi\text{ grad div }\mathbf{A}-\varphi\nabla^2\mathbf{A}=0.$$

Putting $i,j=3,1$ and $1,2$ gives the other components. This can be written

$$\frac{\varepsilon\mu}{c^2}\ddot{\mathbf{A}}-\nabla^2\mathbf{A}+\text{grad}\left(\text{div }\mathbf{A}+\frac{\varepsilon\mu}{c}\dot{V}\right)=0. \qquad \ldots\ldots (24)$$

In order that the propagation of the scalar potential V and the vector potential $\mathbf{A}$ should be consistent, we require

$$\text{div }\mathbf{A}+\frac{\varepsilon\mu}{c}\dot{V}=k, \qquad \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

where k is a constant, usually taken to be zero. This causes (**24**) to be reduced to

$$\nabla^2\mathbf{A}=\frac{\varepsilon\mu}{c^2}\ddot{\mathbf{A}},$$

and on substituting in (**22**) gives

$$\nabla^2 V=\frac{\varepsilon\mu}{c^2}\ddot{V}.$$

The condition (**25**) shows why it was not possible to assume $\partial\varkappa_\mu/\partial x^\mu=0$, as for empty space. This equation must be replaced by the (covariant) statement that all components of the potential function are propagated with the same velocity.

VI. References

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