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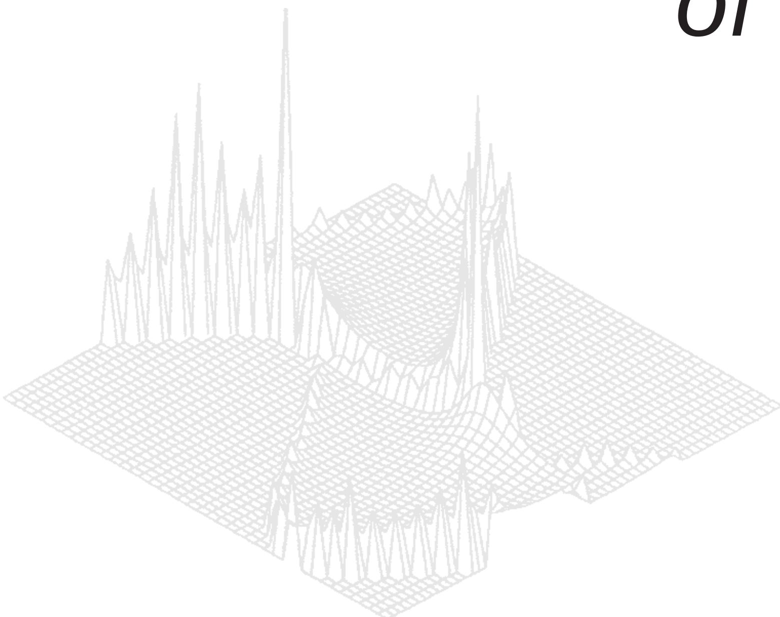

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Finite Temperature Correlation Functions of One-dimensional Interacting Electron Systems*

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Abstract

The well-known bosonisation method of one-dimensional electron systems is extended to finite temperatures. As an example, the correlation functions of the non-interacting case are explicitly calculated. The presentation is pedagogical and is intended to be accessible to graduate students or physicists who are not experts in this field.

1. Introduction

Bosonisation of a quantum field theory describes a method by which fermionic operators in the theory, obeying anti-commutation relations, are replaced by bosonic operators obeying commutation relations.

Replacement of one field theory with another would appear to merely replace the problem of solving the original fermionic field theory with the problem of solving a bosonic field theory. The usefulness of this technique was realised when it was discovered that certain one-dimensional *interacting* fermionic field theories were equivalent to *non-interacting* bosonic field theories. While the solution to interacting field theories is often difficult, and generally only perturbative methods are available to obtain a solution, the solution to a non-interacting field theory is well known and exact (Abrikosov *et al.* 1963; Inkson 1984).

Given the exact solution to the bosonic field theory, the properties of the fermionic theory can be calculated by use of a 'bosonisation dictionary'. For example, correlation functions in terms of fermionic operators can be re-written as expectations for bosonic operators, and solved. During the past decade much attention has been given to the ground state properties of one-dimensional interacting models using bosonisation. However, less attention has been given to understanding the behaviour of the one-dimensional interacting electron systems at finite temperatures. This is our present focus. In the following we present in detail the 'bosonisation dictionary' at finite temperatures. As most of the bosonised models can be described in terms of non-interacting spinless fermionic fixed points, we are going to concentrate mostly on this limit.

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2. Field Theoretical Bosonisation

Bosonisation may be derived by defining a set of boson fields $\phi(x)$, and their conjugate momenta fields $\Pi(x)$, with given commutation relations, and then determining the commutation relations and Green functions of the exponentials of these fields. The Green functions and commutation relations are fermionic in nature, and an identity between the boson fields, $\phi(x)$ and $\Pi(x)$, and fermion fields $\psi(x)$ is made (Shankar 1995; Gulácsi 1997).

Field theoretical bosonisation evolved out of solutions to models in the area of high energy physics. Indeed, many of the models used in the particle physics arena have a direct correspondence to models used in condensed matter physics. Some relevant examples of this correspondence shall be pointed out later on.

Constructive bosonisation, in terms of a Fock space identification of operators, gives a simpler, more transparent derivation of the boson representation of fermion operators. An explicit picture of the physics corresponding to the boson representation may be obtained directly from the formalism.

In this section we derive the Fock space identity between fermions and boson coherent states, closely following the pedagogical articles of Schoenhammer and Meden (1996) and Delft and Schöller (1998). From this derivation we obtain the bosonisation dictionary we require for the finite temperature solution of the models studied.

(2a) Requirements for Bosonisation

The fermionic system must be expressible by a countable set of operators, with unbounded spectrum, that obey fermionic anti-commutation relations. That is, operators $\hat{c}_{k,\sigma}$ with $k \in [-\infty, \infty]$, σ a species identifying label, and

$$\{\hat{c}_{k'\sigma'}, \hat{c}^{\dagger}_{k\sigma}\} = \delta_{kk'}\delta_{\sigma\sigma'}. \tag{1}$$

The species identifying label can be just the electron spin $\sigma = \{\uparrow, \downarrow\}$, or in the Luttinger model the spin combined with the left moving L and right moving R species labels.

The treatment of fermions on different sections of the Fermi surface as different species can be used to generalise this method to higher dimensions. The success of this approach in one dimension hinges on the assumption that there is no mixing of fermion species, due to the large $2k_F$ jump in momentum between the points of the Fermi surface.

At this point there are no requirements on the type of Hamiltonian of the system, as the operator identity is a property of the Fock space.

In order to regularise all the expressions of operators within the theory, a zero particle vacuum state $|\,0\,\rangle$ for the fermions is introduced. This vacuum state is then used to define normal orderings of all the operators. The vacuum state is defined by

$$\begin{aligned} \hat{c}_{k\sigma}|0\rangle &= 0; \qquad \text{for } k > 0, \\ \hat{c}^{\dagger}_{k\sigma}|0\rangle &= 0; \qquad \text{for } k \leq 0. \end{aligned}$$

Hence, for any combination of the fermion operators we may define the normal ordering of such operators by

$$:\hat{A}\hat{B}\hat{C}: = \hat{A}\hat{B}\hat{C} - \langle 0|\hat{A}\hat{B}\hat{C}|0\rangle. \tag{2}$$

(2b) Defining the Boson Operators

The boson operators $\hat{b}^{\dagger}_{q\sigma}$ and $\hat{b}_{q\sigma}$ are defined on the Fock space as linear combinations of particle–hole excitations

$$\hat{b}^{\dagger}_{q\sigma} = \sqrt{\frac{2\pi}{Lq}} \sum_{k=-\infty}^{\infty} \hat{c}^{\dagger}_{k+q,\sigma}\hat{c}_{k\sigma} \qquad \text{and} \qquad \hat{b}_{q\sigma} = \sqrt{\frac{2\pi}{Lq}} \sum_{k=-\infty}^{\infty} \hat{c}^{\dagger}_{k-q,\sigma}\hat{c}_{k\sigma}, \tag{3}$$

where the momentum is $q = 2\pi n/L$ for n a positive integer. The boson operators are proportional to the $q \neq 0$ Fourier components of the fermion density $\hat{\psi}^{\dagger}_{\sigma}(x)\hat{\psi}_{\sigma}(x)$, with the $q = 0$ component related to the fermion number operator $\hat{N}$.

The operators obey the bosonic commutation relations

$$\begin{aligned} [\hat{b}^{\dagger}_{q,\sigma}, \hat{b}^{\dagger}_{q',\sigma'}] = [\hat{b}_{q,\sigma}, \hat{b}_{q',\sigma'}] = 0, \\ [\hat{b}_{q,\sigma}, \hat{b}^{\dagger}_{q',\sigma'}] = \delta_{qq'}\delta_{\sigma\sigma'}. \end{aligned} \tag{4}$$

The last equality requires normal ordering and the existence of the negative k states

$$\begin{aligned} \hat{b}_{q\sigma}\hat{b}^{\dagger}_{q'\sigma'} - \hat{b}^{\dagger}_{q'\sigma'}\hat{b}_{q\sigma} &= \tfrac{2\pi}{Lq}\delta_{\sigma\sigma'} \textstyle\sum_k \left[\hat{c}^{\dagger}_{k+q-q',\sigma}\hat{c}_{k,\sigma} - \hat{c}^{\dagger}_{k+q,\sigma}\hat{c}_{k+q',\sigma}\right] \\ &= \tfrac{2\pi}{Lq}\delta_{\sigma\sigma}\delta_{qq'} \textstyle\sum_k \left[\langle 0|\hat{c}^{\dagger}_{k\sigma}\hat{c}_{k\sigma}|0\rangle - \langle 0|\hat{c}^{\dagger}_{k+q,\sigma}\hat{c}_{k+q,\sigma}|0\rangle\right] \\ &= \delta_{\sigma\sigma'}\delta_{qq'}. \end{aligned} \tag{5}$$

(2c) Partitioning the Fock Space

The fermionic Fock space can be interpreted as a direct sum of subspaces, each of definite particle number N. Excitations within these subspaces are then bosonic particle–hole excitations, with the $\hat{b}_q$ and $\hat{b}^{\dagger}_q$ operators acting as annihilation and creation operators, respectively. As first noted by Overhauser (1965), the bosonic particle–hole excitations form a complete basis of states for each N-particle subspace.

The operators that connect the subspaces together are the unitary Klein operators $\hat{F}^{\dagger}$ and $\hat{F}$, which increase and decrease the total particle number by one, respectively. In this way, the Klein operators act as raising and lowering operators with respect to the fermion number operator. They obey the consequent commutation relations

$$[\hat{N}_{\sigma}, \hat{F}^{\dagger}_{\sigma'}] = \delta_{\sigma\sigma'}\hat{F}^{\dagger}_{\sigma'} \qquad \text{and} \qquad [\hat{N}_{\sigma}, \hat{F}_{\sigma'}] = -\delta_{\sigma\sigma'}\hat{F}_{\sigma'}. \tag{6}$$

The operators were first constructed in terms of the bare fermions by Haldane (1979), and an overview of their properties given by Haldane (1981) and Delft and Schöller (1998). For the purposes of this paper we simply describe the remaining commutation relations obeyed by the Klein operators, which are

$$\begin{aligned} \{\hat{F}^{\dagger}_{\sigma}, \hat{F}_{\sigma'}\} = 2\delta_{\sigma\sigma'}, \\ \{\hat{F}^{\dagger}_{\sigma}, \hat{F}^{\dagger}_{\sigma'}\} = \{\hat{F}_{\sigma}, \hat{F}_{\sigma'}\} = 0, \\ [\hat{b}^{\dagger}_{q\sigma}, \hat{F}^{\dagger}_{\sigma'}] = [\hat{b}_{q\sigma}, \hat{F}^{\dagger}_{\sigma'}] = [\hat{b}^{\dagger}_{q\sigma}, \hat{F}_{\sigma'}] = [\hat{b}_{q\sigma}, \hat{F}_{\sigma'}] = 0. \end{aligned} \tag{7}$$

We also note that because the Klein operators are unitary, we have $\hat{F}_\sigma^\dagger \hat{F}_\sigma = \hat{F}_\sigma \hat{F}_\sigma^\dagger = 1$.

(2d) Fermion Operators as Coherent States

The fermion field operators $\psi_\sigma(x)$ and $\psi_\sigma^\dagger(x)$ are defined in terms of the Fourier components of the $\hat{c}_{k\sigma}$ and $\hat{c}_{k\sigma}^\dagger$ operators, respectively,

$$\psi_\sigma(x) = \sqrt{\frac{2\pi}{L}} \sum_{k=-\infty}^{\infty} e^{-ikx} \hat{c}_{k\sigma} \qquad \text{and} \qquad \psi_\sigma^\dagger(x) = \sqrt{\frac{2\pi}{L}} \sum_{k=-\infty}^{\infty} e^{ikx} \hat{c}_{k\sigma}^\dagger. \tag{8}$$

The fundamental relation that makes bosonisation work is that the state produced by the fermion operator $\psi_\sigma(x)$ on an N-particle ground state $|N_0\rangle$ is an eigenstate of the boson operator $\hat{b}_{q,\sigma}$. This may be derived from the commutation relations

$$\begin{aligned} [\hat{b}_{q\sigma}, \hat{\psi}_{\sigma'}(x)] &= -\delta_{\sigma\sigma'} \sqrt{\tfrac{2\pi}{Lq}} e^{iqx} \hat{\psi}_{\sigma'}(x), \\ [\hat{b}_{q\sigma}^\dagger, \hat{\psi}_{\sigma'}(x)] &= -\delta_{\sigma\sigma'} \sqrt{\tfrac{2\pi}{Lq}} e^{-iqx} \hat{\psi}_{\sigma'}(x), \end{aligned} \tag{9}$$

and the fact that any N-particle ground state $|N_0\rangle$ vanishes when operated on by $\hat{b}_{q,\sigma}$. This allows the representation of the fermion operator $\hat{\psi}_\sigma(x)$ acting on any N particle ground state as a coherent state representation of boson operators. Thus, we have

$$\hat{\psi}_\sigma(x)|N_0\rangle = \hat{F}_\sigma \lambda_\sigma(x) \exp\left(-\sum_{q>0} \sqrt{\frac{2\pi}{Lq}} e^{iqx} \hat{b}_{q\sigma}^\dagger \right) |N_0\rangle, \tag{10}$$

where the Klein operator $\hat{F}_\sigma$ is required to reduce the particle number, and $\lambda_\sigma(x)$ is a phase operator

$$\lambda_\sigma(x) = \sqrt{\frac{2\pi}{L}} \exp\left[-i\frac{2\pi}{L}(\hat{N} - \tfrac{1}{2}\delta_B)x \right]. \tag{11}$$

The boson field operators $\varphi_\sigma(x)$ and $\varphi_\sigma^\dagger(x)$, are introduced to simplify the notation, and are simply

$$\varphi_\sigma(x) = -\sum_{q>0} \sqrt{\frac{2\pi}{Lq}} e^{iqx} \hat{b}_{q\sigma} \qquad \text{and} \qquad \varphi_\sigma^\dagger(x) = -\sum_{q>0} \sqrt{\frac{2\pi}{Lq}} e^{-iqx} \hat{b}_{q\sigma}^\dagger, \tag{12}$$

and allow us to write $\hat{\psi}_\sigma(x)|N_0\rangle = \hat{F}_\sigma \lambda_\sigma(x) e^{\varphi_\sigma^\dagger(x)}|N_0\rangle$.

(2e) The Boson Representation of $\hat{\psi}_\sigma(x)$

Bosonisation is only useful because properties of the fermionic system can be described in terms of certain boson fields, whose properties are simpler to calculate than those of the original fermion fields. The correlation functions, and hence Green functions, of the fermionic system contain combinations of single particle fermion operators, and hence we need to express these operators in terms of a set of boson fields.

In the previous section we showed that the fermion field $\hat{\psi}_\sigma(x)$ can be expressed as a coherent state of boson operators. The generalisation of this boson representation to an arbitrary state of the Fock space is surprisingly straight forward.

As the boson excitations form a complete basis of any N-particle subspace, we can write an arbitrary state $|\psi\rangle$ as some set of boson excitations above an N particle ground state, that is, $|\psi\rangle f(\{\hat{b}^\dagger_{q\sigma}\})|N_0\rangle$.

Using the commutation relations for $\hat{\psi}_\sigma(x)$ and $\hat{b}^\dagger_{q\sigma}$, and the relation

$$e^{-\varphi_\sigma(x)} f(\{\hat{b}^\dagger_{q'\sigma'}\}) e^{\varphi_\sigma(x)} = f\left[\left\{\hat{b}^\dagger_{q'\sigma'} + \delta_{\sigma\sigma'}\sqrt{\frac{2\pi}{Lq}}e^{iqx}\right\}\right], \tag{13}$$

we can derive the bosonisation identity,

$$\hat{\psi}_\sigma(x)|\Psi\rangle = \hat{F}_\sigma \lambda_\sigma(x) e^{\varphi^\dagger_\sigma(x)} e^{-\varphi_\sigma(x)}|\Psi\rangle. \tag{14}$$

The boson representation of the fermion fields allows us to obtain fermion expectation values in terms of the expectation values of the exponentials of the boson fields. In similar treatments (Emery 1979; Mahan 1981; Delft and Schöller 1998), the boson operators $\hat{b}_{q\sigma}$ and $\hat{b}^\dagger_{q\sigma}$ are given factors of $-i$ and i, respectively, which changes the exponentials in equation (14) from $\pm$ to $-i$ for both terms.

The definition of the boson operators as bilinear combinations of fermion operators would suggest that the two particle interaction terms of a fermionic Hamiltonian, which may be expressed as an effective density–density interaction, may be rewritten in terms of a sum of quadratic combinations of boson operators.

If the kinetic term of the fermion Hamiltonian, which in the case of a Fermi gas, only contains a single bilinear combination of fermion operators, was also expressible in terms of quadratic combinations of boson operators, then the model may easily be diagonalised in terms of the boson operators.

By commuting the boson annihilation operator $\hat{b}_{q\sigma}$ with the kinetic term of a fermion Hamiltonian of the form, $\hat{H}_0 = \sum_{k,\sigma} \epsilon(k)\hat{c}^\dagger_{k\sigma}\hat{c}^\dagger_{k\sigma}$,

$$[\hat{H}_0, \hat{b}_{q\sigma}] = \sum_k [\epsilon(k-q) - \epsilon(k)]\hat{c}^\dagger_{k-q,\sigma}\hat{c}_{k\sigma}, \tag{15}$$

we find that, if the quantity $\varepsilon(k-q) - \varepsilon(k)$ is independent of k, the commutation relation is that of a lowering operator $\hat{b}_{q\sigma}$. The kinetic term of the Hamiltonian may then be expressed as a bilinear combination of the operator $\hat{b}_{q\sigma}$ and its adjoint. This identification of the boson and fermion Hamiltonians is known as Krönig's identity (Krönig 1935).

By examination we can see that, for the term $\varepsilon(k-q) - \varepsilon(k)$ to be independent of k, the spectrum must be linear in k, that is, $\varepsilon(k) = Ak + B$, for some constants A and B. For this reason, in the following, we begin our examination of one-dimensional fermions with an infinite linear dispersion.

3. Fermions with Linear Dispersion

One-dimensional fermions will have an infinite linear dispersion $\varepsilon(k) = v_F(k - k_F)$, for v_F the Fermi velocity and k_F the Fermi momentum. This simple model demonstrates the origins of some of the non-Fermi liquid behaviour attributed to one-dimensional systems. The model has an infinite Fermi sea of positron states, but these states do not contribute to the low energy physics and, therefore, the physics is believed to be descriptive of the low energy sector of more realistic one-dimensional models. Restriction of the particles to movement in one direction also has physical realisations. The transport in quantum Hall edge states is an example of one such occurrence.

The construction of a model with infinite linear dispersion provides an exact bosonisation of the model with a non-interacting boson Hamiltonian. All the properties of the fermionic system may then be obtained by direct calculation, using the boson representation. As this model is critical at zero temperature, the scale and Lorentz invariance of the system allows it to be solved using conformal field theory (Christe and Henkel 1993). This is possible due to the Fermi velocity acting as a Lorentz invariant velocity equivalent to the speed of light in conformal field theory models. Models that contain excitations with more than a single velocity may still be solved using a product of Virasoro algebras and a dressed charge matrix (Frahm and Korepin 1990).

In the non-interacting case, $\hat{H}_0 = v_F \sum_{k=-\infty}^{\infty} (k - k_F)\hat{c}^{\dagger}{}_k \hat{c}_k$, the direct solution of the fermion model requires less work than the bosonisation method. However, the bosonisation gives some insight into the structure of one-dimensional fermion systems. The bosonisation solution also extends easily to the interacting case, unlike the direct fermionic solution. Being on a lattice we also have

$$k = \frac{2\pi}{L}\,(n_k - \frac{1}{2}\delta_B)\ , \tag{16}$$

where $n_k \in \mathrm{Z}$, $\delta_B \in [0,2)$, a parameter that determines the boundary conditions, and L is the length of the system. Defining the bosons, for $q > 0$,

$$\hat{b}_q^{\dagger} = \sqrt{\frac{2\pi}{Lq}} \sum_k \hat{c}_{k+q}^{\dagger}\hat{c}_k,$$
$$\hat{b}_q = \sqrt{\frac{2\pi}{Lq}} \sum_k \hat{c}_{k-q}^{\dagger}\hat{c}_k,$$

we then commute the boson operators with the Hamiltonian, to find

$$[\hat{H}_0, \hat{b}_q] = \sqrt{\frac{2\pi}{Lq}} v_F \sum_{k'} \left[(k' - q - k_F)\hat{c}_{k'-q}^{\dagger}\hat{c}_{k'} - (k' - k_F)\hat{c}_{k'-q}^{\dagger}\hat{c}_{k'}\right], \tag{17}$$

which after some algebra simplifies to

$$[\hat{H}_0, \hat{b}_q] = \sqrt{\frac{2\pi}{Lq}} v_F \sum_{k'} (-q)\hat{c}_{k'-q}^{\dagger}\hat{c}_{k'} = -v_F q \hat{b}_q. \tag{18}$$

Using Krönig's (1935) identity we can rewrite the Hamiltonian as a sum over bi-linear boson operators

$$\hat{H}_0 = v_F \sum_q q \hat{b}_q^\dagger \hat{b}_q + \hat{C}, \tag{19}$$

where $\hat{C}$ is an operator that depends on the energies of the bosonic ground states. To determine the operator $\hat{C}$ we apply the fermion Hamiltonian to an arbitrary ground state of N particles,

$$\hat{H}_0|N_0\rangle = \frac{2\pi}{L} v_F \sum_n n - \tfrac{1}{2}\delta_B |N_0\rangle = \frac{\pi v_F}{L} N(N+1-\delta_B)|N_0\rangle. \tag{20}$$

To obtain this result we must include in the bosonised Hamiltonian the same combination of fermion number operators, which must also be normal ordered as N does not include states below the zero particle ground state energy. With the zero particle ground state denoted by $|\,0\,\rangle$, and defined by $\hat{c}_k\,|\,0\,\rangle = 0$ for $k > 0$, and $\hat{c}_k^\dagger\,|\,0\,\rangle = 0$ for $k \le 0$, we define the fermion number operators by

$$\hat{N} = \sum_k \left[\hat{c}_k^\dagger \hat{c}_k - \langle 0|\hat{c}_k^\dagger \hat{c}_k|0\rangle \right]. \tag{21}$$

Hence, we may write the full boson Hamiltonian as

$$\hat{H}_0 = v_F \sum_q q \hat{b}_q^\dagger \hat{b}_q + \frac{\pi v_F}{L} \hat{N}(\hat{N}+1-\delta_B). \tag{22}$$

This Hamiltonian describes a system of free bosons with energies given by $\varepsilon_q = v_F q$ for $q > 0$, with an additional term that gives the bosonic ground state energies for each fixed particle number subspace of the fermion Hilbert space.

(3a) Distribution Function for the Bosons

In the finite temperature calculation of expectation values, we must average over an ensemble of states. We must therefore check that the extra term in the Hamiltonian, which gives the energies of the 'ground states' for the bosons, does not change the distribution function of these bosons at finite temperature.

To begin with we simply calculate the partition function, $\mathcal{Z}$, where

$$\mathcal{Z} = \sum_N \sum_{\{n_q\}} \langle N, \{n_q\}| e^{-\beta \hat{H}_0} |N, \{n_q\}\rangle, \tag{23}$$

which can be written as

$$\mathcal{Z} = \sum_N \exp\left[-\beta \frac{\pi v_F}{L} N(N+1-\delta_B)\right] \sum_{\{n_q\}} \langle N, \{n_q\}| e^{-\beta \sum_q \epsilon_q n_q} |N, \{n_q\}\rangle. \tag{24}$$

First we express the sum in the above equation as a product of exponentials and then, second, we sum over the occupation numbers. Further, we note that the chemical potential for the bosons vanishes, hence, obtaining

$$\mathcal{Z} = \sum_N e^{-\beta \frac{\pi v_F}{L} N(N+1-\delta_B)} / \prod_q 1 - e^{-\beta \epsilon_q}. \tag{25}$$

Similarly, for the boson distribution, we obtain

$$\begin{aligned} \langle \hat{b}_q^\dagger \hat{b}_q \rangle_\beta &= \frac{1}{\mathcal{Z}} \sum_N \sum_{\{n_{q'}\}} \langle N, \{n_{q'}\}| e^{-\beta \hat{H}_0} \hat{b}_q^\dagger \hat{b}_q | N, \{n_{q'}\}\rangle \\ &= 1/(e^{\beta \epsilon_q} - 1). \end{aligned} \tag{26}$$

The terms introduced by the fermion number operator in the Hamiltonian cancel, leaving a free boson distribution function. The lack of dependence of the distribution function on the boson ground states can be understood physically, in that each particle–hole excitation only depends on the difference in energy between the hole and particle, and not on the energies of the particle and hole individually. The partition function, however, relates to the underlying fermion system, and hence is not equal to the partition function of a gas of bosons, and will give fermionic thermodynamic properties. The equality of the fermion and boson partition functions was shown explicitly by Haldane (1981), and is essential in proving the completeness of the boson states.

(3b) Time Dependence of the Boson Fields

For the non-interacting case, the time dependence of the boson operators is trivially obtained due to the non-interacting nature of the boson Hamiltonian. In deriving the time dependence of the boson fields we shall generalise the result for the case of a free boson Hamiltonian with arbitrary dispersion. The boson operators $\hat{b}_q$ have a time dependence given by

$$\hat{b}_q(t) = e^{i\hat{H}t} \hat{b}_q e^{-i\hat{H}t} = e^{-i\epsilon_q t} \hat{b}_q, \tag{27}$$

where ε_q is the dispersion, which in the non-interacting case is simply $\varepsilon_q = v_F\, q$.

We define the boson fields

$$\varphi(x) = -\sum_q \sqrt{\frac{2\pi}{Lq}} e^{iqx} \hat{b}_q$$
$$\varphi^\dagger(x) = -\sum_q \sqrt{\frac{2\pi}{Lq}} e^{-iqx} \hat{b}_q^\dagger,$$

which when commuted with the boson Hamiltonian introduces a factor given by the dispersion ε_q for each $\hat{b}_{q\sigma}$ operator in the sum. Thus, the time dependence of the field operators is easily obtained by moving the time dependent operators inside the sum, to obtain

$$\varphi(x,t) = e^{i\hat{H}t} \hat{\varphi}(x) e^{-i\hat{H}t} = -\sum_q \sqrt{\frac{2\pi}{Lq}} e^{iqx} e^{-i\epsilon_q t} \hat{b}_q, \tag{28}$$

which in the non-interacting case reduces to $\varphi(x,t) = \varphi(x - v_F t)$. Similarly, we find $\varphi^\dagger(x,t) = \varphi^\dagger(x - v_F t)$.

Returning explicitly to the case $\varepsilon_q = v_F q$, the simple time dependence of the $\varphi(x,t)$ boson fields may be shown to extend to the time dependence of the coherent state $\exp(-\varphi(x,\ t)) = \exp(-\varphi(x - v_F t))$. Thus, the evolution in time of the exponentials of the boson fields results in a translation of the fields by $v_F\, t$.

Using a different derivation we can show explicitly how the Hamiltonian acts on the boson field state as a generator of translations. To begin, the commutators of the fields with the Hamiltonian may be rewritten in the form

$$\begin{aligned}
[\hat{H}_0, \varphi(x)] &= i v_F \partial\varphi(x)/\partial x\,, \\
[\hat{H}_0, \varphi^\dagger(x)] &= i v_F \partial\varphi^\dagger(x)/\partial x\,.
\end{aligned}$$

From this we can calculate the commutator of the Hamiltonian with the exponential of the $\varphi(x)$ fields, from which we obtain

$$\begin{aligned}
[\hat{H}_0, e^{-\varphi(x)}] &= e^{-\varphi(x)}\left(-[\hat{H}_0, \varphi(x)] + \frac{1}{2!}[[\hat{H}_0, \varphi(x)], \varphi(x)] + \ldots\right) \\
&= i v_F \partial e^{-\varphi(x)}/\partial x\ . \qquad (29)
\end{aligned}$$

Applying the Baker–Hausdorff identity to the time evolved exponential of the boson field, we get

$$\begin{aligned}
e^{i\hat{H}_0 t} e^{-\varphi(x)} e^{-i\hat{H}_0 t} &= e^{-\varphi(x)} - v_F t \frac{\partial e^{-\varphi(x)}}{\partial x} + (v_F t)^2\, \frac{1}{2!}\frac{\partial^2 e^{-\varphi(x)}}{\partial x^2} + \ldots \\
&= e^{-\varphi(x - v_F t)}. \qquad (30)
\end{aligned}$$

Similarly, we can show that

$$[\hat{H}_0, e^{\varphi^\dagger(x)}] = i v_F \partial e^{\varphi^\dagger(x)}/\partial x, \qquad (31)$$

$$e^{i\hat{H}_0 t} e^{\varphi^\dagger(x)} e^{-i\hat{H}_0 t} = e^{\varphi^\dagger(x - v_F t)}. \qquad (32)$$

The dependence of the fields on the combination $x \pm v_F t$ may also be determined from a Lorentz invariance of the Hamiltonian. The velocity v_F replaces the velocity of light as the fundamental velocity in the theory (Sénéchal 1999).

The Lorentz invariance remains even in the presence of delta function interactions, due to the separation and independence of the charge and spin sectors of the interacting model, and linear spectrum of the excitations. In this case, there exist two independent Lorentz invariant systems with velocities v_ρ and v_χ.

(3c) The Klein Factors

Here we calculate the commutators of the Klein factors $\hat{F}$, $\hat{F}^\dagger$, with the phase factor $\lambda(x)$, and also the time dependence of the Klein factors with the free boson Hamiltonian. First, we calculate the commutation relations between the Klein factors and the phase factor:

$$[\hat{F}, \lambda(x)] = [\hat{F}, \sqrt{\frac{2\pi}{L}} \exp\left(-i\frac{2\pi}{L}(\hat{N} - \frac{1}{2}\delta_B)x\right)]$$
$$= \sqrt{\frac{2\pi}{L}} e^{i\frac{\pi}{L}\delta_B x}[\hat{F}, e^{-i\frac{2\pi}{L}\hat{N}x}]. \tag{33}$$

By expanding the commutator, we obtain the following relation:

$$\hat{F}\lambda(x) = \left(1 + e^{-i\frac{2\pi}{L}x}\right)\lambda(x)\hat{F}. \tag{34}$$

The time dependence of the Klein factors depends only on the number operator part of the free boson Hamiltonian; thus,

$$\hat{F}(t) = e^{i\frac{\pi v_F}{L}\hat{N}(\hat{N}+1-\delta_B)t}\hat{F}e^{-i\frac{\pi v_F}{L}\hat{N}(\hat{N}+1-\delta_B)t} = \sqrt{\frac{L}{2\pi}}e^{-i\frac{2\pi}{L}v_F t}\lambda(v_F t)\hat{F}. \tag{35}$$

These phase factors have all previously been ignored in calculating expectation values of fermion operators (Mattis and Lieb 1965; Voit 1995; Delft and Schöller 1998), when taking the limit $L \to \infty$. In doing so we are making the presumption that the system has a fixed number of particles N, and in the thermodynamic limit $N/L \to \rho$, for some constant ρ.

If we take the phase factor for a fixed N, we can see that the phase term $2\pi N/L$ is just the Fermi wave number k_F. Extrapolating this to the general contribution of the $q = 0$ phase terms, we can now presume these to generate the chemical potential μ in the boson representation. As has been shown in Section 3*a*, the explicit chemical potential vanishes in the boson representation. Our above reasoning shows that it is retained in the formalism, only it is now related to the $q = 0$, or $\hat{N}$ operator, dependence.

To examine the finite size effects on this system, or if the particle number varies over the whole fermion Fock space, these terms must be explicitly included. However, in reducing a one-dimensional system to the linearised model we are only considering excitations close to the Fermi surface, which corresponds to obtaining the large distance behaviour of the correlation functions and Green functions. In this case we simply ignore the contribution of the Klein operators and phase factors, and add a term $e^{ik_F x}$ to account for the chemical potential in the correlation functions.

(3d) Finite Temperature Green Functions

The correlation functions $\langle\psi(x, t)\psi^\dagger(0, 0)\rangle_\beta$ and $\langle\psi^\dagger(0, 0)\psi(x, t)\rangle_\beta$ are easily calculated directly from the time dependent momentum representation of the fermion operators, followed by Fourier transforming over k. From this we obtain

$$\langle\psi(x,t)\psi^\dagger(0,0)\rangle_\beta = \int dk \frac{e^{ikx}e^{-i(\epsilon_k-\mu)t}}{1+e^{-\beta(\epsilon_k-\mu)}} = \frac{i\pi e^{ik_F x}}{\beta v_F \sinh\frac{\pi(x-v_F t)}{\beta v_F}}. \tag{36}$$

The retarded Green function, $G_R^\beta(x, t) = -i\theta(t)\langle[\psi(x, t), \psi^\dagger(0, 0)]_+\rangle_\beta$, can be calculated easily for the free fermion Hamiltonian, and for inverse temperature β. From this calculation we obtain

$$G_R^\beta(x,t) = -i\theta(t)\frac{2\pi}{L}\sum_k\left[\frac{e^{ikx}e^{-i(\epsilon_k-\mu)t}}{1+e^{-\beta(\epsilon_k-\mu)}} + \frac{e^{ikx}e^{-i(\epsilon_k-\mu)t}}{1+e^{\beta(\epsilon_k-\mu)}}\right], \tag{37}$$

and finally

$$G_R^\beta(x,t) = -i\theta(t)\frac{2\pi}{L}\sum_k e^{ikx}e^{-i(\epsilon_k-\mu)t} = -i\theta(t)e^{ik_Fx}\delta(x-v_Ft). \tag{38}$$

The calculation using bosonisation is more complicated for this simple case, but the generalisation to the interacting case is straight forward, whereas the fermionic case is not. By expressing the fermion operators $\hat\psi(x, t)$ in terms of the boson operators, we note that the Klein factors pick up a time dependence due to the number operators in the Hamiltonian, and the exponential of the boson operators obtains a time dependence from the bi-linear boson term; that is,

$$\begin{aligned}\hat\psi(x,t) &= e^{i\hat Ht}\hat\psi(x)e^{-i\hat Ht}\\ &= \sqrt{\frac{L}{2\pi}}e^{-i\frac{2\pi}{L}v_Ft}\lambda(v_Ft)\hat F\lambda(x)e^{\varphi^\dagger(x-v_Ft)}e^{-\varphi(x-v_Ft)}.\end{aligned} \tag{39}$$

The correlation functions are now given by

$$\begin{aligned}&\langle\psi(x,t)\psi^\dagger(0,0)\rangle_\beta \qquad\qquad (39a)\\ &= \left\langle e^{-i\frac{2\pi}{L}v_Ft}\sqrt{\tfrac{L}{2\pi}}\lambda(v_Ft)\hat F\lambda(x)\lambda^\dagger(0)\hat F^\dagger e^{\varphi^\dagger(x-v_Ft)}e^{-\varphi(x-v_Ft)}e^{-\varphi^\dagger(0)}e^{\varphi(0)}\right\rangle_\beta\\ &= \left(1+e^{-i\frac{2\pi}{L}x}\right)e^{-i\frac{2\pi}{L}v_Ft}\left\langle\lambda(x+v_Ft)e^{\varphi^\dagger(x-v_Ft)}e^{-\varphi(x-v_Ft)}e^{-\varphi^\dagger(0)}e^{\varphi(0)}\right\rangle_\beta.\end{aligned}$$

The term involving the fermion number operator will introduce phase factors in the sum over the boson ground states, which will not necessarily cancel with the terms in the partition function, giving phase factors related to the chemical potential. We ignore these for now, and concentrate on the expectation of the boson operators.

As the time dependence may be included by the exchange of x for $x - v_Ft$, from this point we only explicitly write the x dependence.

The expectation of the four exponential boson fields can be simplified by commuting the exponentials to move all the boson creation operators to the left of the annihilation operators. The commutation introduces a term, which we denote by

$$D(x,0) = e^{[\varphi(x),\varphi^\dagger(0)]} = \exp\left(\frac{2\pi}{L}\sum_q\frac{e^{iqx}}{q}\right), \tag{40}$$

that diverges in the limit of large system size. Therefore, this term must be included back into the expectation at a future stage of the calculation in order to obtain correct results.

The expectation of the exponentials of the boson fields can be solved simply by using a relation for expectation values of exponentials, where

$$\left\langle \exp\left(\sum_q A_q \hat{b}_q^\dagger\right) \exp\left(\sum_q B_q \hat{b}_q\right)\right\rangle = \exp\left(\sum_q A_q B_q (e^{\beta\omega_q} - 1)^{-1}\right) \tag{41}$$

in the case where the Hamiltonian is identical to that of a simple harmonic oscillator, so we can follow its solution (Mermin 1966). Ignoring the preceding phase factors generated by the Klein and λ terms, we may express the expectation of the boson fields in the form

$$\begin{aligned}\langle\psi(x)\psi^\dagger(0)\rangle_\beta \quad &\simeq D(x,0)\Big\langle \exp\left(\textstyle\sum_q \sqrt{\frac{2\pi}{Lq}}(1-e^{-iqx})\hat{b}_q^\dagger\right) \\ &\quad \times \exp\left(\textstyle\sum_q \sqrt{\frac{2\pi}{Lq}}(e^{iqx}-1)\hat{b}_q\right)\Big\rangle,\end{aligned} \tag{41a}$$

which gives the A_q and B_q terms in equation (41).

In the continuum limit the sum in the exponential in equation (41) becomes an integral, and the correlation functions can be evaluated explicitly by solving the integrals

$$2\int_0^\infty dq \frac{\cos qx - 1}{q(e^{\beta v_F q} - 1)} + \int_0^\infty \frac{dq}{q} e^{iqx}, \tag{42}$$

where the first integral gives the contribution of the expectation of the fields, and the second integral comes from the $D(x, 0)$ term generated by the commutator.

By combining the two terms we may solve the integral explicitly, using the integral representation of the logarithm of the gamma function given by Ryzhik and Gradstein (1965), and we obtain the expression

$$\int_0^\infty \frac{2dq}{q}\left[\frac{\cos qx - 1}{e^{\beta v_F q} - 1} + \frac{e^{iqx}}{2}\right] = \ln\left[\Gamma\left(-\frac{ix}{\beta v_F}\right)\Gamma\left(1 + \frac{ix}{\beta v_F}\right)\right] + Q, \tag{43}$$

where the divergent constant Q is given by

$$Q = \int_0^\infty \frac{dq}{q} e^{-\beta v_F q}. \tag{44}$$

The divergent constant Q does not depend on x (or t), unlike the term $D(x, t)$. Thus, it may now be treated as a normalisation factor that can be essentially ignored, as it plays no part in the dynamics or spatial dependence of the correlation functions.

The exponential of the right-hand side of equation (43) may be rewritten in the form (Spain and Smith 1970)

$$\langle\psi(x)\psi^\dagger(0)\rangle_\beta \simeq \frac{i\pi}{\beta v_F \sinh \pi x/\beta v_F}. \tag{45}$$

This equation again gives us the fermion correlation function, except for the phase factor $e^{ik_F x}$ contributed by the chemical potential, which we have ignored by omitting the leading phase factors in going from equation (40) to (42).

The commuting of the two fermion operators within the expectation changes the divergent factor $D(x, t)$ to $D(-x, -t)$, which merely results in a change in sign of x (and t) in the integral of equation (43). From this the relation

$$\langle\psi^\dagger(0)\psi(x,t)\rangle_\beta = \langle\psi(-x,-t)\psi^\dagger(0)\rangle_\beta \tag{46}$$

follows immediately. This simplifies the derivation of the retarded Green functions, which contain a sum of these terms.

The time dependent correlation functions, as well as all the finite temperature Green functions, may now be calculated. To obtain the time dependence of any of these functions we simply replace the variable x by $x - v_F t$.

By introducing a convergence term, $e^{-\alpha q}$, the integral may also be solved in closed form as a ratio of gamma functions and a term $i/(x + i\alpha)$, which reduce to the form $\pi/\sinh\pi x$ in the limit $\alpha \to 0$ (Emery 1979). This result uses non-normal ordered terms as boson operators which require the introduction of a term α^{-1} to retain the fermion field anti-commutation relations. The convergence term introduced in the solution of the integral then cancels with the α^{-1} term to produce the desired result.

4. Conclusions

We have shown that the well-known bosonisation method to study one-dimensional electron systems can be easily extended to finite temperatures. The result obtained, see equation (45), was already obtained using conformal field theory (Cardy 1984). It is widely recognised (Christe and Henkel 1993) that equation (45) is one of the most important results of the application of conformal invariance to critical phenomena. The reason for this is that (45) defines a correlation length, $\xi = v_F / 2\pi T x$. Hence, every conformal invariant model is only critical at zero temperature (the quantum critical point). The fact that the gap associated with this correlation length is vanishing as we approach the zero temperature limit may be a simple way to experimentally detect non-Fermi liquids at finite temperature.

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